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# A THREE-CURVES THEOREM FOR VISCOSITY SUBSOLUTIONS OF PARABOLIC EQUATIONS

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ABSTRACT. We prove a three-curves theorem for viscosity subsolutions of fully nonlinear uniformly parabolic equations $F(D^2u, t, x) - u_t = 0$.

## 0. INTRODUCTION

Three-curves theorems play a central role in the qualitative theory of partial differential equations, starting with Hadamard's classical three-circles theorem for the real part of an analytic function. Briefly stated, this theorem says that if $\Delta u \geq 0$ in a domain $\Omega \subset \mathbb{R}^2$ containing two concentric circles of radii $r_1, r_2$ and the region between them and if $M(r)$ denotes the maximum of $u$ on any concentric circle of radius $r$, then $M(r)$ is a convex function of $\log r$. An application of this is Liouville's theorem: functions harmonic in the plane, except possibly at one point and bounded either above or below, are constant. In $n$ dimensions, the three-spheres theorem states that if $\Delta u \geq 0$ in a domain $\Omega \subset \mathbb{R}^n$ containing two concentric spheres of radii $r_1, r_2$ and the region between them and if $M(r)$ denotes the maximum of $u$ on any concentric sphere of radius $r$, then $M(r)$ is a convex function of $r^{2-n}$. A three-cylinders theorem for linear parabolic equations appears in [G].

In this paper we prove the fully nonlinear analogue of a three-curves theorem which appears in [PW] for the 1-dimensional heat equation. Specifically, in Theorem 1.1, we prove the following. Suppose $u$ is a viscosity subsolution of the uniformly parabolic nonlinear equation $F(D^2u, t, x) - u_t = 0$ (with $F(0, \cdot) = 0$) in any region containing two concentric concave paraboloids of opening $2\rho_1^{-2}$ and $2\rho_2^{-2}$ and the region between them (see below for more details). If $M(\rho)$ denotes the maximum of $u$ on any concentric concave paraboloid of opening $2\rho^{-2}$, with $\rho_1 < \rho < \rho_2$, then there exists an a priori function $\psi(\rho)$, such that $M(\rho)$ is a convex function of $\psi(\rho)$.

Let $M > 0$, $x \in \mathbb{R}^n$. We say that $P(x)$ is a paraboloid of *opening* $M$ if $P(x) = \pm\frac{M}{2}|x|^2 + l(x) + l_0$, where $l$ is linear and $l_0$ is constant. $P(x)$ is convex if $+$ appears and concave if $-$ appears. So for $t_0, \rho > 0$, the equation $t = t_0 - \frac{|x|^2}{\rho^2}$ denotes the graph of a concave paraboloid of opening $\frac{2}{\rho^2}$ with vertex at $(t_0, 0) \in \mathbb{R}^{n+1}$, which we will henceforth write as $\rho = \frac{|x|}{\sqrt{t_0 - t}}$. By *concentric* concave paraboloids of opening $2\rho_1^{-2}$ and $2\rho_2^{-2}$, we mean these paraboloids have common vertex $(t_0, 0)$.

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Our region $Q \subset \mathbb{R}^{n+1}$ is described as follows. $Q$ is bounded below by the line $t = 0$ and above by the line $t = t'$, where $t' < t_0$. $Q$ is bounded laterally by the arcs of the paraboloids $\rho_1 = \frac{|x|}{\sqrt{t_0 - t}}$ and $\rho_2 = \frac{|x|}{\sqrt{t_0 - t}}$ of openings $2\rho_1^{-2}$ and $2\rho_2^{-2}$ respectively, with $\rho_1 < \rho_2$. Geometrically, $Q$ is a concave paraboloid shell, truncated just below the vertex $(t_0, 0)$. For $\rho_1 \leq \rho \leq \rho_2$, define the functions

$$\begin{aligned} M_1(\rho) &= \max_{\substack{|x| = \rho\sqrt{t_0 - t} \\ 0 \leq t \leq t'}} u(t, x), \\ M_2 &= \max_{\rho_1\sqrt{t_0} \leq |x| \leq \rho_2\sqrt{t_0}} u(0, x), \\ M(\rho) &= \max\{M_1(\rho), M_2\}. \end{aligned}$$

Hence $M(\rho) = \max_Q u$.

