FINITE RANK OPERATORS IN CLOSED MAXIMAL TRIANGULAR ALGEBRAS II

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Abstract. In this paper, we discuss finite rank operators in a closed maximal triangular algebra $S$. Based on the following result that each finite rank operator of $S$ can be written as a finite sum of rank one operators each belonging to $S$, we prove that $(S \cap \mathcal{F}(H))^w = \{ T \in B(H) : TN \subseteq N_\sim, \forall N \in \mathcal{N} \}$, where $N_\sim = N$, if $\dim N \ominus N_\sim \leq 1$; and $N_\sim = N_\ominus$, if $\dim N \ominus N_\sim = \infty$. We also proved that the Erdos Density Theorem holds in $S$ if and only if $S$ is strongly reducible.

1. Introduction

Finite rank operators and rank one operators are important to the theory of nest algebras. In a nest algebra, each finite rank operator can be written as a finite sum of rank one operators which belong to itself (This result is in [6], but belongs to Ringrose); the $w^*$-closure of all finite rank operators is the whole of the nest algebra ([6], it is known as the famous Erdos Density Theorem). Naturally, we may ask what happens in the case of maximal triangular algebras?

We have proved in [4] that each finite rank operator of a closed maximal triangular algebra $S$ can be represented as a finite sum of rank one operators in $S$. This is first appeared in [4], but for completeness and reader-friendly reasons, we state it in Section 2. In Section 3, using the decomposability of finite rank operators in $S$ and the technique of annihilators, we calculate the $w^*$-closure of all finite rank operators in $S$. In the last section, we give some remarks on Rosenthal’s famous note [10], and obtain a sufficient and necessary condition for which the Erdos Density Theorem holds in $S$.

Now we give some notation and terminology. Let $\mathcal{H}$ be a complex separable infinite-dimensional Hilbert space, $B(\mathcal{H})$ the set of all bounded operators on $\mathcal{H}$ and $\mathcal{F}(\mathcal{H})$ the set of all finite rank operators in $B(\mathcal{H})$. A nest $\mathcal{N}$ is a chain of closed subspaces of Hilbert space $\mathcal{H}$ containing $(0)$ and $\mathcal{H}$ which is closed under intersection and closed span. For $N \in \mathcal{N}$, define

$$N_\sim = \bigvee \{ N' \in \mathcal{N} : N' < N \}.$$
If \( N \neq N_* \), the subspace \( N \oplus N_* \) is called an atom of \( N \). If \( \dim N \oplus N_* \leq 1 \) for any \( N \in \mathcal{N} \), \( \mathcal{N} \) is called a maximal nest. If \( \mathcal{N} \) is a nest, the nest algebra \( T(\mathcal{N}) \) is the set of all operators \( T \) such that \( TN \subseteq N \) for every element \( N \in \mathcal{N} \).

Let \( \mathcal{S} \) be a subalgebra of \( \mathcal{B}(\mathcal{H}) \), and define \( \mathcal{S}^* = \{ A^* : A \in \mathcal{S} \} \). Following Kadison and Singer [8], we shall say that \( \mathcal{S} \) is a triangular algebra if \( \mathcal{D} = \mathcal{S} \cap \mathcal{S}^* \) is a maximal abelian subalgebra of \( \mathcal{B}(\mathcal{H}) \). The maximal abelian \( * \)-algebra \( \mathcal{D} \) is called the diagonal of \( \mathcal{S} \). A maximal triangular algebra is a triangular algebra which is not properly contained in any other such algebra. Applying Zorn’s Lemma, we conclude that any triangular algebra is contained in a maximal triangular algebra with the same diagonal.

Let \( \mathcal{S} \) be a maximal triangular algebra over \( \mathcal{H} \). It is shown in [8], Lemma 2.3.3, that \( \text{Lat}\mathcal{S} \) is totally ordered by inclusion. Hence it forms a nest \( \mathcal{N} \), we shall call \( \mathcal{N} \) the hull nest of \( \mathcal{S} \) and \( T(\mathcal{N}) \) the hull nest algebra of \( \mathcal{S} \). In general, the hull nest \( \mathcal{N} \) is quasi–maximal, that is the subspace \( N \oplus N_* \) has dimension 0, 1 or infinity for any \( N \in \mathcal{N} \) (see [3], Theorem 1). Following [8], we shall say that \( \mathcal{S} \) is irreducible if the hull nest \( \mathcal{N} = \{(0), \mathcal{H}\} \), and that \( \mathcal{S} \) is strongly reducible if \( \mathcal{N} \) is maximal. It is shown in [11] and [12] that not all maximal triangular algebras are norm closed. However, one feels that non-norm-closed maximal triangular algebras are rather pathological and that the proper objects for study should at least be complete. If a triangular algebra is norm-closed, we shall simply say it is closed.

