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# LINEAR PERTURBATIONS OF A NONOSCILLATORY SECOND ORDER DIFFERENTIAL EQUATION II

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ABSTRACT. Let $y_1$ and $y_2$ be principal and nonprincipal solutions of the nonoscillatory differential equation $(r(t)y')' + f(t)y = 0$. In an earlier paper we showed that if $\int^\infty (f-g)y_1y_2\,dt$ converges (perhaps conditionally), and a related improper integral converges absolutely and sufficently rapidly, then the differential equation $(r(t)x')' + g(t)x = 0$ has solutions $x_1$ and $x_2$ that behave asymptotically like $y_1$ and $y_2$. Here we consider the case where $\int^\infty (f-g)y_2^2\,dt$ converges (perhaps conditionally) without any additional assumption requiring absolute convergence.

## 1. Introduction

We consider the differential equation

$$(r(t)x')' + g(t)x = 0 \tag{1}$$

as a perturbation of

$$(r(t)y')' + f(t)y = 0, \tag{2}$$

under the following standing assumption.

**Assumption A.** *Let $r$ and $f$ be real-valued and continuous, with $r > 0$, on $[a, \infty)$. Suppose that (2) is nonoscillatory at infinity. Let $g$ be continuous and possibly complex-valued on $[a, \infty)$.*

It is known [4, p. 355] that since (2) is nonoscillatory at infinity, it has solutions $y_1$ and $y_2$ which are positive on $[b, \infty)$ for some $b \geq a$ and satisfy the following conditions:

$$r(y_1y_2' - y_1'y_2) = 1, \quad t \geq a, \tag{3}$$

$$\lim_{t\to\infty} \frac{y_2(t)}{y_1(t)} = \infty. \tag{4}$$

Without loss of generality we let $b = a$. Henceforth $t \geq a$. It is convenient to define

$$\rho = y_2/y_1. \tag{5}$$

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From (3) and (4),

$$\rho' = 1/ry_1^2 > 0 \quad \text{and} \quad \lim_{t\to\infty} \rho(t) = \infty. \tag{6}$$

We use the Landau symbols "$o$" and "$O$" in the standard way to denote behavior as $t \to \infty$. In [6] we proved the following theorem.

**Theorem 1.** *Suppose that* $\int^\infty (f-g)y_1y_2\,dt$ *converges (perhaps conditionally) and*

$$\sup_{\tau\ge t}\left|\int_\tau^\infty (f-g)y_1y_2\,ds\right| \le \phi(t), \tag{7}$$

*where* $\phi(t) \to 0$ *monotonically as* $t \to \infty$. *Define*

$$G(t) = \int_t^\infty (f-g)y_1^2\,ds, \tag{8}$$

*and suppose that*

$$\int^\infty |G|\phi\rho'\,dt < \infty$$

*and*

$$\limsup_{t\to\infty} (\phi(t))^{-1} \int_t^\infty |G|\phi\rho'\,ds = A < 1/3. \tag{9}$$

*Then (1) has a solution* $x_1$ *such that*

$$x_1 = y_1(1 + O(\phi))$$

*and*

$$(x_1/y_1)' = O(\phi\rho'/\rho),$$

*and a solution* $x_2$ *such that*

$$x_2 = y_2(1 + O(\phi_m))$$

*and*

$$(x_2/y_2)' = O(\phi_m\rho'/\rho),$$

*where*

$$\phi_m = \max\{\phi, \hat{\phi}\}$$

*with*

$$\hat{\phi}(t) = \frac{1}{\rho(t)} \int_a^t \rho'\phi\,ds.$$

This result was an improvement on a theorem of Hartman and Wintner [4, p. 379], and it was subsequently improved by Chen [1] and Šimša [5]. (For more on the Hartman-Wintner problem, see [2] and [3].) In this continuation of [6] we consider the case where $\int^\infty (f-g)y_2^2\,dt$ converges, perhaps conditionally. To motivate the present work, we first apply Theorem 1 under this assumption.

