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DIAGRAMS OF DIVIDE LINKS

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ABSTRACT. Recently N. A'Campo suggested a construction of a link from a generic immersion of a curve into a 2-disk. It is tightly related to the singularity theory. In this paper, we give a simple procedure to draw a diagram of the link from a picture of the curve.

There are several fascinating relations of plane immersed curves and links. One of them which goes through Legendrian links led Arnold [5] to the discovery of three simple invariants J^+, J^-, St of such a curve. N. A'Campo in [2] suggested another construction of a link from a generic immersion of a curve into a 2-disk. It is tightly related to the singularity theory. If $f_{\mathbb{R}} : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ is a germ of an analytic function whose complexification $f_{\mathbb{C}}$ has an isolated critical point, then one can define a link $\mathcal{L}$ of the singularity $f_{\mathbb{C}}$ as an intersection of the zero-level variety of $f_{\mathbb{C}}$ with a small 3-sphere in $\mathbb{C}^2$ centered at the critical point. Such links are sometimes called *algebraic links*. On the other hand, in singularity theory it is useful [1, 9] (see also [6]) to consider a small perturbation D of the real plane singular curve $\{f_{\mathbb{R}} = 0\}$ which is a generic immersed curve with the maximal possible number of double points. A'Campo [2] restored the link $\mathcal{L}$ directly from the curve D.

In this paper we give a simple procedure to draw a diagram of the link $\mathcal{L}$ from a picture of D (Theorem 2.2). It is essentially a particular case of the results of [8]. An advantage of our approach is that we obtain a link diagram directly from a divide picture without deforming it into a so-called ordered Morse signed divide as in [8]. A similar method to draw diagrams was found by M. Hirasawa in [10]. Our diagrams are obviously symmetrical in a sense that the rotation of $\mathbb{R}^3$ by 180° around the x-axis reverses the orientation of $\mathcal{L}$.

1. DIVIDES AND THEIR LINKS

Definition 1.1 ([2, 3]). *A divide D* is the image of a generic immersion of a finite number of copies of the unit interval I=[0,1] in the unit disk $B \subset \mathbb{R}^2$ such that ∂I is embedded in ∂B and double points are the only singularities allowed.

We consider divides up to isotopy of the disk B. The isotopy does not assume to be identical on the boundary ∂B.

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Example 1.2. The curve $x^3 + y^4 = 0$ has a singularity of type E_6 at the origin [6]. A small perturbation of it is a divide which looks as follows.

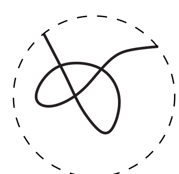
Definition 1.3 ([2, 3])**.** Let x be the horizontal coordinate on the disk B and y be the vertical coordinate. *A divide link* $\mathcal{L}_D$ is a link in the 3-sphere $S^3 = \{(x, y, u, v) \in \mathbb{R}^4 | x^2 + y^2 + u^2 + v^2 = 1\}$ such that (x, y) is a point on D while u and v are the coordinates of a tangent vector to D at the point (x, y).

Therefore each interior point of D has two corresponding points on $\mathcal{L}_D$, and a boundary point of D gives a single point on $\mathcal{L}_D$.

The link $\mathcal{L}_D$ has a natural orientation. Indeed, choose any orientation of every branch of D. Let (u, v) be the tangent vector to D at (x, y) pointing to the direction of the chosen orientation of D. Then the orientation of $\mathcal{L}_D$ is given by the vector $(\dot{x}, \dot{y}, \dot{u}, \dot{v})$. It is easy to see that this orientation of $\mathcal{L}_D$ does not depend on the choice of orientations of branches of D.

The number of components of $\mathcal{L}_D$ equals to the number of branches of the divide D which is the number of the copies of the unit interval I in Definition 1.1. In particular, if D consists of only one branch as in Example 1.2, then $\mathcal{L}_D$ will be a knot.

Remark 1.4. Topological type of a divide link does not change under a regular transversal isotopy of the disk B. Therefore it does not depend on the choice of coordinates in Definition 1.3. Also it does not change under a moving of a piece of the curve D through a triple point [8]. In particular, the following two divides have the same knot type as the one in Example 1.2:



Remark 1.5. In [2] A'Campo proved that all algebraic links are divide links. In [3] he showed that the links $\mathcal{L}_D$ corresponding to a connected divide D are fibered, and computed their monodromy in terms of the combinatorics of divide D. Moreover, he proved that the unknotting number of a one-branch divide knot $\mathcal{L}_D$ equals the number of double points of D. Not all fibered links have the form $\mathcal{L}_D$. Figure eight knot 4_1 is not a divide knot. It is not clear how large is the class of divide links in the class of all fibered links.

Remark 1.6. It has been known for a long time [7] that algebraic knots are classified by the Alexander polynomial. N. A'Campo ([4]) found two different divide knots with the same Alexander polynomial.

2. Diagrams of divide links

Definition 2.1. Let us call a divide *generic* if its points with vertical tangents differ from the double points and the boundary points and their x-coordinates are pairwise different. The divide in Example 1.2 is generic. Any divide can be made generic by a small deformation.

