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ON BIFURCATION POINTS OF A COMPLEX POLYNOMIAL

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Abstract. Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial of degree d. Assume that the set $\tilde{K}_\infty(f) = \{y \in \mathbb{C}$: there is a sequence $x_l \to \infty$ s.t. $f(x_l) \to y$ and $\|df(x_l)\| \to 0\}$ is finite. We prove that the set $\tilde{K}(f) = K_0(f) \cup \tilde{K}_\infty(f)$ of generalized critical values of f (hence in particular the set of bifurcation points of f) has at most $(d-1)^n$ points. Moreover, $\#\tilde{K}_\infty(f) \leq (d-1)^{n-1}$. We also compute the set $\tilde{K}(f)$ effectively.

1. Introduction

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial mapping. It is well-known that f is a fibration outside a finite set. The smallest such set is called *the bifurcation set of* f; we denote it by $B(f)$. It can be proved that the set $K_0(f)$, the set of critical values of f, is contained in $B(f)$. But in general the set $B(f)$ is bigger than $K_0(f)$. It also contains the set $B_\infty(f)$ of bifurcations points at infinity. Briefly speaking the set $B_\infty(f)$ consists of points at which f is not a locally trivial fibration at infinity (i.e., outside a compact set). In the paper [8] we have estimated the number of points in sets $B(f)$ and $B_\infty(f)$. The aim of this paper is to obtain a better estimation, but only for a special class of polynomials (this class coincides with the class of all polynomials for $n = 1, 2$ only). Let

$$\tilde{K}_\infty(f) = \{y \in \mathbb{C} : \textit{there is a sequence } x_l \to \infty \textit{ s.t. } f(x_l) \to y \textit{ and } \|df(x_l)\| \to 0\}.$$

If $c \notin \tilde{K}_\infty(f)$, then we say that f satisfies *Fedoryuk's condition* at c. This set has been studied in [2] and [10]. It is well-known ([10]) that $B_\infty(f) \subset \tilde{K}_\infty(f)$. In particular $B(f) \subset \tilde{K}(f) = K_0(f) \cup \tilde{K}_\infty(f)$. Moreover, if $n = 2$ we have $B_\infty(f) = \tilde{K}_\infty(f)$ and $B(f) = \tilde{K}(f)$ (see [4], [5], [9]). In this paper we give a sharp estimation of the numbers $\#\tilde{K}_\infty(f)$ and $\#\tilde{K}(f)$ (and hence also the numbers $\#B_\infty(f)$ and $\#B(f)$), provided $\#\tilde{K}_\infty(f) < \infty$. We also give an effective method to compute the set $\tilde{K}(f)$. Our main result is:

Theorem 1.1. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial of degree $d > 0$. Assume that the set $\tilde{K}_\infty(f)$ is finite. Let $a = \#\tilde{K}_\infty(f)$ and $b = \#\tilde{K}(f)$. Then:*
1) $(d-1)a + b \leq d(d-1)^{n-1}$,
2) $a \leq (d-1)^{n-1}$ *and* $b \leq (d-1)^n$,

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3) *if* $\tilde{K}_\infty(f) \neq \emptyset$, *then* $b \leq \max\{1,\ (d-1)^n - d + 1\}$,

4) *if* e *denotes the number of isolated critical points of* f, *then* $a + e \leq (d-1)^n$ *and* $da + e \leq d(d-1)^{n-1}$,

5) *moreover, if* $a > 0$, *then* $a + e \leq \max\{1,\ (d-1)^n - d + 1\}$.

Corollary 1.1. *Let* $f : \mathbb{C}^n \to \mathbb{C}$ *be a polynomial of degree* $d > 0$. *If* $\#\tilde{K}_\infty(f) = (d-1)^{n-1}$, *then* $\tilde{K}(f) = \tilde{K}_\infty(f)$ *and* f *has no isolated critical points.*

Proof. Indeed, we have $(d-1)a + b \leq d(d-1)^{n-1}$ and $a \leq b$, hence $a = b$. Moreover, since $da + e \leq d(d-1)^{n-1}$, we obtain $e = 0$. □

Remark 1.1. Let us note that for $n = 2$ the set $\tilde{K}_\infty(f)$ is always finite and $B_\infty(f) = \tilde{K}_\infty(f)$ (see [4], [5], [9]). In particular, for $n = 2$ we recover a well-known fact ([3], [9]) that $\#B_\infty(f) \leq d - 1$. Moreover, we get a sharp estimation of numbers $\#B(f)$ and $\#B_\infty(f)$ in the class of all polynomials $f \in \mathbb{C}[x, y]$ of degree d.

