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EXTREMAL METRICS FOR THE FIRST EIGENVALUE OF THE LAPLACIAN IN A CONFORMAL CLASS

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Abstract. Let M be a compact manifold. First, we give necessary and sufficient conditions for a Riemannian metric on M to be extremal for λ_1 with respect to conformal deformations of fixed volume. In particular, these conditions show that for any lattice Γ of $\mathbb{R}^n$, the flat metric g_Γ induced on $\mathbb{R}^n/\Gamma$ from the standard metric of $\mathbb{R}^n$ is extremal (in the previous sense). In the second part, we give, for any Γ, an upper bound of λ_1 on the conformal class of g_Γ and exhibit a class of lattices Γ for which the metric g_Γ maximizes λ_1 on its conformal class.

1. Introduction

Let M be a compact differentiable manifold of dimension $n \geq 2$ and let λ_1 be the functional which assigns to each Riemannian metric g on M the first positive eigenvalue of the Laplacian Δ_g of g. As λ_1 is not invariant under dilatations, we will restrict ourselves to Riemannian metrics having a given volume. It is known that if $n \geq 3$, then the functional λ_1 is unbounded (see [3]). However, for any metric g on M, the restriction of λ_1 to the conformal class $C(g) = \{\phi g \ ; \ \phi > 0$ and $V(\phi g) = V(g)\}$ of g is bounded (see [6] and [10]). Here $V(g)$ denotes the Riemannian volume of g.

This paper consists of two sections. The first one deals with the (constrained) critical points of λ_1 restricted to a conformal class $C(g)$. Such a critical metric will be called C-extremal (here criticity is defined in a generalized sense because λ_1 is not differentiable in general). The results we obtain give both a necessary condition and a sufficient one for a Riemannian metric g to be C-extremal:

- *Necessary condition* (Theorem 2.1): "There exists a finite family $\{f_1, ..., f_k\}$ of first eigenfunctions of Δ_g such that $\sum_{i\leq k} f_i^2 = 1$."
- *Sufficient condition* (Theorem 2.2): "There exists an $L_2(g)$-orthonormal basis $\{f_1, ..., f_m\}$ of the first eigenspace of Δ_g such that $\sum_{i\leq m} f_i^2 = 1$."

Note that conditions of this type concerning the "unconstrained" critical metrics were obtained by Nadirashvili [6] in dimension 2 and by the authors [7] in all dimensions. The necessary condition means that there exists a harmonic map f from (M, g) to a unit sphere with energy density $|df|^2 = \lambda_1(g)$ (see [5]). A

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direct consequence of the sufficient condition is that the metric of any compact homogeneous Riemannian space is C-extremal (Corollary 2.1).

This last assertion gives rise to many natural examples of C-extremal metrics. In particular, for any lattice Γ of $\mathbb{R}^n$, the flat metric g_Γ induced on the torus $\mathbf{T}^n_\Gamma = \mathbb{R}^n/\Gamma$ from the Euclidean metric of $\mathbb{R}^n$ is C-extremal. A natural question is then to know for which lattices $\Gamma \subset \mathbb{R}^n$ the flat metric g_Γ is a global maximizer for λ_1 on $C(g_\Gamma)$. In the second section we exhibit a class of lattices for which this property holds (Corollary 3.1). This class contains some well known lattices as the square and the hexagonal ones. But, the main result of this section is Theorem 3.1 which gives, for any lattice Γ, an upper bound of λ_1 on $C(g_\Gamma)$ in terms of an invariant of Γ:

$$\sup_{g\in C(g_\Gamma)} \lambda_1(g) \leq \frac{4\pi^2}{n} c(\Gamma^\star)$$

where $c(\Gamma^\star) = \inf\{\sum_{1\leq i\leq n} |\tau_i|^2;\ \{\tau_1, ..., \tau_n\}$ is a basis of the dual lattice $\Gamma^\star$ of $\Gamma\}$.

Note that in [9], Ros and the authors have obtained a 2-dimensional version of this last result. It should also be mentioned that, in the particular case where Γ is the square lattice of $\mathbb{R}^2$, this result is due to Li and Yau [10]. On the other hand, Nadirashvili [11] showed that the flat metric corresponding to the equilateral lattice maximizes λ_1 among all the metrics of the same volume on the 2-torus $\mathbf{T}^2$.