Let

$$H(t) = \int_t^\infty (f-g)y_1y_2\,ds, \tag{10}$$

and recall from (7) that

$$\sup_{\tau\ge t}\{|H(\tau)|\} \le \phi(t).$$

Let

$$I(t)=\int_t^\infty (f-g)y_2^2\,ds, \tag{11}$$

and suppose that

$$\sup_{\tau\ge t}\{|I(\tau)|\}\le \sigma(t), \tag{12}$$

where $\sigma(t)\to 0$ monotonically as $t\to\infty$. From (8), (10), and (11),

$$H(t)=-\int_t^\infty \frac{I'}{\rho}\,ds=\frac{I(t)}{\rho(t)}+\int_t^\infty I\left(\frac{1}{\rho}\right)'\,ds \tag{13}$$

and

$$G(t)=-\int_t^\infty \frac{I'}{\rho^2}\,ds=\frac{I(t)}{\rho^2(t)}+\int_t^\infty I\left(\frac{1}{\rho^2}\right)'\,ds,$$

so

$$|H(t)|\le 2\sigma(t)/\rho(t)\quad\text{and}\quad |G(t)|\le 2\sigma(t)/\rho^2(t). \tag{14}$$

It is straightforward to verify that (9) holds with $\phi=\sigma/\rho$ and $A=0$. Therefore Theorem 1 implies that (1) has solutions $x_1$ and $x_2$ such that

$$x_1=y_1(1+O(\sigma/\rho)), \tag{15}$$

$$(x_1/y_1)'=O(\sigma\rho'/\rho^2), \tag{16}$$

$$x_2=y_2(1+O(\hat\phi)), \tag{17}$$

and

$$(x_2/y_2)'=O(\hat\phi\rho'/\rho), \tag{18}$$

with

$$\hat\phi(t)=\frac{1}{\rho(t)}\int_a^t \frac{\sigma\rho'}{\rho}\,ds.$$

At best, (17) and (18) imply that

$$x_2=y_2(1+O(1/\rho))$$

and

$$(x_2/y_2)'=O(\rho'/\rho^2)$$

if $\int_a^\infty \sigma\rho'/\rho\,ds<\infty$, which may be false. Among other things, we will show that (17) and (18) can be replaced by

$$x_2=y_2(1+O(\sigma/\rho)) \tag{19}$$

and

$$(x_2/y_2)'=O(\sigma\rho'/\rho^2). \tag{20}$$

These two equations are improvements over (17) and (18), since $\lim_{t\to\infty}\rho(t)\hat\phi(t)/\sigma(t)=\infty$ in any case. In fact, it can be seen from (15), (16), (19), and (20) that $(x_i/y_i)-1$, $i=1$, 2, approach zero at the same rate as $t\to\infty$, as do $(x_i/y_i)'$, $i=1$, 2. We also note that the results of these four equations can be written as

$$x_i/y_i=1+O(\sigma y_1/y_2)\quad\text{and}\quad (x_i/y_i)'=O(\sigma/ry_2^2),\quad i=1,2.$$

## 2. Main results

**Theorem 2.** *Suppose that* $\int^{\infty}(f-g)y_2^2\,dt$ *converges. Let* $I$ *and* $\sigma$ *be as in (11) and (12). Then (1) has a solution* $x_1$ *that satisfies (15) and (16), and a solution* $x_2$ *such that*

$$\frac{x_2-y_2}{y_1}=O(\sigma) \tag{21}$$

*and*

$$\left(\frac{x_2-y_2}{y_1}\right)'=O\left(\frac{\sigma\rho'}{\rho}\right). \tag{22}$$