2. Preliminaries

Let us recall that a mapping $f : \mathbb{C}^n \to \mathbb{C}^m$ is *not proper* at a point $y \in \mathbb{C}^m$ if there is no neighborhood U of y such that $f^{-1}(\overline{U}))$ is compact. In other words, f is not proper at y if there is a sequence $x_l \to \infty$ such that $f(x_l) \to y$. Let S_f denote the set of points at which the mapping f is not proper. We have the following characterization of the set S_f (see [6], [7]):

Theorem 2.1. *Let* $F = (F_1, ..., F_m) : \mathbb{C}^n \to \mathbb{C}^m$ *be a generically-finite polynomial mapping. Then the set* S_F *is an algebraic subset of* $\mathbb{C}^m$ *and it is either empty or it has pure dimension* $n - 1$. *Moreover, if* $n = m$ *we have*

$$\deg S_F \leq \frac{(\prod_{i=1}^n \deg F_i) - \mu(F)}{\min_{1 \leq i \leq n} \deg F_i},$$

where $\mu(F)$ *denotes the geometric degree of* F *(i.e., it is a number of points in a generic fiber of* F*).*

In the proof of Theorem 1.1 we need the following technical lemmas. The first lemma follows from the Bezout theorem in the version of Vogel.

Lemma 2.1. *Let* A *be an irreducible algebraic subvariety of* $\mathbb{C}^N$ *and let* H *be a linear subspace of* $\mathbb{C}^N$. *Assume that the set* $H \cap A = \{x_1, ..., x_r\}$ *is finite. Then* $\deg A \geq r$. *More precisely, if germ* $\mathbf{A}_{x_i}$ *have* m_i *irreducible components, for* $i = 1, ..., r$, *then* $\deg A \geq \sum_{i=1}^r m_i$.

The next lemma is:

Lemma 2.2. *Let* $B \subset A$ *be algebraic subsets of* $\mathbb{C}^{N+1}$, $\dim B < \dim A = n$. *Let* L *be a line and* M *a linear subspace of* $\mathbb{C}^{N+1}$, *which contains* L, $\dim M = n$. *Assume that* $L \not\subset B$. *Then there exists a linear projection* $p : \mathbb{C}^{N+1} \to M$ *such that* p *restricted to* A *is finite and* $L \not\subset p(B)$. *In particular* p *is proper on* A.

Proof. Take a point $a \in L \backslash B$. Let Λ be the Zariski closure of the cone $\bigcup \overline{ax},\ x \in B$. It is easy to see that $\dim \Lambda \leq n$. Let H_∞ be the hyperplane at infinity of $\mathbb{C} \times \mathbb{C}^N$. For any $Z \subset \mathbb{C}^N$ denote by $\tilde{Z}$ the projective closure of Z. Observe that

$$\dim H_\infty \cap (\tilde{\Lambda} \cup \tilde{\Gamma} \cap \tilde{M}) \leq n - 1.$$

Thus, there is a projective subspace $Q \subset H_\infty$ of dimension $N-n$, which is disjoint with $(\tilde{\Lambda} \cup \tilde{A} \cap \tilde{M})$. Denote by $p_Q : \mathbb{P}^{N+1} \setminus Q \to \tilde{M}$ the linear projection determined by the subspace Q.

Now, let $p : \mathbb{C}^{N+1} \to M$ be the restriction of p_Q to $\mathbb{C}^{N+1}$. It is easily seen that p has desired properties, i.e., $p : A \to M$ is a finite mapping and $a \notin L \cap p(B)$. □

Lemma 2.3. *Let $\Phi = (\phi_1, ..., \phi_n) : \mathbb{C}^n \to \mathbb{C}^m$ be a polynomial mapping with* deg $\phi_i = d$, *for $i = 1, ..., m$. Let $r = dim\ \Phi(\mathbb{C}^n)$. Assume that there is a variety W, which contains $\mathbb{C}^n$ as a dense subset and a polynomial proper mapping $\overline{\Phi} : W \to \mathbb{C}^n$, such that $\Phi = res_{\mathbb{C}^n}\overline{\Phi}$. Let q be a maximal number of connected components of fibers of $\overline{\Phi}$. Then $q \le d^r$. Moreover, if $r = n$ and the mapping Φ is not proper, then $q \le d^n - d$.*

Proof. First, taking the normalization we can assume that the variety W is normal. Let $\Gamma = cl(\Phi(\mathbb{C}^n))$ and let $p : \Gamma \to \mathbb{C}^r$ be a finite linear projection. Take $\Phi' = p \circ \Phi$. If q' denotes a maximal number of connected components of fibers of $\overline{\Phi'}$, then it is easy to see that $q' \ge q$. Moreover, if a projection p is sufficiently general, then we have $\Phi' = (\phi'_1, ..., \phi'_r)$, where deg $\phi'_i = d$, for $i = 1, ..., r$.