2. The functional λ_1 restricted to a conformal class of metrics

Let (M, g) be a compact connected smooth Riemannian manifold of dimension $n \geq 2$ and volume $V(g)$. Denote by $C(g)$ the set of Riemannian metrics conformal to g and having the same volume:

$$C(g) = \left\{ \phi g\ ;\ \phi > 0 \quad \text{and} \quad \int_M \phi^{n/2} \nu_g = V(g) \right\},$$

where ν_g is the Riemannian volume element of g. We consider the functional $\lambda_1 : C(g) \to \mathbb{R}$ where, $\forall g' \in C(g)$, $\lambda_1(g')$ is the first positive eigenvalue of the Laplace-Beltrami operator $\Delta_{g'}$ of g'. For any one parameter analytic family $(g_t)_t$ of $C(g)$, the function $t \to \lambda_1(g_t)$ has right and left derivatives w.r.t. t (see [7]) and we have

$$\frac{d}{dt}\lambda_1(g_t)\ |_{t=0^+} \leq \frac{d}{dt}\lambda_1(g_t)\ |_{t=0^-}.$$

Definition 2.1. The metric g is said to be C-extremal for the λ_1 functional if for any analytic conformal deformation $(g_t)_t \subset C(g)$ with $g_0 = g$, we have

$$\frac{d}{dt}\lambda_1(g_t)\ |_{t=0^+} \leq 0 \leq \frac{d}{dt}\lambda_1(g_t)\ |_{t=0^-}.$$

It is easy to check that this last condition is equivalent to

$$\lambda_1(g_t) \leq \lambda_1(g) + o(t), \quad \text{as} \quad t \to 0.$$

In the sequel we will denote by $E_1(g)$ the eigenspace associated to $\lambda_1(g)$.

Theorem 2.1. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 2$. If the metric g is C-extremal for λ_1, then there exists a finite family of first eigenfunctions $\{f_1, ..., f_k\} \subset E_1(g)$ satisfying $\sum_{i\leq k} f_i^2 = 1$.*

Remark. An equivalent formulation of Theorem 2.1 is: *If g is C-extremal, then there exists a harmonic map f from (M,g) to a unit sphere $\mathbb{S}^d$ satisfying $|df|^2 = \lambda_1(g)$. Indeed, a map $f = (f_1, ..., f_{d+1}) : (M,g) \to \mathbb{S}^d$ is harmonic if and only if $\Delta_g f_i = |df|^2 f_i$ for all $i \le d+1$ (see* [5]*).*

Reciprocally, we have the following theorem:

Theorem 2.2. *Let (M,g) be a compact Riemannian manifold of dimension n. If there exists an $L_2(g)$-orthonormal basis $\{f_1, ..., f_k\}$ of $E_1(g)$ such that $\sum_{i\le k} f_i^2$ is constant, then g is a C-extremal metric for λ_1.*

An immediate consequence of this theorem is the following:

Corollary 2.1. *If (M,g) is a compact Riemannian homogeneous space, then g is C-extremal for λ_1.*

Proof of Corollary 2.1. Let $f_1, ..., f_k$ be an $L_2(g)$-orthonormal basis of $E_1(g)$. An elementary argument shows that the function $\sum_{i\le k} f_i^2$ is invariant under the isometry group of (M,g). Thus it is constant and we can apply Theorem 2.2. □

Proof of Theorem 2.1. Let $g_t = \phi_t g \in C(g)$ be an analytic deformation of g and set $\alpha = \frac{d}{dt}\phi_t \mid_{t=0}$. One can easily deduce from Berger's work [2] (see also [1] and [7]) that there exists an $L_2(g)$-orthonormal basis $\{u_1, ..., u_k\}$ of $E_1(g)$ such that

$$\frac{d}{dt}\lambda_1(g_t) \mid_{t=0^+} = -\max_{i\le k} \int_M \alpha q(u_i)\nu_g \tag{1}$$

and

$$\frac{d}{dt}\lambda_1(g_t) \mid_{t=0^-} = -\min_{i\le k} \int_M \alpha q(u_i)\nu_g \tag{2}$$

where $q(u) = |du|^2 + \frac{n}{4}\Delta_g(u^2)$.