Theorem 2.2. *For a generic divide D a link diagram of $\mathcal{L}_D$ can be drawn in the following way:*

(1) *Consider a horizontal line below the disk B, say the line $\{y = -1.5\}$. Let s be the symmetry with respect to the line. We are going to draw the link diagram of $\mathcal{L}_D$ by modifications of the union $D \cup s(D)$.*

(2) *Replace each double point of the union by a crossing of the type .*

(3) *Connect each boundary point p of D with $s(p)$ by a vertical string.*

(4) *Replace a small piece of our curve near each point p with a vertical tangent by two vertical strings connecting the upper and lower parts of the picture as in the examples below. The strings make a positive half-twist at the line of the reflection s. Note that at the upper part of the picture the right string goes below all intersected intervals of D while the left string goes above the intervals. Correspondingly at the lower part of the picture the right string goes above the intervals of $s(D)$ and the left string goes below the intervals. Examples 2.3 and 2.4 demonstrate this.*

Example 2.3. For the divide of Example 1.2 the theorem gives the diagram shown in Figure 1.

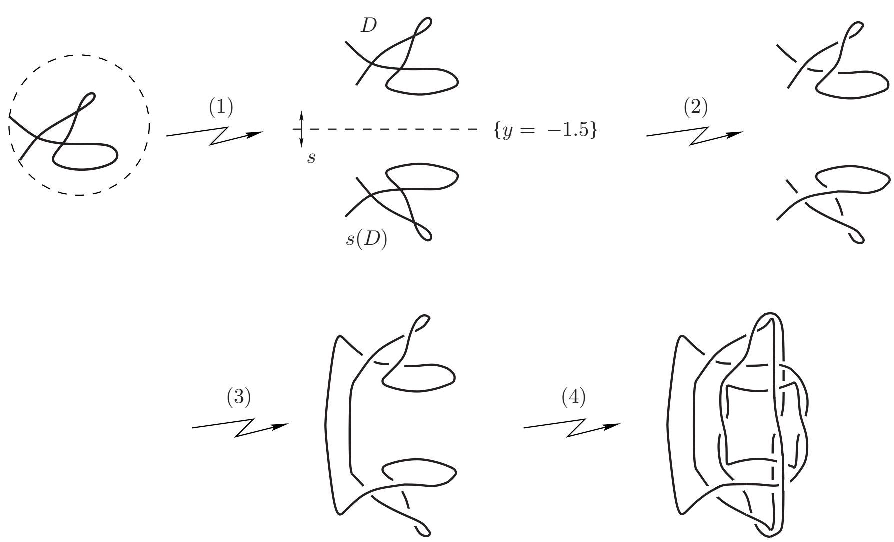
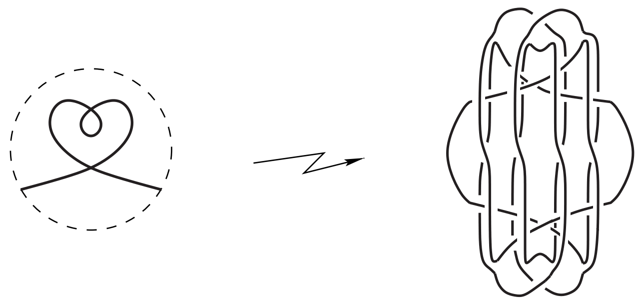


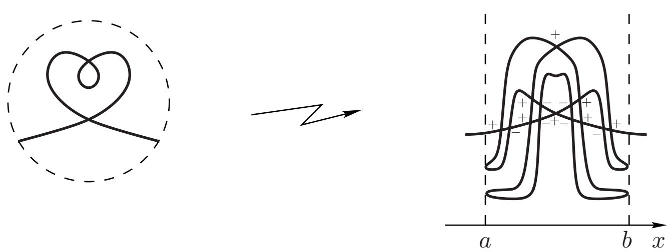
Figure 1.

Example 2.4. We borrowed this divide from [3]. The corresponding knot is 10_{145}.

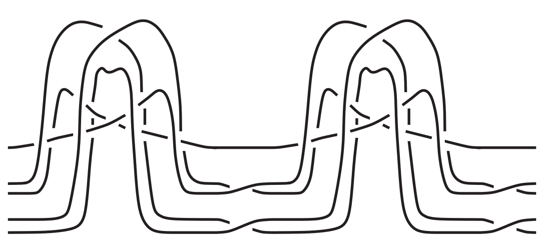


Remark 2.5. As it was noted in [8] the theorem is also valid in the situation when we allow closed immersed components in the definition of a divide.

2.1. **Proof of the Theorem 2.2.** The theorem follows from the results [8] which give a representative braid for an *ordered Morse signed divide* (OMS). An OMS is a divide D such that the x-coordinate as a function on D has only two critical values: a as the minimum critical value and b as the maximum critical value; and the x-coordinate of each double point is between a and b. Besides this a sign $+$ or $-$ is attached to each double point of D. See the details in [8]. An OMS is not a generic divide in our sense. Also in [8] there is an algorithm transforming an arbitrary divide to OMS (and attaching signs to double points). We use such a particular transformation pulling down a narrow tail near each critical point of the function $x|_D$ and then move a minimum (resp. maximum) left (resp. right) to the level a (resp. b). For the divide of Example 2.4 this transformation gives the following OMS:



Then applying Proposition 4.2 from [8] we get the following representative braid:



The closure of this braid gives a knot diagram isotopic to ours. It is clear that the same arguments work for any generic divide as well.

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