Suppose that \( \mathcal{S} \) is a subspace of \( \mathcal{B}(\mathcal{H}) \), if \( \mathcal{S} \cap \mathcal{F}(\mathcal{H}) \) is weakly dense in \( \mathcal{S} \), we say that the Erdos Density Theorem holds in \( \mathcal{S} \).

2. Finite rank operators

Definition 2.1. Let \( \mathcal{A} \) be a subalgebra of \( \mathcal{B}(\mathcal{H}) \), and let \( n \) be a positive integer. \( \mathcal{A} \) is \( n \)-fold transitive if for any choice of elements \( x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathcal{H} \) with \( \{x_i\}_{i=1}^n \) linearly independent, there exists a sequence \( \{A_k\} \subseteq \mathcal{A} \) such that

\[
\lim_k A_k x_i = y_i, \quad \forall 1 \leq i \leq n.
\]

Thus \( \mathcal{A} \) is 1-fold transitive if and only if \( \text{Lat}\mathcal{A} = \{(0), \mathcal{H}\} \).

Lemma 2.2. Let \( \mathcal{S} \) be a closed irreducible triangular algebra, then \( \mathcal{S} \) is \( n \)-fold transitive, \( \forall n \geq 1 \).

Proof. Since the Hilbert space \( \mathcal{H} \) is separable infinite–dimensional, then the diagonal \( \mathcal{D} = \mathcal{S} \cap \mathcal{S}^* \) is a countably decomposable maximal abelian \( * \)-subalgebra of \( \mathcal{B}(\mathcal{H}) \), and since \( \mathcal{S} \) is irreducible, so by [1], Theorem 3.3, \( \mathcal{S} \) is strongly dense in \( \mathcal{B}(\mathcal{H}) \).

Suppose that \( x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathcal{H} \) with \( \{x_i\}_{i=1}^n \) linearly independent. By the Hahn–Banach Theorem, we can choose bounded operators \( F_1, \ldots, F_n \) such that \( F_i(x_j) = \delta_{ij} \). Set

\[
Tx = \sum_{i=1}^n F_i(x)y_i.
\]

Then \( T \in \mathcal{B}(\mathcal{H}) \) and \( Tx_i = y_i \). Since \( \mathcal{S} \) is strongly dense in \( \mathcal{B}(\mathcal{H}) \), we can find, for each \( k \geq 1 \), an \( A_k \in \mathcal{S} \) such that

\[
\| A_k x_i - Tx_i \| \leq 1/k, \quad i = 1, 2, \ldots, n.
\]

Hence \( \lim_k A_k x_i = Tx_i = y_i \), proving that \( \mathcal{S} \) is \( n \)-fold transitive. \( \Box \)
If $x, y$ are nonzero vectors in $\mathcal{H}$, we define the rank one operator $x \otimes y$ by
$$(x \otimes y)(z) = (z, y)x, \quad \forall z \in \mathcal{H}.$$  

**Lemma 2.3** (F.Y. Lu [10]). Let $S$ be a norm-closed subalgebra of $\mathcal{B}(\mathcal{H})$ that satisfies the following conditions:
(1) $I \in S$;
(2) $\text{Lat}S = \{(0), \mathcal{H}\}$;
(3) $S \cap S^*$ abelian.
Then $S$ contains no rank one operators.

**Proof.** Suppose that there is a nonzero rank one operator $x \otimes y \in S$. Since $\text{Lat}S = \{(0), \mathcal{H}\}$ and $I \in S$, it follows that $[Sx] = \mathcal{H}$. Hence for any $z \in \mathcal{H}$, there exists $\{S_\alpha\} \subseteq S$ such that $\lim_\alpha S_\alpha x = z$. Since $S$ is norm-closed, it follows that
$$z \otimes y = (\lim_\alpha S_\alpha x) \otimes y = \lim_\alpha S_\alpha (x \otimes y) \in S.$$  
Since $\text{Lat}S^*$ is also trivial, similarly, for any $w \in \mathcal{H}$ there exists $\{S_\beta\} \subseteq S$ such that $\lim_\beta S_\beta y = w$. Hence,
$$z \otimes w = \lim_\beta z \otimes (S_\beta^* y) = \lim_\beta (z \otimes y) S_\beta \in S.$$  
Thus $S$ contains all rank one operators in $\mathcal{B}(\mathcal{H})$.

Now suppose that $u, v$ are linearly independent vectors in $\mathcal{H}$ and $(u, v) \neq 0$. Then the self-adjoint rank one operators $u \otimes u$ and $v \otimes v$ belong to $S \cap S^*$. However,
$$(u \otimes u)(v \otimes v) = (v, u)u \otimes v \neq (u, v)v \otimes u = (v \otimes v)(u \otimes u);$$  
this contradicts condition (3). □

**Proposition 2.4.** Let $S$ be a closed irreducible triangular algebra, then $S$ contains no nonzero finite rank operators.