First step. There exists a finite family $\{f_1, ..., f_k\} \subset E_1(g)$ such that $\sum_{i\le k} q(f_i) = 1$.

Proof. This assertion is equivalent to the fact that the constant function 1 belongs to the convex hull K of the set $\{q(f)\ ;\ f \in E_1(g)\} \subset C^\infty(M)$. Let us then assume that $1 \notin K$. Since K is a convex cone and $K \cup \{1\}$ is contained in a finite dimensional subspace of $C^\infty(M)$ (endowed with the $L_2(g)$-inner product), there exists, by the classical separation results (see for instance [12]), a function β such that $\int_M \beta\nu_g > 0$ and, for any $f \in K$, $\int_M \beta f \nu_g < 0$. Now, we let $\beta_0 = \beta - \frac{1}{V(g)}\int_M \beta\nu_g$ and consider the deformation $g_t = \phi_t g \in C(g)$, where $\phi_t = (\frac{V(g)}{V(e^{t\beta_0}g)})^{2/n} e^{t\beta_0}$. Since $\frac{d}{dt}V\left(e^{t\beta_0}g\right) \mid_{t=0} = \frac{n}{2}\int_M \beta_0\nu_g = 0$, we get $\frac{d}{dt}\phi_t \mid_{t=0} = \beta_0$. On the other hand, for any $u \in E_1(g)$, $u \neq 0$, we have

$$\begin{aligned}\int_M \beta_0 q(u)\nu_g &= \int_M \beta q(u)\nu_g - \frac{1}{V(g)}\left(\int_M \beta\nu_g\right)\left(\int_M q(u)\nu_g\right)\\ &= \int_M \beta q(u)\nu_g - \frac{1}{V(g)}\left(\int_M \beta\nu_g\right)\left(\int_M |du|^2\nu_g\right) < 0.\end{aligned}$$

Therefore, applying (1) and (2) we get $\frac{d}{dt}\lambda_1(g_t) \mid_{t=0^-} \ge \frac{d}{dt}\lambda_1(g_t) \mid_{t=0^+} > 0$ which contradicts the C-extremality of g.

Second step. $\sum_{i\le k} f_i^2 = 1/\lambda_1(g)$.

Proof. Let $f : M \to \mathbb{R}^k$ be the map given by $f = (f_1, ..., f_k)$. As $\sum_{i\le k} q(f_i) = 1$ we obtain

$$\frac{n}{4}\Delta_g |f|^2 = \frac{n}{4}\Delta_g \left(\sum_{i\le k} f_i^2\right) = \sum_{i\le k} \left(q(f_i) - |df_i|^2\right) = 1 - |df|^2.$$

Now, for any $i \le k$, we have

$$|df_i|^2 = -\frac{1}{2}\Delta_g f_i^2 + f_i \Delta_g f_i = -\frac{1}{2}\Delta_g f_i^2 + \lambda_1(g) f_i^2.$$

Hence, the previous equation becomes

$$\frac{n-2}{4}\Delta_g |f|^2 = 1 - \lambda_1(g)|f|^2,$$

and then

$$\frac{n-2}{4}\Delta_g \left(|f|^2 - \frac{1}{\lambda_1(g)}\right) = -\lambda_1(g)\left(|f|^2 - \frac{1}{\lambda_1(g)}\right).$$

It follows from the positivity of Δ_g that $|f|^2 - \frac{1}{\lambda_1(g)} = 0$.

In conclusion, the family $\left\{\sqrt{\lambda_1(g)} f_1, ..., \sqrt{\lambda_1(g)} f_k\right\}$ satisfies the statement of Theorem 2.1.