Consequently we can assume that $\Phi' = \Phi$, i.e., that the mapping Φ is a dominant mapping. By Bezout's Theorem we have that a generic fiber of Φ has at most d^r irreducible components. It implies that a generic fiber of the mapping $\overline{\Phi}$ also has at most d^r irreducible components. By the Stein Factorization Theorem there exist a normal variety S, and regular surjective mappings $p : W \to S$, $q : S \to \mathbb{C}^r$, such that $\overline{\Phi} = q \circ p$, where p has only connected fibers and q is finite. Moreover, it is easy to see that the geometric degree $\mu(q)$ of the mapping q is estimated by d^r. Since varieties $S, \mathbb{C}^r$ are normal and the mapping q is finite, we have that every fiber of the mapping q has at most d^r points. Consequently, we obtain that every fiber of $\overline{\Phi}$ has at most d^r connected components.

Now assume that $r = n$ and the mapping Φ is not proper. In particular $S_\Phi \ne \emptyset$. By Theorem 2.1 we get that the geometric degree $\mu(\Phi)$ of the mapping Φ is estimated by $d^n - d(\deg S_\Phi) \le d^n - d$. In particular a generic (and consequently every) fiber of $\overline{\Phi}$ has at most $d^n - d$ connected components. □

3. Estimations

Now we can pass to the proof of Theorem 1.1. In fact we prove slightly more general results. Let $a = \#\tilde{K}_\infty(f)$ and $b = \#\tilde{K}(f)$. We begin with:

Theorem 3.1. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial of degree $d > 0$. Assume that the set $\tilde{K}_\infty(f)$ is finite. Let $\Phi = (\frac{\partial f}{\partial x_1}, ... \frac{\partial f}{\partial x_n})$ and let $r = dim\ \Phi(\mathbb{C}^n)$. Then $b \le (d-1)^r$. Moreover, if $\tilde{K}_\infty(f) \ne \emptyset$, then we have better estimation $b \le \max\{1,\ (d-1)^n - d + 1\}$. Finally, if e denotes the number of isolated critical points of f, then $a + e \le (d-1)^n$. If $\tilde{K}_\infty(f) \ne \emptyset$, then we have better estimation $a + e \le \max\{1,\ (d-1)^n - d + 1\}$.*

Proof. For $n = 1$ the theorem is obviously true. Let $n > 1$. Consider the polynomial mapping $\Phi = (\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}) : \mathbb{C}^n \to \mathbb{C}^n$. Let $r = \dim\ \Phi(\mathbb{C}^n)$. It is well-known (see e.g., [7]) that there is a normal variety W, which contains $\mathbb{C}^n$ as a dense subset and a polynomial proper mapping $\overline{\Phi} : W \to \mathbb{C}^n$, such that $\Phi = res_{\mathbb{C}^n}\overline{\Phi}$. By a proper modification of W we can assume that the mapping f has a regular extension $\overline{f} : W \to \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$. Let $A = \overline{\Phi}^{-1}(0)$. It is easy to see that

$\tilde{K}(f) = \overline{f}(A) \setminus \{\infty\}$. It means that $\#\tilde{K}(f)$ is estimated by the number of connected components of the set A. Consequently, by Lemma 2.3 we have that $b \leq (d-1)^r$.