**Proof.** Suppose that there exists a rank $n$ operator $F$ in $S$. Set
$$F = \sum_{i=1}^n x_i \otimes z_i,$$
where $\{x_i\}_{i=1}^n$ and $\{z_i\}_{i=1}^n$ are both linearly independent.
Following Lemma 2.2, $S$ is $n$-fold transitive. So there exists a sequence $\{A_k\} \subseteq S$ such that
$$\lim_k A_k x_1 = x_1 \quad \text{and} \quad \lim_k A_k x_i = 0, \quad 1 < i \leq n.$$  
Since $S$ is norm-closed, then
$$x_1 \otimes z_1 = \lim_k A_k \left( \sum_{i=1}^n x_i \otimes z_i \right) = \lim_k A_k F \in S.$$  
This is a contradiction to Lemma 2.3. Hence $S$ does not contain nonzero finite rank operators. □

**Lemma 2.5.** Let $S$ be a maximal triangular algebra with hull nest $\mathcal{N}$. If $N \in \mathcal{N}$ and $\dim(N \ominus N_\perp) \leq 1$, then $P(N)TP(N_\perp)^\perp \in S$, $\forall T \in \mathcal{B}(\mathcal{H})$.

**Proof.** Following the proof of [15], Lemma 5.2. □

For the purpose of this paper, we give another form of [10], Theorem 5.2.3.
Lemma 2.6. Suppose that $S$ is a closed maximal triangular algebra. Then a rank one operator $x \otimes y \in S$ if and only if there exists an element $N$ in $N$ such that:
(1) if $\dim N \cap N_+ \leq 1$, $x \in N$ and $y \in N_+^\perp$;
(2) if $\dim N \cap N_+ = \infty$, $x \in N$, $y \in N_+^\perp$; or $x \in N_+$, $y \in N_+^\perp$.

Proof. Sufficiency. It follows from Lemma 2.5 and [8], Lemma 2.3.2. Necessity. Since $x \otimes y \in \mathcal{S}$, there exists an element $N \in \mathcal{N}$ such that $x \in N$ and $y \in N_+^\perp$. Write
$$x = x_1 + x_2 \in N_+ \oplus (N \cap N_-),$$
$$y = y_1 + y_2 \in (N \cap N_-) \oplus N_+^\perp,$$ then
$$x \otimes y = x_1 \otimes y_1 + x_2 \otimes y_2 + x_2 \otimes y_1.$$ It follows from [8], Lemma 2.3.2 that $x_1 \otimes y_1$ and $x_2 \otimes y_2$ belong to $S$; thus, $x_2 \otimes y_1$ also belongs to $S$.

If $\dim N \cap N_+ = \infty$, following the proof of [5], Theorem 1, $P(N \cap N_-)SP(N \cap N_-)$ is a closed irreducible triangular algebra in $B(N \cap N_-)$. Thus by Proposition 2.4,
$$x_2 \otimes y_1 = P(N \cap N_-) (x_2 \otimes y_1) P(N \cap N_-) = 0.$$ Then $x_2 = 0$ or $y_1 = 0$. If $x_2 = 0$, $x \in N_-$ and $y \in N_+^\perp$; if $y_1 = 0$, $x \in N$ and $y \in N_+^\perp$.

If $\dim N \cap N_- \leq 1$, $x \in N$ and $y \in N_+^\perp$. \hfill \qed

Theorem 2.7. Suppose that $S$ is a closed maximal triangular algebra, and $F$ is a finite rank operator in $S$, then $F$ can be written as a finite sum of rank one operators each belonging to $S$, and the number of rank one operators necessary to form $F$ is bounded above 3 times the rank of $F$.

Proof. Set $\mathcal{N}$ to be the hull nest of $S$, and let $F$ be a rank $n$ operator in $S$. Since $F \in \mathcal{S} \subseteq \mathcal{T}(\mathcal{N})$, then by [6], Theorem 1, there exist \{\{N_i\}_{i=1}^n \subseteq \mathcal{N}\} and \{\{y_i\}_{i=1}^n \subseteq \mathcal{N}\} with $x_i \in N_i$, $y_i \in N_i^\perp$, $i = 1, 2, \ldots, n$ such that
$$F = x_1 \otimes y_1 + x_2 \otimes y_2 + \cdots + x_n \otimes y_n.$$ Write
$$x_i = x_i^1 + x_i^2 \in N_i \oplus (N_i \cap N_-),$$
$$y_i = y_i^1 + y_i^2 \in (N_i \cap N_-) \oplus N_i^\perp;$$ then
$$F = \sum_{i=1}^n (x_i^1 \otimes y_i^1 + x_i \otimes y_i^2 + x_i^2 \otimes y_i^1) = F_1 + F_2$$
with $F_1 = \sum_{i=1}^n (x_i^1 \otimes y_i^1 + x_i \otimes y_i^2)$, $F_2 = \sum_{i=1}^n (x_i^2 \otimes y_i^1)$. Following [5], Lemma 2.3.2, the rank one operators $x_i^1 \otimes y_i^1$ and $x_i \otimes y_i^2$ belong to $S$. Hence $F_1 \in S$, $F_2 \in S$. In the following, we shall prove that $x_i^2 \otimes y_i^1 \in \mathcal{S}$, $i = 1, 2, \ldots, n$.