Proof of Theorem 2.2. Let $\{f_1, ..., f_k\}$ be an $L_2(g)$-orthonormal basis of $E_1(g)$ such that $\sum_{i\le k} f_i^2 = 1$. For any $L_2(g)$-orthonormal basis $\{u_1, ..., u_k\}$ of $E_1(g)$ and any function $\beta \in C^\infty(M)$ we have

$$\sum_{i\le k} \int_M \beta q(u_i)\nu_g = \sum_{i\le k} \int_M \beta q(f_i)\nu_g.$$

Indeed, this sum is nothing but the $L_2(g)$-trace of the quadratic form Q defined on $E_1(g)$ by $Q(u) = \int_M \beta q(u)\nu_g$. Now, as $\sum_{i\le k} f_i^2 = 1$, we have

$$\sum_{i\le k} |df_i|^2 = -\frac{1}{2}\Delta_g \left(\sum_{i\le k} f_i^2\right) + \sum_{i\le k} \lambda_1(g) f_i^2 = \lambda_1(g)$$

and then

$$\sum_{i\le k} q(f_i) = \sum_{i\le k} |df_i|^2 + \frac{n}{4}\Delta_g \left(\sum_{i\le k} f_i^2\right) = \lambda_1(g).$$

Therefore,

$$\sum_{i\le k} \int_M \beta q(u_i)\nu_g = \lambda_1(g) \int_M \beta\nu_g.$$

Now let $g_t = \phi_t g \in C(g)$ be a conformal deformation of g and let $\alpha = \frac{d}{dt}\phi_t\mid_{t=0}$. As $V(g_t)$ is constant w.r.t. t, it follows that $\int_M \alpha\nu_g = 0$. Thus, for any $L_2(g)$-orthonormal basis $\{u_1, ..., u_k\}$ of $E_1(g)$, we have

$$\sum_{i\le k} \int_M \alpha q(u_i)\nu_g = 0,$$

and, therefore, $\min_{i\le k} \int_M \alpha q(u_i)\nu_g \le 0$ and $\max_{i\le k} \int_M \alpha q(u_i)\nu_g \ge 0$. Applying (1) and (2), we obtain the C-extremality of g. $\square$

3. A conformal upper bound for λ_1 on tori

Let Γ be a lattice of $\mathbb{R}^n$ and denote by g_Γ the flat metric induced on the torus $\mathbf{T}^n_\Gamma = \mathbb{R}^n/\Gamma$ from the Euclidean metric of $\mathbb{R}^n$. The purpose of this section is to give an explicit upper bound of λ_1 on $C(g_\Gamma)$. Recall that the spectrum of the Laplacian of g_Γ is given by: $Sp(g_\Gamma) = \{4\pi^2|\tau|^2 \ ; \ \tau \in \Gamma^\star\}$, where $\Gamma^\star = \{\tau \in \mathbb{R}^n \ ; \ \forall x \in \Gamma, \langle x, \tau\rangle \in \mathbf{Z}\}$ is the dual lattice of Γ. The first eigenvalue of g_Γ is then $\lambda_1(g_\Gamma) = 4\pi^2|\tau_0|^2$, where τ_0 is an element of minimal length in $\Gamma^\star\backslash\{0\}$. We denote by $c(\Gamma^\star)$ the infimum of $\sum_{i\leq n}|\tau_i|^2$, where $\{\tau_1, ..., \tau_n\}$ runs over all the basis of $\Gamma^\star$.

Theorem 3.1. *If (M, g) is a Riemannian manifold conformally equivalent to $(\mathbf{T}^n_\Gamma, g_\Gamma)$ with $V(g) = V(g_\Gamma)$, then*

$$\lambda_1(g) \leq \frac{4\pi^2}{n} c(\Gamma^\star).$$

Equality holds if and only if there exists a basis $\{\tau_1, ..., \tau_n\}$ of $\Gamma^\star$ satisfying $|\tau_1|^2 = ... = |\tau_n|^2 = c(\Gamma^\star)/n$, and if (M, g) is isometric to $(\mathbf{T}^n_\Gamma, g_\Gamma)$.

It follows from corollary 2.1 that for any lattice Γ, the flat metric g_Γ is an extremal metric for the functional λ_1 restricted to $C(g_\Gamma)$. The following corollary shows that for certain lattices Γ, the flat metric g_Γ corresponds in fact to the absolute maximum of λ_1 on $C(g_\Gamma)$.

Corollary 3.1. *Assume that $\Gamma^\star$ is generated by a basis $\{\tau_1, ..., \tau_n\}$ such that $|\tau_1|^2 = ... = |\tau_n|^2 = c(\Gamma^\star)/n$. Then, for any metric $g \in C(g_\Gamma)$,*

$$\lambda_1(g) \leq \lambda_1(g_\Gamma),$$

equality holding if and only if $g = g_\Gamma$.