We now make a few brief comments about viscosity subsolutions of parabolic equations. For $f \in C(Q)$ and positive constants $\lambda \leq \Lambda$, $\underline{S}(\lambda, \Lambda, f)$ denotes the class of viscosity subsolutions of the equation $\mathcal{M}^+(D^2u, \lambda, \Lambda) - u_t = f(t, x)$. That is, $u \in C(Q)$ and satisfies $\mathcal{M}^+(D^2u, \lambda, \Lambda) - u_t \geq f(t, x)$ *in the viscosity sense*, where for any real $n \times n$ symmetric matrix $M$

$$\mathcal{M}^+(M, \lambda, \Lambda) = \mathcal{M}^+(M) = \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i,$$

where $e_i = e_i(M)$ are the eigenvalues of $M$. By diagonalizing $M$, it can be shown that $\mathcal{M}^+$ is subadditive. That is, $\mathcal{M}^+(M + N) \leq \mathcal{M}^+(M) + \mathcal{M}^+(N)$ for any symmetric matrices $M, N$.

In general, a function $u$, continuous in a bounded domain $Q \subset \mathbb{R}^{n+1}$, is a *viscosity subsolution* of the fully nonlinear parabolic equation

$$F(D^2u(t, x), t, x) - u_t(t, x) = f(t, x), \quad (t, x) \in Q,$$

if the following condition holds: if $(t_0, x_0) \in Q$, $\psi \in C^2(Q)$ and $u - \psi$ has a local maximum at $(t_0, x_0)$ (i.e., $\psi$ *touches* $u$ *from above* at $(t_0, x_0)$), then

$$F(D^2\psi(t_0, x_0), t_0, x_0) - \psi_t(t_0, x_0) \geq f(t_0, x_0).$$

Finally, it is known (see Proposition 2.13 [CC]) that viscosity subsolutions of $F(D^2u, t, x) - u_t = f(t, x)$ belong to the class $\underline{S}(\frac{\lambda}{n}, \Lambda, f(t, x) - F(0, t, x))$. So if $u$ is a viscosity subsolution of the uniformly parabolic nonlinear equation $F(D^2u, t, x) - u_t = 0$ and $F(0, \cdot) = 0$, then $u \in \underline{S}(\frac{\lambda}{n}, \Lambda, 0)$. Our Theorem 1.1 applies to this class of functions. See [CC] (Chapter 2) and [W] (Chapter 3) for a complete discussion about viscosity solutions of fully nonlinear equations.

We will need the following lemma, which appears in [CC] for the elliptic case and in [W] for the parabolic case.

**Lemma 0.1.** *Let $u \in \underline{S}(\lambda, \Lambda, f)$, $\varphi \in C^2(Q)$ and suppose $\mathcal{M}^+(D^2\varphi(z), \lambda, \Lambda) - \varphi_t(z) \leq g(z)\ \forall\, z = (t, x) \in Q$. Then $u - \varphi \in \underline{S}(\lambda, \Lambda, f - g)$ in $Q$.*

*Proof.* Let $\psi$ be any $C^2(Q)$ function touching the graph of $u - \varphi$ from above at the point $z_0 = (t_0, x_0) \in Q$. Then $\psi + \varphi \in C^2(Q)$ and touches the graph of $u$ from above at $z_0$. Since $u \in \underline{S}(\lambda, \Lambda, f)$, we have $\mathcal{M}^+(D^2(\psi + \varphi)(z_0)) - (\psi + \varphi)_t(z_0) \geq f(z_0)$. By the subadditivity of $\mathcal{M}^+$, this gives $\mathcal{M}^+(D^2\psi(z_0)) + \mathcal{M}^+(\varphi(z_0)) - \psi_t(z_0) - \varphi_t(z_0) \geq f(z_0)$, which by assumption on $\varphi$ yields $\mathcal{M}^+(D^2\psi(z_0)) - \psi_t(z_0) \geq f(z_0) - g(z_0)$. $\square$

## 1. Main theorem

Before we state Theorem 1.1, we make some comments concerning the maximum principle which relate to our theorem. For simplicity, we make these remarks for the linear setting, $Lu - u_t \geq 0$, where $L := a^{ij}(t,x)\frac{\partial}{\partial x^i \partial x^j}$, the $a^{ij}(t,x)$ are measurable and satisfy $\lambda|\xi|^2 \leq a^{ij}(t,x)\xi^i\xi^j \leq \Lambda|\xi|^2$, $\forall\, \xi \in \mathbb{R}^n$. The same comments hold true for the class $\underline{S}(\lambda, \Lambda, 0)$.