Without loss of generality, let $N_1 \leq N_2 \leq \cdots \leq N_n$.

If $N_1 = N_1^\perp$, then $x_1^2 = y_1^1 = 0$. So we can suppose that $N_i \neq N_1^\perp$, $\forall 1 \leq i \leq n$. For a fixed $i$, suppose that $N_{i-q+1} < N_{i-q} = \cdots = N_i = \cdots = N_{i+p} < N_{i+p+1}$.
Thus by Proposition 2.4, 

$$\dim N' \leq \dim N.$$ 

Now we distinguish two cases.

Case 1. $\dim N_i \oplus N_i = \infty$. Following the proof of [4], Theorem 1, we have that $P(N_i \oplus N_i SP(N_i \oplus N_i)$ is a closed irreducible triangular algebra in $B(N_i \oplus N_i)$. Thus by Proposition 2.4, $P(N_i \oplus N_i SP(N_i \oplus N_i)$ does not contain any nonzero finite rank operators. Hence, if $\sum_{j=i-q}^{i+p} x_j \otimes y_j \neq 0$, we have

$$\sum_{j=i-q}^{i+p} x_j \otimes y_j \notin S.$$ 

This is a contradiction, so $\sum_{j=i-q}^{i+p} x_j \otimes y_j = 0$.

Case 2. $\dim N_i \oplus N_i = 1$. Following Lemma 2.5, we have

$$x_j \otimes y_j \in S, \quad j = i - q, \ldots, i + p.$$ 

Since the hull nest is quasi-maximal, the two cases are jointly exhaustive. Since $i$ is arbitrary, we obtain that $F_2$ is also a finite sum of rank one operators in $S$. So any rank $n$ operator can be written as a finite sum of rank one operators each belonging to $S$. \hfill \Box

### 3. The $w^*$-Closure of Finite Rank Operators

In this section, we will describe the $w^*$-closure of finite rank operators in $S$. Set

$$W = \{X \in B(H) : XN \subseteq \tilde{N}, \forall N \in \mathcal{N}\},$$

where $\tilde{N} = N_e$ if $\dim N \oplus N_e \leq 1$; and $\tilde{N} = N$ if $\dim N \oplus N_e = \infty$.

**Lemma 3.1.** $W$ is a weakly closed $T(\mathcal{N})$–ideal determined by the order homomorphism $N \to \tilde{N}$ of $\mathcal{N}$ into itself; and a rank one operator $x \otimes y \in W$ if and only if there exists an element $N$ in $\mathcal{N}$ such that $x \in N, y \in N_e^-$, where $N_e = N$, if $\dim N \oplus N_e \leq 1$; $N_e = N_e$, if $\dim N \oplus N_e = \infty$.

**Proof.** The fact that $W$ is a weakly closed $T(\mathcal{N})$–ideal is obvious from the definition of $W$.

By virtue of [7], Lemma 1.1, a rank one operator $x \otimes y \in W$ if and only if there exists an element $N \in \mathcal{N}$ such that $x \in N, y \in N_e^-$. In the following, we will compute $N_e$. For any $N \in \mathcal{N}$, we consider separately three cases. Recall that $N_e = \bigvee\{N': \tilde{N}' < N\}$ defined in [7].

Case 1. $\dim N \ominus N_e = 1$. In this case, $\tilde{N} = N_e < N$. If $N' > N, N_e ^- \geq N$. Thus $\tilde{N}' \geq N$. So $N_e = N$.

Case 2. $\dim N \oplus N_e = \infty$. In this case, $\tilde{N} = N$. Since $\tilde{N}_e \leq N_e < N$, $N_e = N_e$.

Case 3. $\dim N \ominus N_e = 0$. Thus, $\tilde{N} = N_e = N$. In this case, we can prove that

$$\{N' \in \mathcal{N} : N' < N\} = \{N' \in \mathcal{N} : \tilde{N}' < N\}.$$
Indeed, since \( \tilde{N} \leq N' \), we have that \( \{N' \in \mathcal{N} : N' < N\} \subseteq \{N' \in \mathcal{N} : \tilde{N} < N\} \).
Conversely, if \( N' \notin \{N' \in \mathcal{N} : N' < N\} \), that is \( N' \geq N \) and \( \tilde{N} \geq \tilde{N} = N \). So \( N' \notin \{N' \in \mathcal{N} : \tilde{N} < N\} \). Hence \( \{N' \in \mathcal{N} : N' < N\} \supseteq \{N' \in \mathcal{N} : \tilde{N} < N\} \).
Therefore, \( N_\infty = \bigvee \{N' \in \mathcal{N} : \tilde{N} < N\} = \bigvee \{N' \in \mathcal{N} : N' < N\} = N_\infty = N \).
Since the hull nest \( \mathcal{N} \) is quasi-maximal, the three cases are jointly exhaustive. This completes the proof. \( \square \)

Set \( \mathcal{C}_1(\mathcal{H}) \) as the ideal of all trace class operators in \( \mathcal{B}(\mathcal{H}) \).