Remarks. (1) Any lattice Γ of $\mathbb{R}^2$ is, up to isometry, of the form $\Gamma = \mathbf{Z}(c, 0) + \mathbf{Z}(a, b)$ with $c > 0$, $b > 0$, $0 \leq a \leq \frac{c}{2}$ and $a^2 + b^2 \geq c^2$. Its dual is $\Gamma^\star = \mathbf{Z}\left(\frac{1}{c}, -\frac{a}{bc}\right) \oplus \mathbf{Z}\left(0, \frac{1}{b}\right)$. In this case, the hypothesis in Corollary 3.1 simply means that $a^2 + b^2 = c^2$. This 2-dimensional case was studied in [9].

(2) In higher dimensions, there exist many lattices known for their special properties in number theory or sphere packings and which satisfy the hypothesis of the corollary (see [4] for details and examples). Among them we may mention:

- *The square lattice* $\mathbf{Z}^n$.
- *The Hexagonal lattice* $A_n = \mathbf{Z}^{n+1} \cap E$, where $E = \{(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1}, x_1 + ... + x_{n+1} = 0\}$. Its dual $A_n^\star$ is the lattice of E generated by the n vectors $\tau_j = \frac{1}{n+1}\sum_{k\leq n+1} e_k - e_j$, $\{e_j\}_{j\leq n+1}$ being the standard basis of $\mathbb{R}^{n+1}$. All the τ_j are of minimal length $\frac{n}{n+1}$ and $c(A_n^\star) = \frac{n^2}{n+1}$ (note that A_2 is, up to homothety, just the equilateral lattice of $\mathbb{R}^2$).
- *The lattice* Γ_n with $n = d^2$ for some integer $d \geq 2$, generated by the vectors $a_j = e_j - e_n$ for $j \leq n-1$ and $a_n = de_n$. Its dual $\Gamma_n^\star$ is generated by $\tau_j = e_j$ for $j \leq n-1$ and $\tau_n = \frac{1}{d}\sum_{k\leq n} e_k$. All these τ_j are of minimal length 1 and $c(\Gamma_n^\star) = n = d^2$.

Before starting the proof of Theorem 3.1, let us recall that the n-energy $E_n(\psi)$ of a map ψ from an n-dimensional compact Riemannian manifold (M, g) into $\mathbb{S}^N$ is $E_n(\psi) = \int_M |d\psi|^n \nu_g$, where $|d\psi|$ is the Hilbert-Schmidt norm of $d\psi$ associated to the metrics g and can. It is important to note that $E_n(\psi)$ is invariant under conformal changes of the metric g. Furthermore, for conformal immersions, the n-energy is proportional to the volume $E_n(\psi) = n^{n/2} V(\psi^\star can)$. In the sequel the group of conformal diffeomorphisms of $\mathbb{S}^N$ will be denoted by $G(N)$.

An important ingredient in the proof of Theorem 3.1 is the following result that we established in [6] (see also [8]):

Theorem 3.2 ([6])**.** *If $\psi : (M, g) \to \mathbb{S}^N$ is conformal and minimal, then*

$$\sup_{\gamma \in G(N)} E_n(\gamma \circ \psi) = E_n(\psi).$$

Moreover, if the equality $E_n(\gamma \circ \psi) = E_n(\psi)$ holds for a non-isometric $\gamma \in G(N)$, then $\psi(M)$ is a totally geodesic sphere of $\mathbb{S}^N$.

Note that in dimension 2, the first part of this theorem corresponds to a well-known result of Li and Yau [10] concerning the conformal area of minimal surfaces of spheres.

Proof of Theorem 3.1. Let $\{\tau_1, ..., \tau_n\}$ be a basis of $\Gamma^\star$ such that $\sum_i |\tau_i|^2 = c(\Gamma^\star)$ and let $\phi : \mathbf{T}^n_\Gamma \to \mathbb{S}^{2n-1}$ be the map induced from

$$\phi(x) = \frac{1}{\sqrt{n}}(f_1(x), g_1(x), ..., f_n(x), g_n(x))$$

where $f_j(x) + i g_j(x) = \exp 2i\pi\langle x, \tau_j\rangle$. The functions f_j and g_j are in fact eigenfunctions of the Laplacian of g_Γ associated to the eigenvalue $4\pi^2|\tau_j|^2$.

