

## KODAIRA DIMENSION OF SYMMETRIC POWERS

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ABSTRACT. We compute the plurigenera and the Kodaira dimension of the  $d$ th symmetric power  $S^d X$  of a smooth projective variety  $X$ . As an application we obtain genus estimates for the curves lying on  $X$ .

We work over the complex numbers. When  $X$  is a smooth projective curve of genus  $g$ , elementary arguments show that the  $d$ th symmetric power  $S^d X$  is uniruled as soon as  $d > g$ , and therefore that the plurigenera vanish. When the dimension of  $X$  is greater than one, the situation is quite different. Using ideas of Roitman [Ro1, Ro2] and Reid [Re], we prove:

**Theorem 1.** *Let  $X$  be smooth projective variety with  $n = \dim X > 1$ . Let  $\Sigma_d$  be a desingularization of  $S^d X$ . Then there are isomorphisms*

$$S^d H^0(X, \omega_X^{\otimes m}) \cong H^0(\Sigma_d, \omega_{\Sigma_d}^{\otimes m})$$

whenever  $mn$  is even.

**Corollary 1.** *With the previous assumptions, the  $m$ th plurigenus*

$$P_m(\Sigma_d) = \binom{d + P_m(X) - 1}{d}$$

whenever  $mn$  is even. The Kodaira dimension  $\kappa(\Sigma_d) = d\kappa(X)$ .

*Proof.* The first formula is an immediate consequence of the theorem. It implies that

$$P_m(\Sigma_d) = O(P_m(X)^d) = O(m^{d\kappa(X)})$$

which yields the second formula.  $\square$

(D. Huybrechts pointed out to us that these statements were already known in the case  $\dim X = 2$  and were used to compute the Kodaira dimension of the Hilbert scheme of  $d$  points on  $X$ ; see [HL], theorem 11.1.2.)

Recall that a projective variety  $Z$  is uniruled provided there exists a variety  $Z'$  and dominant rational map  $Z' \times \mathbb{P}^1 \dashrightarrow Z$  which is nontrivial on the second factor. The reference [K] is more than adequate for standard properties of uniruled varieties.

**Corollary 2.** *If  $X$  has nonnegative Kodaira dimension, then  $S^d X$  is not uniruled for any  $d$ .*

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*Proof.* Since uniruledness is a birational property, it is enough to observe that  $\Sigma_d$  is not uniruled because it has nonnegative Kodaira dimension.  $\square$

The most interesting corollaries involve genus estimates for curves lying on  $X$ . The phrase “ $d$  general points of  $X$  lie on an irreducible curve with genus  $g$  normalization” will mean that there is an irreducible quasiprojective family  $\mathcal{C} \rightarrow T$  of smooth projective genus  $g$  curves and a morphism  $\mathcal{C} \rightarrow X$  which is a generically one-to-one on the fibers  $\mathcal{C}_t$  and such that the morphism from the relative symmetric power

$$S^d\mathcal{C} := \mathcal{C} \times_T \mathcal{C} \times_T \dots \mathcal{C} / S_d$$

to  $S^dX$  is dominant.

**Corollary 3.** *Suppose that the Kodaira dimension of  $X$  is nonnegative and that  $d$  general points lie on an irreducible curve with genus  $g$  normalization. Then  $g \geq d$ .*

*Proof.* Assume the contrary that  $g < d$ , and let  $\mathcal{C} \rightarrow T$  be the corresponding family. Then each fiber  $S^d\mathcal{C}_t$  is a projective space bundle over the Jacobian  $J(\mathcal{C}_t)$  by Abel-Jacobi; in particular, it is uniruled. Therefore  $S^d\mathcal{C}$  and hence  $S^dX$  are uniruled, but this contradicts the previous corollary.  $\square$

**Corollary 4.** *Suppose that  $X$  has general type and that  $d$  general points lie on an irreducible curve with genus  $g$  normalization. Then  $g > d$ .*