Moreover, if $\tilde{K}_\infty(f) \neq \emptyset$ and $r = n$, then we have better estimation $b \leq (d-1)^n - d + 1$. If $r < n$, then $b \leq (d-1)^r \leq \max\{1,\ (d-1)^n - d + 1\}$. Finally, if e denotes the number of isolated critical points of f, then $a + e \leq (d-1)^n$ and again, if $\tilde{K}_\infty(f) \neq \emptyset$, then $a + e \leq \max\{1,\ (d-1)^n - d + 1\}$. □

Corollary 3.1. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial of degree $d > 0$. Assume that the set $\tilde{K}_\infty(f)$ is finite. Then*

$$b \leq (d-1)^n.$$

Theorem 3.2. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial of degree $d > 0$. Assume that the set $\tilde{K}_\infty(f)$ is finite. Then*

$$(d-1)a + b \leq d(d-1)^{n-1}.$$

Proof. For $n = 1$ the theorem is obviously true. Let $n > 1$. Let us define a polynomial mapping $\Psi : \mathbb{C}^n \to \mathbb{C} \times \mathbb{C}^n$ by

$$\Psi = (f, \frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}).$$

Denote $\Gamma = \Psi(\mathbb{C}^n)$, and by $\overline{\Gamma}$ its Zariski closure. Let $r = \dim \overline{\Gamma}$. Consider the line $L := \mathbb{C} \times \{(0, ..., 0)\} \subset \mathbb{C} \times \mathbb{C}^n$. We further identify this line with a copy of $\mathbb{C}$. By definition of $\tilde{K}(f)$ we have

$$\tilde{K}(f) = L \cap \overline{\Gamma}.$$

We further identify this line with a copy of $\mathbb{C}$. We have two possibilities:

1) $r = n$, i.e., Ψ is a generically finite mapping,

2) $r < n$, i.e., Ψ is not a generically-finite mapping.

Let us consider case 1). By the definition of $\tilde{K}_\infty(f)$ and Ψ we have

$$\tilde{K}_\infty(f) = L \cap S_\Psi,$$

where S_Ψ denotes the set of points at which the mapping Ψ is not proper. Recall that by the assumption the set $\tilde{K}_\infty(f)$ is finite, hence also $\#L \cap S_\Psi < \infty$. Choose a linear space M of dimension n, which contains the line L. Lemma 2.2 applied to $A = \overline{\Gamma}$ and $B = S_\Psi$ yields a projection $p : \mathbb{C}^{n+1} \to M$ which is finite on $\overline{\Gamma}$ and such that $L \not\subset p(S_\Psi)$. Denote $X = p(S_\Psi)$. Then $\tilde{K}_\infty(f) \subset X$ and $L \not\subset X$. Since p is proper on $\overline{\Gamma}$, we obtain that $X = S_F$, where $F = p \circ \Psi$.

Moreover, we have $F_i = a_{i0} f + \sum_{k=1}^n a_{ik} \frac{\partial f}{\partial x_i}$. If we take a projection p to be sufficiently general, then by a linear change of coordinates

$$T(x_1, ..., x_n) = (x_1, x_2 - (a_{20}/a_{10})x_1), ..., x_n - (a_{n0}/a_{10})x_1),$$

we get that $T \circ F = (F_1, ..., F_n)$, where $\deg F_1 = d$, $\deg F_i = d - 1$ for $i > 1$. Hence we can assume that $F = (F_1, ..., F_n)$, where $\deg F_1 = d$, $\deg F_i = d - 1$ for $i > 1$.

Let us estimate the geometric degree $\mu(F)$ of F. We have $\mu(F) = \mu(p \circ \Psi) \geq \mu(res_{\overline{\Gamma}} p) = \deg\ \overline{\Gamma}$. Let us estimate the degree of $\overline{\Gamma}$. Consider a linear subspace $H = L = \mathbb{C} \times \{0, 0, ..., 0\}$ and take $A = \overline{\Gamma}$. It is easy to see that $H \cap A = \tilde{K}(f)$. By Lemma 2.1 we have $\deg A \geq b$, consequently $\mu(F) \geq b$. Now Theorem 2.1 yields that the degree of the variety $X \subset M$ is bounded by $(d(d-1)^{n-1} - b)/(d-1)$. So, the set $X \cap L$ has no more than $(d(d-1)^n - b)/(d-1)$ points. Finally we obtain that $a \leq (d(d-1)^n - b)/(d-1)$ and that $(d-1)a + b \leq d(d-1)^n$.