First step. $|d\phi|^2 = \left(\frac{4\pi^2}{n} c(\Gamma^\star)\right)$ and then $E_n(\phi) = \left(\frac{4\pi^2}{n} c(\Gamma^\star)\right)^{\frac{n}{2}} V(g_\Gamma)$.

Proof. As for every $j \leq n$, $f_j^2 + g_j^2 = 1$, we have

$$\begin{aligned} 0 &= \frac{1}{2}\Delta\left(f_j^2 + g_j^2\right) = f_j \Delta f_j + g_j \Delta g_j - \left(|df_j|^2 + |dg_j|^2\right) \\ &= 4\pi^2|\tau_j|^2\left(f_j^2 + g_j^2\right) - \left(|df_j|^2 + |dg_j|^2\right) = 4\pi^2|\tau_j|^2 - \left(|df_j|^2 + |dg_j|^2\right). \end{aligned}$$

Summing up, we obtain

$$|d\phi|^2 = \frac{1}{n}\sum_{1\leq j\leq n}\left(|df_j|^2 + |dg_j|^2\right) = \frac{4\pi^2}{n}\sum_{1\leq j\leq n}|\tau_j|^2 = \frac{4\pi^2}{n}c(\Gamma^\star).$$

This proves the claim.

Although the map ϕ is in general neither minimal nor conformal, it will maximize the n-energy on its $G(2n-1)$-orbit:

Second step. For every non-isometric $\gamma \in G(2n-1)$, we have $E_n(\gamma \circ \phi) < E_n(\phi)$.

Proof. The standard embedding ϕ_{cl} of the Clifford torus ($\mathbf{T}^n_{cl} = \mathbb{R}^n/\mathbf{Z}^n$, $g_{cl} = g_{\mathbf{Z}^n}$) is given by $\phi_{cl}(x) = \frac{1}{\sqrt{n}}(\exp 2i\pi x_1, ..., \exp 2i\pi x_n)$. This embedding is minimal and homothetic. Hence, by Theorem 3.2, for any non-isometric $\gamma \in G(2n-1)$, $E_n(\gamma \circ \phi_{cl}) < E_n(\phi_{cl})$. The assertion of this step will then follow from the fact that, for any $\gamma \in G(2n-1)$,

$$E_n(\gamma \circ \phi)/E_n(\phi) = E_n(\gamma \circ \phi_{cl})/E_n(\phi_{cl}).$$

Indeed, for any $\gamma \in G(2n-1)$,

$$\begin{aligned} E_n(\gamma\circ\phi) = \int_{\mathbf{T}^n_\Gamma} |d\gamma|^n \circ \phi |d\phi|^n \nu_{g_\Gamma} &= \left(\frac{4\pi^2}{n} c(\Gamma^\star)\right)^{\frac{n}{2}} \int_{\mathbf{T}^n_\Gamma} |d\gamma|^n \circ \phi \nu_{g_\Gamma} \\ &= \frac{E_n(\phi)}{V(g_\Gamma)} \int_{\mathbf{T}^n_\Gamma} |d\gamma|^n \circ \phi \nu_{g_\Gamma}. \end{aligned}$$

Likewise, as $V(g_{cl}) = 1$, $E_n(\gamma\circ\phi_{cl}) = E_n(\phi_{cl}) \int_{\mathbf{T}^n_{cl}} |d\gamma|^n \circ \phi_{cl} \nu_{g_{cl}}$.