*Proof.* We assume that  $g \leq d$  for some family  $\mathcal{C} \rightarrow T$ . By the previous corollary, we may suppose that  $g = d$ . Denote the maps  $S^d\mathcal{C} \rightarrow S^dX$  and  $S^d\mathcal{C} \rightarrow T$  by  $p$  and  $\pi$ , respectively. The map  $p$  is dominant and generically injective on the fibers of  $\pi$ . If a general fiber  $p^{-1}(Z)$  has positive dimension, then an irreducible hyperplane section  $H \subset T$  meets  $\pi(p^{-1}(Z))$ . Therefore  $S^d\mathcal{C} \times_T H \rightarrow S^dX$  is still dominant, and we may replace  $T$  by  $H$  and  $S^d\mathcal{C}$  by the fiber product. By continuing in this way, we can assume that  $p$  is generically finite. Choose a desingularization  $\overline{T}$  of a compactification of  $T$ , and a nonsingular compactification  $S$  of  $S^d\mathcal{C} \times_T \overline{T}$  such that  $p$  extends to a morphism of  $S$  to a desingularization  $\Sigma_d$  of  $S^dX$ . We then have  $\kappa(\Sigma_d) \leq \kappa(S)$  which implies that  $S$  has general type. On the other hand the general fiber of  $S \rightarrow \overline{T}$  is  $S^d\mathcal{C}_t$  is birational to an Abelian variety by the theorems of Abel and Jacobi. This implies that

$$\kappa(S) \leq \dim\overline{T} + \kappa(S^d\mathcal{C}_t) = \dim\overline{T} < \dim S$$

by [Mo, 2.3], but this is impossible since  $S$  has general type.  $\square$

### 1. PROOF OF THE MAIN THEOREM

Recall [Re] that a variety  $Y$  has canonical singularities provided that

- (1)  $Y$  is normal.
- (2)  $\omega_Y^{[r]} := (\omega_Y^{\otimes r})^{**}$  is locally free for some  $r > 0$ , where  $\omega_Y = (\Omega_Y^{\dim Y})^{**}$ .
- (3) If  $f : Y' \rightarrow Y$  is a resolution of singularities, then  $f_*\omega_{Y'}^{\otimes r} = \omega_Y^{[r]}$ .

The smallest such  $r$  is called the index. If  $Y$  is canonical, then the third condition holds for all  $r \geq 1$  [Re, 1.3]. Thus the index is the smallest  $r$  for which  $\omega_Y^{[r]}$  is locally free. It is enough to test the last condition for a particular resolution of singularities. This condition is equivalent to a more widely used condition involving pullbacks of canonical divisors.

**Lemma 1.** *Let  $Z$  be a smooth variety on which a finite group  $G$  acts. Let  $Y = Z/G$ . If  $Y$  has canonical singularities of index dividing  $r$ , then*

$$H^0(Y, \omega_Y^{[r]}) \subseteq H^0(Z, \omega_Z^{\otimes r})^G.$$

*If the fix point locus has codimension greater than one, equality holds.*

*Proof.* Construct a commutative diagram

$$\begin{array}{ccc} Z' & \longrightarrow & Z \\ g' \downarrow & & \downarrow g \\ Y' & \xrightarrow{f} & Y \end{array}$$

where  $Y' \rightarrow Y$  is a desingularization, and  $Z'$  is a  $G$ -equivariant desingularization of the fiber product. Then there are inclusions

$$\omega_Y^{[r]} = f_* \omega_{Y'}^{\otimes r} \subseteq (f \circ g')_*(\omega_{Z'}^{\otimes r})^G = g_*(\omega_Z^{\otimes r})^G.$$

This implies the first part of the lemma. The inclusion  $\omega_Y^{[r]} \subseteq g_*(\omega_Z^{\otimes r})^G$  is an equality on the complement of the fixed point locus. Since  $\omega_Y^{[r]}$  is locally free (hence reflexive) and  $g_*(\omega_Z^{\otimes r})^G$  is torsion free, the second statement follows.  $\square$

**Proposition 1.** *Let  $X$  be a smooth variety of dimension  $n > 1$ . Then  $S^d X$  has canonical singularities of index 1 if  $n$  is even, and canonical singularities of index at most 2 if  $n$  is odd.*