*Proof.* We have already proved the assertion concerning $x_1$. For the assertion concerning $x_2$, we use the contraction mapping theorem. If

$$x_2(t)=y_2(t)+\int_t^{\infty}(y_2(s)y_1(t)-y_1(s)y_2(t))(f(s)-g(s))x_2(s)\,ds, \tag{23}$$

then $x_2$ satisfies (1). Although this suggests a transformation to work with, it is better to use a transformation with the fixed point $\zeta$, where

$$\zeta=(x_2-y_2)/y_1.$$

Rewriting (23) in terms of $\zeta$ yields

$$\begin{aligned}\zeta(t) \quad &= \quad \int_t^{\infty}(y_2(s)-y_1(s)\rho(t))(f(s)-g(s))y_2(s)\,ds\\ &\quad + \int_t^{\infty}(y_2(s)-y_1(s)\rho(t))(f(s)-g(s))y_1(s)\zeta(s)\,ds.\end{aligned}$$

We use the transformation $\mathcal{T}z=Q+\mathcal{L}z$, where

$$Q(t)=\int_t^{\infty}(y_2(s)-y_1(s)\rho(t))(f(s)-g(s))y_2(s)\,ds$$

and

$$(\mathcal{L}z)(t)=\int_t^{\infty}(y_2(s)-y_1(s)\rho(t))(f(s)-g(s))y_1(s)z(s)\,ds.$$

From (10), (11), and (13),

$$Q(t)=I(t)-\rho(t)H(t)=-\rho(t)\int_t^{\infty}I(1/\rho)'\,ds,$$

so $|Q(t)|\le\sigma(t)$, from (12). Moreover,

$$Q'=I'-\rho H'-H\rho'=-H\rho',$$

so

$$|Q'(t)|\le 2\sigma(t)\rho'(t)/\rho(t),$$

from (14). Therefore we let $\mathcal{T}$ act on the Banach space $\mathcal{B}$ of functions $z$ on $[t_0,\infty)$ such that

$$z=O(\sigma)\quad\text{and}\quad z'=O(\sigma\rho'/\rho),$$

with norm

$$\|z\|=\sup_{t\ge t_0}\left\{\max\left\{\frac{|z|}{\sigma},\frac{\rho|z'|}{\sigma\rho'}\right\}\right\}. \tag{24}$$

We will show that $\mathcal{T}$ maps $\mathcal{B}$ into $\mathcal{B}$, and is a contraction if $t_0$ is sufficiently large. Since $Q \in \mathcal{B}$, it suffices to show that $\mathcal{L}$ is a contraction of $\mathcal{B}$ if $t_0$ is sufficiently large. To this end, suppose $z \in \mathcal{B}$ and $t_0 \le t < T$, and consider the finite integral

$$w_T(t;z) = \int_t^T (y_2(s) - y_1(s)\rho(t))(f(s) - g(s))y_1(s)z(s)\,ds.$$

From (5) and (8),

$$\begin{aligned} w_T(t;z) &= -\int_t^T (\rho(s) - \rho(t))z(s)G'(s)\,ds \\ &= -(\rho(T) - \rho(t))z(T)G(T) \\ &\quad + \int_t^T (\rho(s) - \rho(t))G(s)z'(s)\,ds \\ &\quad + \int_t^T z(s)G(s)\rho'(s)\,ds. \end{aligned} \tag{25}$$

From (14) and (24),

$$|(\rho(T) - \rho(t))z(T)G(T)| < 2\|z\|\sigma^2(T)/\rho(T) \to 0 \text{ as } T \to \infty,$$

$$|(\rho(s) - \rho(t))G(s)z'(s)| \le 2\|z\|\sigma^2(s)\rho'(s)/\rho^2(s), \quad s \ge t,$$

and

$$|z(s)G(s)\rho'(s)| \le 2\|z\|\sigma^2(s)\rho'(s)/\rho^2(s).$$

Therefore we can let $T \to \infty$ in (25) and conclude that

$$(\mathcal{L}z)(t) = -\int_t^\infty (\rho(s) - \rho(t))z(s)G'(s)\,ds \tag{26}$$

exists and satisfies the inequality

$$|(\mathcal{L}z)(t)| < 4\|z\| \int_t^\infty \frac{\sigma^2\rho'}{\rho^2}\,ds < 4\|z\|\frac{\sigma^2(t)}{\rho(t)}. \tag{27}$$