Let $M(\rho)$ be defined as above. If $u$ is nonconstant and satisfies $Lu - u_t \geq 0$ in $Q$, then by the maximum principle, $M(\rho)$ cannot be constant in any interval, nor have an interior maximum. Moreover, $M(\rho)$ cannot have a relative maximum (since $u$ is a subsolution) and so has at most one minimum. Hence $M(\rho)$ either always increases, always decreases or first decreases and then increases.

Three-curves theorems rely heavily on the maximum principle. In our three-paraboloids theorem, we use the maximum principle in the following way. Suppose $Lu - u_t \geq 0$ in Q. We define a function $\varphi(\rho) = a + b\psi(p)$, where constants $a$, $b$ (with $b > 0$) are chosen so that $\varphi(\rho_1) = M(\rho_1)$, $\varphi(\rho_2) = M(\rho_2)$ and $L\varphi - \varphi_t \leq 0$ in $Q$. This gives $L\varphi - \varphi_t \leq Lu - u_t$ in $Q$ and $u \leq \varphi$ on $\partial' Q$. By the maximum principle, $u \leq \varphi$ in $Q$ and hence $M(\rho) \leq \varphi(\rho)$ for $\rho \in (\rho_1, \rho_2)$.

But to do this, since $L\varphi - \varphi_t = b(L\psi - \psi_t)$ and $b > 0$, we need $\psi$ to satisfy $L\psi - \psi_t \leq 0$. Yet $b = \frac{M(\rho_2) - M(\rho_1)}{\psi(\rho_2) - \psi(\rho_1)}$ and $b > 0$ implies that $\psi(\rho)$ is increasing or decreasing with $M(\rho)$. Thus we need to find a function $\psi(\rho)$ which is an increasing supersolution and another function $\psi(\rho)$ which is a decreasing supersolution. We denote the increasing supersolution by $\psi_+(\rho)$ and the decreasing supersolution by $\psi_-(\rho)$. The explicit forms of $\psi_+$, $\psi_-$ in the fully nonlinear setting are given in equations (3) and (4). Hence in our nonlinear setting, it is not a single function $\psi$ but a pair $(\psi_+, \psi_-)$ which satisfies the conclusion of our Theorem 1.1. This unavoidable feature occurs even in the linear case for subsolutions of uniformly *elliptic* equations with measurable coefficients $Lu := a^{ij}(x)u_{x^i x^j} = 0$ in the simple case of spheres $|x| = r$, where $r \in (r_1, r_2)$. See Chapter 2.12 in [PW] for a complete discussion of three-curves theorems for elliptic equations.

Of course, if $\psi$ is a *solution* to the differential equation, then so is $\varphi$ (independent of the sign of $b$) and the single function $\psi$ will satisfy the desired convexity inequality. It is this situation that lends itself most easily to applications. In particular, for the three-spheres theorem for $\Delta u \geq 0$ in a spherical region in $\mathbb{R}^n$ $(n \geq 3)$, $\psi(r) = r^{2-n}$, while for the three-paraboloids theorem for $\Delta u - u_t \geq 0$, the single $\psi$ that works is $\psi(\rho) = \int_\alpha^\rho \frac{e^{r^2/4}}{r^{n-1}} dr$. See equation (6) in our proof of Tychonov's theorem, which is an application of the three-paraboloids theorem for the heat equation.

**Theorem 1.1.** *Let $u \in \underline{S} = \underline{S}(\lambda, \Lambda, 0)$ in a domain $Q \subset \mathbb{R}^{n+1}$ containing two concave concentric parabaloids of opening $2\rho_1^{-2}$ and $2\rho_2^{-2}$ and the region between them. If $M(\rho)$ denotes the maximum of $u$ on any concentric concave paraboloid of opening $2\rho^{-2}$, with $\rho_1 < \rho < \rho_2$, then there exists a differentiable function $\psi(\rho)$, depending only $n, \lambda, \Lambda$ and $\rho$, such that*

$$M(\rho) \leq \frac{M(\rho_1)(\psi(\rho_2) - \psi(\rho)) + M(\rho_2)(\psi(\rho) - \psi(\rho_1))}{\psi(\rho_2) - \psi(\rho_1)}. \tag{1}$$