*Proof.* As the result is local analytic for  $X$ , we may replace it by  $\mathbb{C}^n$ . Consider the action of the symmetric group  $S_d$  on  $\mathbb{C}^{nd} = \mathbb{C}^n \times \dots \times \mathbb{C}^n$  by permutation of factors, and let  $h : S_d \rightarrow GL_{nd}(\mathbb{C})$  be the corresponding homomorphism. This action is equivalent to a direct sum of  $n$  copies of the standard representation  $\mathbb{C}^d$  where  $S_d$  acts via permutation matrices. Therefore  $h(S_d)$  does not contain any quasi-reflections (because  $n > 1$ ) and  $\det(h(\sigma)) = \text{sign}(\sigma)^n$ . When  $n$  is even,  $h(S_d) \subset SL_{nd}(\mathbb{C})$ . This implies that  $S^d X = \mathbb{C}^{nd}/S_d$  is Gorenstein by [W], and therefore canonical of index one [Re, 1.8].

The case when  $n$  is odd is more laborious. For any element  $\sigma \in S_d$  of order  $r$ , define  $S(\sigma)$  as follows: choose a primitive  $r$ th root of unity  $\epsilon$  and express the eigenvalues of  $h(\sigma)$  as  $\lambda_i = \epsilon^{a_i}$  where  $0 \leq a_i < r$ , set  $S(\sigma) = \sum a_i$ . By [Re, 3.1], to prove that  $S^d X$  is canonical it will suffice to verify that  $S(\sigma) \geq r$  for every element  $\sigma$  of order  $r$ . Let  $\mathbb{C}^d$  be the permutation representation of  $S_d$ . If  $e_1, \dots, e_d$  is the standard basis then  $\sigma \cdot e_i = e_{\sigma(i)}$ . If  $\epsilon$  is a primitive  $r$ th root of unity, then it is easy to see that the eigenvectors of the cycle  $\sigma = (12 \dots r)$  acting on  $\mathbb{C}^d$  are  $e_1 + \epsilon^i e_2 + \dots + \epsilon^{(r-1)i} e_r$  and  $e_{r+1}, \dots, e_d$ . Therefore the nonunit eigenvalues are  $\epsilon, \dots, \epsilon^{r-1}$  and these occur with multiplicity one. Hence  $S(\sigma) = nr(r-1)/2 \geq r$  as required. The general case is similar. Let  $\sigma$  be a permutation of order  $r$  and  $\epsilon$  as before. Write  $\sigma$  as a product of disjoint cycles of length  $r_i$ . Therefore  $r$  is the least common multiple of the  $r_i$ , and let  $r'_i = r/r_i$ . A list (with possible repetitions) of the nonunit eigenvalues of  $\sigma$  acting on  $\mathbb{C}^d$  is

$$\epsilon^{r'_1}, \dots, \epsilon^{r'_1(r_1-1)}, \epsilon^{r'_2}, \dots, \epsilon^{r'_2(r_2-1)}, \dots$$

Therefore

$$S(\sigma) = \frac{n}{2} [r'_1 r_1 (r_1 - 1) + r'_2 r_2 (r_2 - 1) + \dots] \geq r$$

and this proves that  $S^d X$  is canonical. It remains to check that the index is at most 2. For this it suffices to observe that if  $x_i$  are coordinates on  $\mathbb{C}^{nd}$ , then

$$(dx_1 \wedge \dots \wedge dx_{nd})^{\otimes 2}$$

is  $S_d$  invariant. This determines a generator of  $\omega_{S^d X}^{[2]}(X)$ , which shows that this module is free.  $\square$

*Proof of the main theorem.* Let  $m$  be an integer such that  $mn$  is even (hence a multiple of the index of  $S^d X$ ). Then

$$H^0(\omega_{\Sigma_d}^{\otimes m}) = H^0(\omega_{S^d X}^{[m]}) = H^0(\omega_{X^d}^{\otimes m})^{S_d}.$$

By Künneth's formula, this equals

$$[H^0(\omega_X^{\otimes m}) \otimes \dots \otimes H^0(\omega_X^{\otimes m})]^{S_d} = S^d H^0(\omega_X^{\otimes m}).$$

$\square$

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