Now let us consider case 2). It is easy to see that $a = b$. Choose a linear space $M \cong \mathbb{C}^{r+1}$, which contains the line L. Lemma 2.2 applied to $A = \overline{\Gamma}$ and $B = S_\Psi$ yields a projection $p : \mathbb{C}^{n+1} \to M$ which is finite on $\overline{\Gamma}$ and such that $L \not\subset p(\overline{\Gamma})$. Denote $X = p(\overline{\Gamma})$. Then $\tilde{K}(f) \subset X$ and $L \not\subset X$. Let $F = (F_0, F_1, ..., F_r) = p \circ \Psi$. We have $F_i = a_{i0} f + \sum_{k=1}^n a_{ik} \frac{\partial f}{\partial x_k}$, where $i = 0, 1, ..., r$. Moreover, we can assume that $F_0 = f$. By a linear change of coordinates $T(x_0, ..., x_n) = (x_0, x_1 - a_{10}x_1, ..., x_n - a_{n0}x_1)$ we get that

$$T \circ F = (f, \sum_{k=1}^n a_{k1} \frac{\partial f}{\partial x_k}, ..., \sum_{k=1}^n a_{kr} \frac{\partial f}{\partial x_k}).$$

In particular we can assume that $F = (f, F_1, ..., F_r)$. Take a mapping $\Lambda : \mathbb{C}^r \ni (t_1, ..., t_r) \to (\sum_{k=1}^r a_{1k}t_k, ..., \sum_{k=1}^r a_{nk}t_k)$. Taking a projection p (and hence values a_{ij}) sufficiently general, we can assume that the linear subspace $\Lambda(\mathbb{C}^r)$ meets the fiber $F^{-1}(0)$ in the finite and non-empty set. This means that a mapping $G := F \circ \Lambda : \mathbb{C}^r \to X$ is generically-finite, in particular it must be dominant. By the construction we have $G = (g, \frac{\partial g}{\partial t_1}, ..., \frac{\partial g}{\partial t_r})$, where $g = f \circ \Lambda$. Moreover, $\tilde{K}(f) \subset X \cap L = \overline{G(\mathbb{C}^r)} \cap L = \tilde{K}(g)$ and we can use Theorem 3.1. Consequently $b \leq (d-1)^r \leq (d-1)^{n-1}$. Since $a = b$, we have $(d-1)a + b = db \leq d(d-1)^{n-1}$. This finishes the proof of Theorem 3.2. □

We can summarize our results as:

Corollary 3.2. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial of degree $d > 0$. Assume that the set $\tilde{K}_\infty(f)$ is finite. Let $\Phi = (\frac{\partial f}{\partial x_1}, ... \frac{\partial f}{\partial x_n})$ and $r = dim\ \Phi(\mathbb{C}^n)$. If $r = n$, then $a \leq (d-1)^{n-1}$ and $b \leq (d-1)^n$. If $r < n$, then $a = b \leq (d-1)^r$.*

Proof. Indeed, if $r < n$, then it is easy to see that $a = b$ (there is no isolated critical points) and the corollary follows from Theorem 3.1. Let $r = n$. We have $(d-1)a + b \leq d(d-1)^{n-1}$ and $a \leq b$. Consequently $da \leq d(d-1)^{n-1}$ and finally $a \leq (d-1)^{n-1}$. Moreover, $b \leq (d-1)^n$ by Theorem 3.1. □

Our last result is the following:

Theorem 3.3. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial of degree $d > 0$. Assume that the set $\tilde{K}_\infty(f)$ is finite. If f has e isolated critical points, then*

$$da + e \leq d(d-1)^{n-1}.$$

Proof. The proof goes along similar lines as the first part of the proof of Theorem 3.1. If $e = 0$, the result follows from Corollary 3.2. Hence, we can assume that $e > 0$, in particular we can assume that the mapping Ψ (we take the notation from the proof of Theorem 3.2) is generically finite. Let us consider mappings p, F and set X as above. Note that X is exactly the set of points at which the mapping $F = p \circ \Psi$ is not proper. As above we can assume that $F = (F_1, ..., F_n)$, where deg $F_1 = d$, deg $F_i = d - 1$ for $i > 1$.

Now let us estimate the geometric degree $\mu(F)$ of F more precisely. We have $\mu(F) = \mu(p \circ \Psi) = \mu(\Psi)\mu(res_{\overline{\Gamma}}p) = \mu(\Psi) \deg \overline{\Gamma}$. Let us estimate the degree of $\overline{\Gamma}$. Consider a linear subspace $H = L = \mathbb{C} \times \{0, 0, ..., 0\}$ and take $A = \overline{\Gamma}$. It is easy to see that $H \cap A = \tilde{K}(f) := \{a_1, ..., a_k\}$. Let m_i denote the number of irreducible components of (an analytic) germ $\mathbf{A}_{a_i} = \bigcup_{j=1}^{m_i} \mathbf{B}_{ij}$. By Lemma 2.1 we have deg $A \geq \sum_{i=1}^k m_i$. Let c be an isolated critical point of f. We say that c lies over an irreducible component $\mathbf{B}_{ij}$ of germ $\mathbf{A}_{a_i}$ if there is a small bal U around c, such that $\Psi(U) \subset \mathbf{B}_{ij}$. It is easy to see that every critical point lies over some $\mathbf{B}_{ij}$ and for a fixed component $\mathbf{B}_{ij}$, there is at most $\mu(\Psi)$ critical points which lie over it. In particular $e \le (\sum_{i=1}^k m_i)\mu(\Psi) \le (\deg A)\mu(\Psi) = \mu(F)$. In fact, if we also consider the points at infinity, which correspond to asymptotic values, we have stronger inequality