*Proof.* For $\rho = \frac{|x|}{\sqrt{t_0 - t}}$, define the function $\varphi(\rho) = a + b\psi(\rho)$, where constants $a$, $b$ $(b > 0)$ are chosen so that $\varphi(\rho_1) = M(\rho_1)$ and $\varphi(\rho_2) = M(\rho_2)$. We will find $\psi$

such that $v = u - \varphi \in \underline{S}(\lambda, \Lambda, 0)$ and then apply the maximum principle to $v$ on $Q$. Since $u \in \underline{S}(\lambda, \Lambda, 0)$ and $\varphi \in C^2(Q)$, by Lemma 0.1, we need only show that $\mathcal{M}^+(D^2\varphi(t,x), \lambda, \Lambda) - \varphi_t(t,x) \leq 0,\ \forall\, (t,x) \in Q$.

From $\varphi_{x_i x_j} = b\left\{\psi''(\rho)\rho_{x_i}\rho_{x_j} + \psi'(\rho)\rho_{x_i x_j}\right\}$ and $\varphi_t = b\,\psi'(\rho)\rho_t$, direct calculation gives

(2)

$$\varphi_{x_i x_j}(t,x) = \frac{b}{|x|^2(t_0 - t)}\left\{\psi'' x_i x_j + \frac{\psi'}{\rho}\left(\delta_{ij}|x|^2 - x_i x_j\right)\right\}, \quad \varphi_t(t,x) = \frac{b\psi' \cdot \rho}{2(t_0 - t)}.$$

That is,

$$D^2\varphi(t,x) = \frac{b}{|x|^2(t_0 - t)}\left\{x^T x\left(\psi'' - \frac{\psi'}{\rho}\right) + \frac{\psi'}{\rho}|x|^2 I\right\}$$

and for the matrix inside the braces, $\frac{\psi'}{\rho}|x|^2$ is an eigenvalue of multiplicity $n-1$, while $|x|^2\psi''$ is an eigenvalue of multiplicity 1. Say $\psi' \geq 0$. Then if $\psi'' \geq 0$,

$$\begin{aligned}\mathcal{M}^+(D^2\varphi(t,x)) &= \frac{b}{|x|^2(t_0 - t)}\left\{\Lambda(n-1)\frac{\psi'}{\rho}|x|^2 + \Lambda|x|^2\psi''\right\}\\ &= \frac{b\Lambda}{t_0 - t}\left\{(n-1)\frac{\psi'}{\rho} + \psi''\right\},\end{aligned}$$

and hence

$$\begin{aligned}\mathcal{M}^+(D^2\varphi(t,x)) - \varphi_t(t,x) &= \frac{b\Lambda}{t_0 - t}\left\{(n-1)\frac{\psi'}{\rho} + \psi'' - \frac{\psi'\rho}{2\Lambda}\right\}\\ &= \frac{b\Lambda}{t_0 - t}\left\{\psi'' + \psi'\left(\frac{n-1}{\rho} - \frac{\rho}{2\Lambda}\right)\right\},\end{aligned}$$

while, if $\psi'' < 0$,

$$\begin{aligned}\mathcal{M}^+(D^2\varphi(t,x)) &= \frac{b}{|x|^2(t_0 - t)}\left\{\Lambda(n-1)\frac{\psi'}{\rho}|x|^2 + \lambda|x|^2\psi''\right\}\\ &= \frac{b\lambda}{t_0 - t}\left\{\psi'' + \frac{\Lambda(n-1)}{\lambda}\cdot\frac{\psi'}{\rho}\right\},\end{aligned}$$

and hence

$$\begin{aligned}\mathcal{M}^+(D^2\varphi(t,x)) - \varphi_t(t,x) &= \frac{b\lambda}{t_0 - t}\left\{\psi'' + \frac{\Lambda(n-1)}{\lambda}\cdot\frac{\psi'}{\rho} - \frac{\psi'\rho}{2\lambda}\right\}\\ &= \frac{b\lambda}{t_0 - t}\left\{\psi'' + \psi'\left(\frac{c_1}{\rho} - \frac{\rho}{2\lambda}\right)\right\},\end{aligned}$$

where $c_1 = \frac{\Lambda(n-1)}{\lambda}$. Since $n - 1 \leq c_1$, both cases for $\psi' \geq 0$ give