From (26),

$$(\mathcal{L}z)'(t) = \rho'(t)\int_t^\infty zG'\,ds = -\rho'(t)\left(z(t)G(t) + \int_t^\infty Gz'\,ds\right).$$

From (14) and (24), the last integral converges absolutely and

$$|(\mathcal{L}z)'(t)| \le 2\|z\|\rho'(t)\left(\frac{\sigma^2(t)}{\rho^2(t)} + \int_t^\infty \frac{\sigma^2\rho'}{\rho^3}\,ds\right) < 4\|z\|\frac{\sigma^2(t)\rho'(t)}{\rho^2(t)}.$$

From this and (27),

$$\|(\mathcal{L}z)\| < 4\|z\|\sigma(t)/\rho(t).$$

Hence $\mathcal{L}$ (and consequently $\mathcal{T}$) is a contraction of $\mathcal{B}$ if $\sigma(t_0)/\rho(t_0) < 1/4$. Therefore there is a (unique) $\zeta \in \mathcal{B}$ such that $\mathcal{T}\zeta = \zeta$, and the function $x_2$ defined by $x_2 = y_2 + y_1\zeta$ $(t \ge t_0)$ is a solution of (1) that satisfies (21) and (22). We can extend the definition of $x_2$ back to $t = a$. $\square$

**Corollary 1.** *Under the assumptions of Theorem 2, $x_2$ satisfies (19) and (20).*

*Proof.* Since $y_2/y_1 = \rho$, (21) implies that $y_2$ satisfies (19) and

$$x_2/y_1 = \rho + O(\sigma).$$

From (22),

$$(x_2/y_1)' = \rho' \left(1 + O\left(\sigma/\rho\right)\right).$$

Therefore

$$\begin{aligned}\left(\frac{x_2}{y_2}\right)' &= \left(\frac{x_2}{y_1\rho}\right)' = \left(\frac{x_2}{y_1}\right)' \frac{1}{\rho} - \frac{x_2}{y_1}\frac{\rho'}{\rho^2} \\ &= \frac{\rho'}{\rho}(1 + O(\sigma/\rho)) - \frac{\rho'}{\rho^2}(\rho + O(\sigma)) = O\left(\frac{\sigma\rho'}{\rho^2}\right).\end{aligned}$$

□

It is natural to ask whether the convergence of $\int^\infty (f-g)y_2^2\,dt$ is necessary for the existence of a solution $x_2$ of (1) such that

$$x_2 = y_2(1 + o(1/\rho)) \quad \text{and} \quad (x_2/y_2)' = o(\rho'/\rho^2).$$

Although we do not know the answer to this question, we offer the following related theorem.

**Theorem 3.** *If (1) has a solution $x_2$ that satisfies (19) and (20) for some positive monotonic function $\sigma$ such that $\lim_{t\to\infty}\sigma(t) = 0$, then*

$$\int_t^\infty (f-g)y_1y_2\,dt = O(\sigma/\rho). \tag{28}$$

*Moreover, if*

$$\int^\infty \frac{\sigma\rho'}{\rho}\,dt < \infty, \tag{29}$$

*then $\int^\infty (f-g)y_2^2\,dt$ converges.*

*Proof.* From (20), $R(t) = \int_t^\infty (x_2/y_2)'\,dt$ converges absolutely and

$$R = O(\sigma/\rho). \tag{30}$$

If $t > T$, define

$$R_T(t) = \int_t^T \left(\frac{x_2}{y_2}\right)' ds.$$

From (5) and (6),

$$\left(\frac{x_2}{y_2}\right)' = \frac{y_2x_2' - x_2y_2'}{y_2^2} = u\frac{\rho'}{\rho^2}, \tag{31}$$

where

$$u = r(y_2x_2' - x_2y_2').$$

From (1) and (2),

$$u' = (f-g)y_2x_2.$$

Therefore

$$R_T(t) = \frac{u(t)}{\rho(t)} - \frac{u(T)}{\rho(T)} + \int_t^T (f-g)y_1x_2\,ds.$$