**Theorem 3.2.** Suppose that \( \mathcal{S} \) is a closed maximal triangular algebra with hull nest \( \mathcal{N} \), then \( \rho \in \mathcal{B}(\mathcal{H}_*) \) annihilates \( \mathcal{S} \cap \mathcal{F}(\mathcal{H}) \) if and only if \( \rho \) is of the form \( \rho(\cdot) = tr(X\cdot) \),
where \( X \) is a trace class operator in \( \mathcal{W} \).

**Proof.** *Necessity.* If \( \rho \in \mathcal{B}(\mathcal{H}_*) \cong \mathcal{C}_1(\mathcal{H}) \), there exists an operator \( X \in \mathcal{C}_1(\mathcal{H}) \) such that \( \rho(\cdot) = tr(X\cdot) \) and \( \rho \) annihilates \( \mathcal{S} \cap \mathcal{F}(\mathcal{H}) \). For any \( Y \in \mathcal{F}(\mathcal{H}) \) and \( N \in \mathcal{N} \), by [5], Lemma 2.3.2 and Lemma 2.5, the operator \( P(N)YP(\tilde{N})^\perp \in \mathcal{S} \cap \mathcal{F}(\mathcal{H}) \). Thus
\[
tr(P(\tilde{N})^\perp XP(N)Y) = tr(XP(N)YP(\tilde{N})^\perp) = 0, \quad \forall Y \in \mathcal{F}(\mathcal{H}).
\]
From \( \mathcal{F}(\mathcal{H})^w = \mathcal{B}(\mathcal{H}) \) and the \( w^* \)-continuity of the map \( tr(P(\tilde{N})^\perp XP(N)\cdot) \) it follows that
\[
tr(P(\tilde{N})^\perp XP(N)Y) = 0, \quad \forall Y \in \mathcal{B}(\mathcal{H}).
\]
Then
\[
P(\tilde{N})^\perp XP(N) = 0, \quad \forall N \in \mathcal{N}.
\]
So
\[
X \in \mathcal{W} \cap \mathcal{C}_1(\mathcal{H}).
\]

*Sufficiency.* If \( X \in \mathcal{W} \cap \mathcal{C}_1(\mathcal{H}) \), let \( x \otimes y \) be any rank one operator of \( \mathcal{S} \). Then, by Lemma 2.6, there exists an element \( N \in \mathcal{N} \) such that:
(1) if \( dim N \cap N_\infty \leq 1 \), then \( x \in N \) and \( y \in N_\perp \). Since \( \tilde{N} = N_\infty \), we have that
\[
tr(X(x \otimes y)) = tr(XP(N)(x \otimes y)P(N_\perp)) = tr(P(N_\perp)^\perp XP(N)(x \otimes y)) = 0.
\]
(2) if \( dim N \cap N_\infty = \infty \), we distinguish two cases.
Case 1. \( x \in N, y \in N_\perp \). Since \( \tilde{N} = N \),
\[
tr(X(x \otimes y)) = tr(P(N_\perp)^\perp XP(N)(x \otimes y)) = 0.
\]
Case 2. \( x \in N_\infty, y \in N_\perp \). Since \( X \in \mathcal{W} \), \( XN_\infty \subseteq \tilde{N}_\infty \subseteq N_\infty \). Thus,
\[
tr(X(x \otimes y)) = tr(P(N_\perp)^\perp XP(N_\infty)(x \otimes y)) = 0.
\]
Therefore the map \( tr(X\cdot) \) annihilates any rank one operators in \( \mathcal{S} \). Since the map \( tr(X\cdot) \) is linear, it follows from Theorem 2.7 that
\[
tr(XF) = 0, \quad \forall F \in \mathcal{S} \cap \mathcal{F}(\mathcal{H}).
\]
So \( \rho(\cdot) = tr(X\cdot) \) annihilates \( \mathcal{S} \cap \mathcal{F}(\mathcal{H}) \). \( \square \)
Theorem 3.2 tells us that \((S \cap F(H))_+ = W \cap C_1(H)\). Since \((S \cap F(H)) \cap K(H) = S \cap F(H)\), Theorem 3.2 also shows that \((S \cap F(H))^\perp = W \cap C_1(H)\).

In order to calculate the annihilator of \(W \cap C_1(H)\), we need some results about weakly closed \(T(N)\)-modules. These results have their own interest. Note that the symbol “~” in the following results 3.3–3.5 is not the same as that defined in the beginning of Section 3.