$$\mathcal{M}^+(D^2\varphi(t,x)) - \varphi_t(t,x) \leq \frac{bK}{t_0 - t}\left\{\psi'' + \psi'\left(\frac{c_1}{\rho} - \frac{\rho}{2\Lambda}\right)\right\} = 0 \tag{3}$$

for

$$\psi = \psi_+(\rho) := \int_\alpha^\rho \frac{e^{r^2/4\Lambda}}{r^{c_1}}\,dr$$

and $K$ is either $\lambda$ or $\Lambda$. Now suppose $\psi' \leq 0$. If $\psi'' \geq 0$, then as before

$$\mathcal{M}^+(D^2\varphi) = \frac{b\Lambda}{t_0 - t}\left\{\psi'' + \frac{\lambda(n-1)}{\Lambda}\cdot\frac{\psi'}{\rho}\right\},$$

and hence

$$\mathcal{M}^+(D^2\varphi) - \varphi_t = \frac{b\Lambda}{t_0 - t}\left\{\psi'' + \psi'\left(\frac{c_2}{\rho} - \frac{\rho}{2\Lambda}\right)\right\},$$

where $c_2 = \frac{\lambda(n-1)}{\Lambda}$, while, if $\psi'' < 0$,

$$\mathcal{M}^+(D^2\varphi) = \frac{b\lambda}{t_0 - t}\left\{\psi'' + (n-1)\frac{\psi'}{\rho}\right\},$$

thus

$$\mathcal{M}^+(D^2\varphi) - \varphi_t = \frac{b\lambda}{t_0 - t}\left\{\psi'' + \psi'\left(\frac{n-1}{\rho} - \frac{\rho}{2\lambda}\right)\right\}.$$

Since $c_2 \le n-1$, both cases for $\psi' \le 0$ yield

$$\mathcal{M}^+(D^2\varphi(t,x)) - \varphi_t(t,x) \le \frac{bK}{t_0 - t}\left\{\psi'' + \psi'\left(\frac{c_2}{\rho} - \frac{\rho}{2\lambda}\right)\right\} = 0 \tag{4}$$

for

$$\psi = \psi_-(\rho) := \int_\rho^\beta \frac{e^{r^2/4\lambda}}{r^{c_2}}\,dr.$$

Thus in all cases, we have a function $\psi(\rho) = \psi(\rho, n, \lambda, \Lambda)$ for which $\mathcal{M}^+(D^2\varphi, \lambda, \Lambda) - \varphi_t \le 0$ in $Q$, which setting $v = u - \varphi$, gives $v \in \underline{S}(0)$ in $Q$. We now show $v \le 0$ on $\partial' Q$. Recall that $M(\rho) = \max\{M_1(\rho), M_2\}$, where for $\rho_1 \le \rho \le \rho_2$,

$$M_1(\rho) = \max_{\substack{|x| = \rho\sqrt{t_0 - t}\\ 0 \le t \le t'}} u(t,x), \qquad M_2 = \max_{\rho_1\sqrt{t_0} \le |x| \le \rho_2\sqrt{t_0}} u(0,x).$$

On $|x| = \rho_1\sqrt{t_0 - t}$, $v = u - \varphi(\rho_1) \le M_1(\rho_1) - \varphi(\rho_1) \le M(\rho_1) - \varphi(\rho_1) = 0$. The same inequalities show that $v \le 0$ on $|x| = \rho_2\sqrt{t_0 - t}$. Finally, on $\{t = 0\} \cap Q$, we have $v(0,x) = u(0,x) - \varphi(\rho) \le M_2 - \varphi(\rho) \le 0$. Thus $v \le 0$ on $\partial' Q$ and hence by the maximum principle for viscosity subsolutions, $v \le 0$ in $Q$. That is, $u \le \varphi$ in $Q$. Hence $M(\rho) \le \varphi(\rho)$, which gives us (1). □

If $u \in \overline{S}(\lambda, \Lambda, 0)$, Theorem 1.1, applied to $-u$, along with the identity $\max(-w) = -\min w$, immediately yields (1) with the inequality reversed and $m(\rho)$ in place of $M(\rho)$, where $m(\rho) = \min_Q u$. Since $S(\lambda, \Lambda, 0) = \underline{S}(\lambda, \Lambda, 0) \cap \overline{S}(\lambda, \Lambda, 0)$, setting $\omega(\rho) = M(\rho) - m(\rho)$ and adding these inequalities gives the following convexity inequality for the oscillation of viscosity *solutions*.