From (20) and (31), $u = o(\sigma)$, so we can let $T \to \infty$ and invoke (30) to conclude that

$$\hat{R}(t) \stackrel{\mathrm{df}}{=} \int_t^\infty (f-g)y_1x_2\,ds = O(\sigma/\rho). \tag{32}$$

Now let

$$\begin{aligned} S_T(t) &= \int_t^T (f-g)y_1y_2\,ds = -\int_t^T \frac{y_2}{x_2}\hat{R}'\,ds \\ &= \frac{y_2(t)}{x_2(t)}\hat{R}(t) - \frac{y_2(T)}{x_2(T)}\hat{R}(T) + \int_t^T \hat{R}\left(\frac{y_2}{x_2}\right)'\,ds. \end{aligned} \tag{33}$$

But

$$\left(\frac{y_2}{x_2}\right)' = -\frac{y_2^2}{x_2^2}\left(\frac{x_2}{y_2}\right)' = O\left(\frac{\sigma\rho'}{\rho^2}\right)$$

from (19) and (20). From this and (32), we can let $T \to \infty$ in (33) to conclude that

$$S(t) \stackrel{\mathrm{df}}{=} \int_t^\infty (f-g)y_1y_2 = O(\sigma/\rho). \tag{34}$$

This verifies (28). If (29) holds and $T > a$, then

$$\int_a^T (f-g)y_2^2\,dt = -\int_a^T \rho S'\,dt = \rho(a)S(a) - \rho(T)S(T) + \int_a^T S\rho'\,dt. \tag{35}$$

Since (34) implies that $\lim_{T\to\infty}\rho(T)S(T) = 0$ and (29) and (34) together imply that $\int^\infty S\rho'\,dt$ converges, (35) implies that $\int^\infty (f-g)y_2^2\,dt$ converges. □

## 3. Examples

Examples illustrating our results can be constructed by letting

$$g(t) = f(t) + \frac{u(t)S(t)}{y_2^2(t)}, \quad t \ge a,$$

where $u$ and $S$ are continuously differentiable and $S$ has a bounded antiderivative $C$ on $[a,\infty)$, while $\lim_{t\to\infty} u(t) = 0$ and $\int^\infty |u'(t)|\,dt < \infty$. Then

$$\int_t^\infty (f(s)-g(s))y_2^2(s)\,ds = -\int_t^\infty u(s)S(s)\,ds = -u(s)C(s)\Big|_t^\infty + \int_t^\infty u'(s)C(s)\,ds$$

converges, and the convergence may be conditional. Here we may take

$$\sigma(t) = M\sup_{\tau\ge t}\left(|u(\tau)| + \int_\tau^\infty |u'(s)|\,ds\right),$$

where $M$ is an upper bound for $C$ on $[a,\infty)$.

For a specific example, consider the equation

$$x'' + \frac{\sin t}{t^2(\log t)^\alpha}x = 0, \quad t \ge a > 0 \quad (\alpha > 0), \tag{36}$$

as a perturbation of $y'' = 0$. Our results imply that (36) has solutions $x_1$ and $x_2$ such that

$$x_1(t) = 1 + O\left(t^{-1}(\log t)^{-\alpha}\right), \quad x_1'(t) = O\left(t^{-2}(\log t)^{-\alpha}\right)$$

and

$$x_2(t) = t + O((\log t)^{-\alpha}), \quad x_2'(t) = 1 + O(t^{-1}(\log t)^{-\alpha}).$$

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