**Lemma 3.3.** Suppose that \(E, \widetilde{E}\) are comparable projections in \(B(H)\). If \(A \in C_1(H)\) and \((I - \widetilde{E})AE = 0\), then \(A\) can be decomposed as \(A = A_1 + A_2\) such that

1) \((I - \widetilde{E})A_1 = 0, \hspace{0.5em} A_2E = 0\);
2) \(\| A \|_1 = \| A_1 \|_1 + \| A_2 \|_1\).

**Proof.** We consider separately two cases.

Case 1. \(\widetilde{E} \leq E\). We decompose \(H\) as \(E \oplus (E \ominus \widetilde{E}) \oplus E^\perp\). Since \((I - \widetilde{E})AE = 0\), corresponding to the decomposition of \(H\), the trace class operator \(A\) has the matrix form

\[
A = \begin{pmatrix}
B_{11} & B_{12} & B_{13} \\
0 & 0 & B_{23} \\
0 & 0 & B_{33}
\end{pmatrix}.
\]

Thus, following [3], Lemma 3.3, \(A\) can be written as

\[
A = \begin{pmatrix}
B_{11} & B_{12} & C \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & 0 & D \\
0 & 0 & B_{23} \\
0 & 0 & B_{33}
\end{pmatrix} = A_1 + A_2
\]

and \(\| A \|_1 = \| A_1 \|_1 + \| A_2 \|_1\). It follows from the matrix form of \(A_1, A_2\) that \((I - \widetilde{E})A_1 = 0\) and \(A_2E = 0\).

Case 2. \(E \leq \widetilde{E}\). Decompose \(H\) as \(E \oplus (\widetilde{E} \ominus E) \oplus \widetilde{E}^\perp\). In this case \(A\) has the matrix form

\[
A = \begin{pmatrix}
B_{11} & B_{12} & B_{13} \\
B_{21} & B_{22} & B_{23} \\
0 & B_{32} & B_{33}
\end{pmatrix}.
\]

Similarly, by [3], Lemma 3.3, we have

\[
A = \begin{pmatrix}
B_{11} & C_{12} & C_{13} \\
B_{21} & C_{22} & C_{23} \\
0 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & D_{12} & D_{13} \\
0 & D_{22} & D_{23} \\
0 & B_{32} & B_{33}
\end{pmatrix} = A_1 + A_2
\]

and \(\| A \|_1 = \| A_1 \|_1 + \| A_2 \|_1\). Following the matrix form of \(A_1, A_2\), we have that \((I - \widetilde{E})A_1 = 0\) and \(A_2E = 0\). □

**Lemma 3.4.** Let \(U = \{ X \in B(H) : XN \subseteq \overline{N}, \forall N \in N \}\), where the map \(N \rightarrow \overline{N}\) is an order homomorphism of \(N\) into \(\overline{N}\). Then \(P(\overline{N})TP(\overline{N}) \subseteq U\), for any \(N \in N, T \in B(H)\).

**Proof.** The proof is routine. □

**Proposition 3.5.** Suppose that \(U\) is a weakly closed \(T(N)\)-module determined by the order homomorphism \(N \rightarrow \overline{N}\), then each extreme point of the unit ball \(b(U \cap C_1(H))\) is a norm-one rank one operator in \(U \cap C_1(H)\).
Proof. Suppose that $A$ is an extreme point of $b_1(\mathcal{U} \cap \mathcal{C}_1(\mathcal{H}))$. First, we shall prove that there exists an element $N_0 \in \mathcal{N}$ such that $A = P(N_0^*)AP(N_0)^\perp$. Recall that $N_0 = \big\{ N : N > N_0, \forall N \in \mathcal{N} \big\}$ defined in [7].

For $N \in \mathcal{N}$, suppose that $A \neq P(\bar{N})A$ and $AP(N) \neq 0$. Since $A \in \mathcal{U}$ and $N, \bar{N} \in \mathcal{N}$, we have that $(I - P(\bar{N}))AP(N) = 0$ and $P(N), P(\bar{N})$ are comparable projections. Thus, it follows from Lemma 3.3 that the trace class operator $A$ can be decomposed as $A = A_1 + A_2$ and $P(\bar{N})^\perp A_1 = 0, A_2 P(N) = 0$. So $A_1 P(N) = AP(N), P(\bar{N})^\perp A_2 = P(\bar{N})^\perp A$. Owing to the hypothesis $A \neq P(\bar{N})A$ and $AP(N) \neq 0$, we have that $A_2 \neq 0$ and $A_1 \neq 0$. Now we shall prove that $A_1, A_2 \in \mathcal{U} \cap \mathcal{C}_1(\mathcal{H})$. Following Lemma 3.3 (2), we only need to prove $A_1, A_2 \in \mathcal{U}$. Since $P(\bar{N})^\perp A_1 = 0$,

$$A_1 = P(\bar{N})A_1 = P(\bar{N})A_1 P(N) + P(\bar{N})A_1 P(N)^\perp = P(\bar{N})AP(N) + P(\bar{N})A_1 P(N)^\perp.$$ 