**Corollary 1.2.** *Let $u \in S(\lambda, \Lambda, 0)$ in a domain $Q \subset \mathbb{R}^{n+1}$ containing two concave concentric paraboloids of opening $2\rho_1^{-2}$ and $2\rho_2^{-2}$ and the region between them. If $\omega(\rho)$ denotes the oscillation of $u$ on any concentric concave paraboloid of opening $2\rho^{-2}$, with $\rho_1 < \rho < \rho_2$, then*

$$\omega(\rho) \le \frac{\omega(\rho_1)(\psi(\rho_2) - \psi(\rho)) + \omega(\rho_2)(\psi(\rho) - \psi(\rho_1))}{\psi(\rho_2) - \psi(\rho_1)}. \tag{5}$$

In the linear setting, a simplified version of Theorem 1.1 yields a uniqueness result for *slowly increasing* solutions of the nonhomogeneous Dirichlet problem

$$\begin{cases} \Delta u - u_t = f, & (t,x) \in (0,T) \times \mathbb{R}^n, \\ u(0,x) = g(x), & x \in \mathbb{R}^n, \end{cases}$$

originally due to Tychonov. Our proof is a generalization of an argument which appears in [PW].

**Theorem 1.3.** *Let $u, w \in C(\overline{Q})$ be solutions of $\Delta u - u_t = f$ in the strip $Q = (0,T) \times \mathbb{R}^n$ with $u(0,x) = w(0,x) = g(x)$. If there are constants $c_1, c_2$ such that*

$$|u(t,x)|,\, |w(t,x)| \leq c_1 e^{c_2|x|^2} \quad \textit{uniformly for } t \in [0,T],$$

*then $u \equiv w$ in $Q$.*

*Proof.* If $v$ satisfies $\Delta v - v_t = 0$ in the paraboloid region $Q$ of Theorem 1.1, then setting $\varphi(\rho) = a + b\psi(\rho)$, an easy calculation using (2) shows

$$\Delta\varphi(t,x) - \varphi_t(t,x) = \frac{b}{t_0 - t}\left\{\psi'' + \psi'\left(\frac{n-1}{\rho} - \frac{\rho}{2}\right)\right\} = 0 \tag{6}$$

for

$$\psi(\rho) = \int_\alpha^\rho \frac{e^{r^2/4}}{r^{n-1}}\, dr,$$

and thus we obtain convexity inequality (5) for $\omega(\rho) = \operatorname{osc}_Q v$ and $\psi(\rho)$. So for $u, w$ in our theorem, set $v = u - w$, put $t_0 < \frac{1}{4c_2}$ and apply inequality (5) to $v$ in $Q_1 = [0, \frac{t_0}{2}] \times \mathbb{R}^n$, where $\Delta v - v_t = 0$ and $v(0,x) = 0$. Now let $\rho_2 \to \infty$ in (5). From the trivial inequality $\operatorname{osc} v \leq 2 \max v$ we have $\omega(\rho_2) \leq 4c_1 e^{c_2\rho_2^2(t_0 - t)}$. Since $\psi'(\rho_2) = \rho_2^{1-n} e^{\frac{\rho_2^2}{4}}$ with $c_2(t_0 - t) - \frac{1}{4} < 0$, we have $\lim\limits_{\rho_2 \to \infty} \frac{\omega(\rho_2)}{\psi(\rho_2)} = 0$, which by (5) yields $\omega(\rho) \leq \omega(\rho_1)$. Letting $\rho_1 \to 0$, we see that the oscillation of $v$ in $Q_1$ occurs on the hyperplane $x = 0$, which by the maximum principle implies $\omega \equiv 0$ in $Q_1$. Hence $v$ is constant in $Q_1$. But $v(0,x) = 0$ implies this constant must be 0, so $v \equiv 0$ in $Q_1$. Repeating this process, now using $t = \frac{t_0}{2}$ as the initial line, we find that $v \equiv 0$ in $Q_2 = [\frac{t_0}{2}, t_0] \times \mathbb{R}^n$. After a finite number of steps, we get $v \equiv 0$ in $Q$ and hence $u \equiv w$ in $Q$. □

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