$$e + a \le (\sum_{i=1}^{k} m_i)\mu(\Psi) \le (\deg A)\mu(\Psi) = \mu(F).$$

Now the degree of the variety $X \subset M$ is bounded by $(d(d-1)^{n-1} - e - a)/(d-1)$ by Theorem 2.1. So, the set $X \cap L$ has no more than $(d(d-1)^n - e - a)/(d-1)$ points. Finally we obtain that $a \le (d(d-1)^n - e - a)/(d-1)$ and that $da + e \le d(d-1)^n$. □

Example 3.1 (see [8])**.** We show that our estimate is sharp to both $\tilde{K}_\infty(f)$ and $B_\infty(f)$. More precisely, we have:

For every $d > 0$ there are polynomials $g_n \in \mathbb{C}[x_1, ..., x_n]$; $n = 1, 2, ...,$ and $f_n \in \mathbb{C}[x_1, ..., x_n]$; $n = 2, 3, ...,$ of degree d, with finite sets $\tilde{K}_\infty(g_n)$ and $\tilde{K}_\infty(f_n)$ such that:

1) $\#\tilde{K}(g_n) = \#B(g_n) = (d-1)^n$;
2) $\#\tilde{K}_\infty(f_n) = \#B_\infty(f_n) = (d-1)^{n-1}$.

First we construct a polynomial g_n. Let us consider a polynomial of one variable $h(t) := t^d/d - t$ and take

$$g_n = \sum_{i=1}^{n} A_i h(x_i),$$

where numbers A_i are sufficiently general. It is easy to check that $\#K_0(g_n) = (d-1)^n$. Put $f_n(x_1, ..., x_n) := g_{n-1}(x_1, ..., x_{n-1})$. It is easy to see that $K_0(g_{n-1}) = \tilde{K}_\infty(f_n) = B_\infty(f_n)$ and consequently $\#\tilde{K}_\infty(f_n) = \#B_\infty(f_n) = (d-1)^{n-1}$.

Remark 3.1. It is worth mentioning that the set $\tilde{K}(f)$ can be computed effectively. In particular we are in a position to effectively check whether the set $\tilde{K}_\infty(f)$ is finite. Indeed, let us recall that $\Psi = (f, \frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}) = (\psi_1, ..., \psi_{n+1})$, $\Gamma = \Psi(\mathbb{C}^n)$, $L = \mathbb{C} \times \{0, ..., 0\}$ and $\tilde{K}(f) = L \cap \overline{\Gamma}$. Hence, it is enough to produce equations for the hypersurface $\overline{\Gamma}$. It can be done by using the Gröbner bases techniques.

Let us consider the ideal I given by polynomials $\{y_i - \psi_i(x)\}_{i=1,...,n+1}$ in the ring

$$R = \mathbb{C}[x_1, ..., x_n, y_1, ..., y_{n+1}].$$

In R we consider the lexicographic order, i.e., $x_1 > x_2 > ... > x_n > y_1 > ... > y_{n+1}$. Now compute a Gröbner basis $\mathcal{A}$ of the ideal I in R and then take $\mathcal{B} = \mathcal{A} \cap \mathbb{C}[y_1, ..., y_{n+1}]$. It is a standard fact that $\mathcal{B}$ is the Gröbner basis of the ideal $I(\overline{\Gamma})$ of the hypersurface $\overline{\Gamma}$. Consequently, we have $\tilde{K}(f) = \{y_1 \in \mathbb{C} : h(y_1, 0, ..., 0) = 0, \ for\ every\ h \in \mathcal{B}\}$. In particular, the set $\tilde{K}_\infty(f)$ is finite iff there exists a polynomial $h \in \mathcal{B}$, such that $h(y_1, 0, ..., 0) \not\equiv 0$.

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