Since $A \in \mathcal{U}, P(\bar{N}), P(N) \in T(\mathcal{N})$ and $\mathcal{U}$ is a weakly closed $T(\mathcal{N})$-module, the operator $P(\bar{N})AP(N) \in \mathcal{U};$ and by virtue of Lemma 3.4, $P(\bar{N})A_1 P(N)^\perp \in \mathcal{U}$. Hence $A_1 \in \mathcal{U}$. Similarly, we can prove $A_2 \in \mathcal{U}$. Thus, it follows from Lemma 3.3 (2) that $A$ is not an extreme point of $b_1(\mathcal{U} \cap \mathcal{C}_1(\mathcal{H}))$. This is a contradiction. Therefore, for any $N \in \mathcal{N}$, either $A = P(\bar{N})A$ or $AP(N) = 0$. Set

$$N_0 = \bigvee \{ N \in \mathcal{N} : AP(N) = 0 \}.$$ 

Naturally, $AP(N_0) = 0$ and for any $N > N_0, A = P(\bar{N})A$. Thus we have

$$A = P(\bar{N})AP(N_0)^\perp, \quad \forall N > N_0.$$ 

Taking a limit, it follows from the definition of $N_0$ that

$$A = P(N_0^*)AP(N_0)^\perp.$$ 

In the following, we shall prove that $A$ is a rank one operator. Since $A \in \mathcal{C}_1(\mathcal{H}) \subseteq \mathcal{K}(\mathcal{H})$, $A$ can be written as

$$A = \sum_{k=1}^{+\infty} \lambda_k e_k \otimes f_k,$$

where $\sum$ is convergent according to the norm topology, $\{ \lambda_k \}$ are s-numbers of $A$ and $\| e_k \| = \| f_k \| = 1$. Thus,

$$A = P(N_0^*)AP(N_0)^\perp = \sum_{k=1}^{+\infty} \lambda_k P(N_0^*)(e_k \otimes f_k)P(N_0)^\perp.$$ 

Since $\| A \|_1 = \sum_{k=1}^{+\infty} \lambda_k \leq \sum_{k=1}^{+\infty} \lambda_k \| P(N_0^*)(e_k \otimes f_k)P(N_0)^\perp \|_1$, we have

$$\lambda_k = \lambda_k \| P(N_0^*)(e_k \otimes f_k)P(N_0)^\perp \|_1, \quad \forall k = 1, 2, \ldots .$$ 

Therefore, if $\lambda_k \neq 0$, we have $\| P(N_0^*)e_k \|. \| P(N_0)^\perp f_k \| = 1$. Hence $e_k \in N_0^*$ and $f_k \in N_0^\perp$.

By [7], Lemma 1.1, for any $k \geq 1$, $e_k \otimes f_k \in \mathcal{U} \cap \mathcal{C}_1(\mathcal{H})$. Since $A$ is an extreme point of $b_1(\mathcal{U} \cap \mathcal{C}_1(\mathcal{H}))$,

$$\lambda_2 = \lambda_3 = \cdots = 0.$$ 

Thus $A$ is a norm-one rank one operator. \( \square \)
We come back to study finite rank operators in $S$.

Lemma 3.6. The unit ball $b_1(W \cap C_1(H))$ is the norm-closed convex hull of its extreme points, where $W$ is defined in the beginning of Section 3.

Proof. Following Theorem 3.2 and $S \cap F(H) = (S \cap F(H)) \cap K(H)$, we obtain

$(S \cap F(H))_{γ} = (S \cap F(H))_{γ} = W \cap C_1(H)$.

Hence,

$W \cap C_1(H) = (K(H)/S \cap F(H))_{γ}$.

By virtue of the Krein-Milman Theorem, $b_1(W \cap C_1(H))$ is the $w^*$-closed convex hull of its extreme points. It follows from [3], Corollary 16.4, that the boundary points of $b_1(W \cap C_1(H))$ belong to the norm-closed convex hull of its extreme points. Therefore $b_1(W \cap C_1(H))$ is the norm-closed convex hull of its extreme points. □

Now we are in the position to compute the $(W \cap C_1(H))_{γ}$. Set

$V = \{ T \in B(H) : T N \subseteq N_{∞}, \forall N \in N \}$,

where $N_∞ = N$, if dim$N \cap N_{∞} \leq 1$; and $N_∞ = N_{∞}$, if dim$N \cap N_{∞} = ∞$.

Theorem 3.7. $(W \cap C_1(H))_{γ} = V$.

Proof. Suppose that $T \in (W \cap C_1(H))_{γ}$. For any $N \in N$, nonzero vectors $x \in N$ and $y \in N_{∞}$. By virtue of Lemma 3.1, the rank one operator $x ⊗ y$ belongs to $W \cap F(H) \subseteq W \cap C_1(H)$. Hence

$0 = tr(Tx ⊗ y) = (Tx, y)$, ∀$x \in N$, $y \in N_{∞}$.

So $TN \subseteq N_{∞}$ for any $N \in N$, and $T \in V$. Thus, $(W \cap C_1(H))_{γ} \subseteq V$.

Conversely, let $T \in V$. For any $N \in N$, $x \in N$ and $y \in N_{∞}$, we have that

$tr(Tx ⊗ y) = (Tx, y) = (P(N_∞)T^*P(N)x, y) = 0$.

Thus, $T$ annihilates all rank-one operators in $W$. It follows from Lemma 3.1 and Proposition 3.5 that $T$ annihilates all extreme points of $b_1(W \cap C_1(H))$. Thus by Lemma 3.6, $T$ annihilates $b_1(W \cap C_1(H))$ and $T \in (W \cap C_1(H))_{γ}$. Therefore $(W \cap C_1(H))_{γ} = V$. □

Theorem 3.8. $(S \cap F(H))_{w^*} = V = \{ T \in B(H) : TN \subseteq N_{∞}, \forall N \in N \}$, where $N_∞ = N$, if dim$N \cap N_{∞} \leq 1$; $N_∞ = N_{∞}$, if dim$N \cap N_{∞} = ∞$.

Proof. It follows from Theorem 3.2 and Theorem 3.7 that

$(S \cap F(H))_{w^*} = [(S \cap F(H))_{γ}]_{γ} = (W \cap C_1(H))_{γ} = V$. □

Corollary 3.9. $(S \cap F(H))_{w} = (S \cap F(H))_{w^*} = (S \cap F(H))_{w^*} = V$.

Proof. Since $V$ is weakly closed and $(S \cap F(H))_{w^*} = V$, we have $(S \cap F(H))_{w} = V$. Owing to the convexity of $S \cap F(H)$, $(S \cap F(H))_{w} = (S \cap F(H))_{w^*}$.
4. The Erdos Density Theorem in $S$

In this section, we will prove that the Erdos Density Theorem holds in $S$ if and only if $S$ is strongly reducible.

**Proposition 4.1.** Suppose that $S$ is a maximal triangular algebra, and that $N$ is the hull nest of $S$. Then $S$ is weakly dense in $T(N)$.

**Proof.** Set $A = S^w$. It is easy to show that $LatA = LatS = N$ and $A \supseteq S \cap S^*$, so $A$ is a weakly closed algebra which contains a m.a.s.a and $LatA$ is completely ordered. Following [13], Theorem 9.24, $A$ is a reflexive algebra. Hence $A = AlglatA = AlgN = T(N)$, that is, $S^w = T(N)$. □

Note that in Proposition 4.1, $S$ is not assumed to be closed. Following Proposition 4.1, we can obtain [16] Rosenthal’s famous result: a weakly closed maximal triangular algebra is hyper-reducible, that is, $S^w = T(N)$. If a maximal triangular algebra $S$ is not weakly closed, we have $S^w \supset S$. Owing to the maximality of $S$, $S^w$ is not a triangular algebra. Hence Rosenthal’s result does not imply Proposition 4.1, which is more general. Now we give an application of Proposition 4.1.

**Corollary 4.2.** Let $S$ be a maximal triangular algebra, then $S' = CI$.

**Proof.** Following [2], Lemma 3.6, the commutant of a nest algebra is trivial. So $S' = (S^w)' = T(N)' = CI$. □

**Theorem 4.3.** Suppose that $S$ is a closed maximal triangular algebra, then $S \cap \mathcal{F}(H)$ is weakly dense in $S$ if and only if $S$ is strongly reducible.

**Proof.** If $S$ is strongly reducible, it follows from Corollary 3.9 that $(S \cap \mathcal{F}(H))^w = T(N)$. Thus $S \cap \mathcal{F}(H)$ is weakly dense in $S$.

Suppose, on the contrary, that $S \cap \mathcal{F}(H)$ is weakly dense in $S$. It follows from Corollary 3.9 and Proposition 4.1 that $V = (S \cap \mathcal{F}(H))^w = S^w = T(N)$.

Thus $T(N) = V = \{T \in B(H) : TN \subseteq N, \forall N \in \mathcal{N}\}$, where $N_\sim = N$, if $\dim N \oplus N_\sim \leq 1$; and $N_\sim = N_\sim$, if $\dim N \oplus N_\sim = \infty$. It is easy to prove that $\dim N \oplus N_\sim \leq 1$ for any $N \in \mathcal{N}$. Indeed, suppose that there exists an element $N$ in $\mathcal{N}$ such that $\dim N \oplus N_\sim = \infty$. In this case, $N_\sim = N_\sim$. So the identity operator $I \in T(N)$ and $I \notin V$. This contradicts $T(N) = V$. Hence for any $N \in \mathcal{N}$, $\dim N \oplus N_\sim \leq 1$. Thus $S$ is strongly reducible. □

**References**


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