Let $\{a_1, ..., a_n\}$ be the basis of Γ defined by $\langle a_j, \tau_k\rangle = \delta_{jk}$ (i.e. $\{a_j\}_{j\le n}$ is the dual basis of $\{\tau_j\}_{j\le n}$). The linear automorphism of $\mathbb{R}^n$ which sends the canonical basis $\{e_j\}_{j\le n}$ of $\mathbb{R}^n$ to the basis $\{a_j\}_{j\le n}$ induces a diffeomorphism ρ between $\mathbf{T}^n_{cl}$ and $\mathbf{T}^n_\Gamma$. This ρ satisfies $\phi\circ\rho = \phi_{cl}$ so that we have (by the change of variables rule)

$$\int_{\mathbf{T}^n_\Gamma} |d\gamma|^n \circ \phi \nu_{g_\Gamma} = |\det \rho| \int_{\mathbf{T}^n_{cl}} |d\gamma|^n \circ \phi_{cl} \nu_{g_{cl}} = V(g_\Gamma) \int_{\mathbf{T}^n_{cl}} |d\gamma|^n \circ \phi_{cl} \nu_{g_{cl}}.$$

It follows that $E_n(\gamma\circ\phi)/E_n(\phi) = E_n(\gamma\circ\phi_{cl})/E_n(\phi_{cl})$.

Third step. For any $g \in C(g_\Gamma)$, we have

$$\lambda_1(g) \le \left(\frac{1}{V(g)} E_n(\phi)\right)^{2/n} = \frac{4\pi^2}{n} c(\Gamma^\star).$$

Proof. Let $g \in C(g_\Gamma)$. By a standard argument (see [10]) there exists $\gamma \in G(2n-1)$ such that for any $j \le 2n$, $\int_{\mathbf{T}^n_\Gamma} (\gamma\circ\phi)_j \nu_g = 0$. Using the min-max principle we get, for any $j \le 2n$,

$$\lambda_1(g) \int_{\mathbf{T}^n_\Gamma} (\gamma\circ\phi)_j^2 \nu_g \le \int_{\mathbf{T}^n_\Gamma} |d(\gamma\circ\phi)_j|^2 \nu_g.$$

Summing up and applying Hölder inequality and the previous steps we obtain

$$\begin{aligned} \lambda_1(g)V(g) &\le \sum_i \int_{\mathbf{T}^n_\Gamma} |d(\gamma\circ\phi)_j|^2 \nu_g \le \left(\int_{\mathbf{T}^n_\Gamma} |d(\gamma\circ\phi)|^n \nu_g\right)^{2/n} V(g)^{1-2/n} \\ &= E_n(\gamma\circ\phi)^{2/n} V(g)^{1-2/n} \le E_n(\phi)^{2/n} V(g)^{1-2/n} \\ &= \frac{4\pi^2}{n} c(\Gamma^\star) V(g_\Gamma)^{2/n} V(g)^{1-2/n} = \frac{4\pi^2}{n} c(\Gamma^\star) V(g). \end{aligned}$$

This proves the desired inequality.

Fourth step (*equality case*). Let $g \in C(g_\Gamma)$. The equality $\lambda_1(g) = \frac{4\pi^2}{n} c(\Gamma^\star)$ holds if and only if $|\tau_1| = ... = |\tau_n|^2 = c(\Gamma^\star)/n$ and $g = g_\Gamma$.

Proof. Suppose that $|\tau_1| = ... = |\tau_n|^2 = c(\Gamma^\star)/n$. Then it is easy to see that

$$\lambda_1(g_\Gamma) = \inf\{4\pi^2|\tau|^2 \ \ , \ \ \tau \in \Gamma^\star \quad \text{and} \quad \tau \ne 0\} = \frac{4\pi^2}{n} c(\Gamma^\star).$$

Now, assume that the equality $\lambda_1(g) = \frac{4\pi^2}{n} c(\Gamma^\star)$ holds for a metric $g = (\exp 2\alpha) g_\Gamma \in C(g_\Gamma)$. This implies that equality holds in the inequalities of the proof of the Third step. In particular, $E_n(\gamma \circ \phi) = E_n(\phi)$ and then (Second step) γ is an isometry of $\mathbb{S}^{2n-1}$ which can be chosen as the identity map. Moreover, for any $j \leq 2n$, we have

$$\lambda_1(g) = \int_{\mathbf{T}^n_\Gamma} |d\phi_j|^2 \nu_g / \int_{\mathbf{T}^n_\Gamma} \phi_j^2 \nu_g.$$

Then, for any $j \leq 2n$,

$$\Delta_g f_j = \lambda_1(g) f_j \quad \text{and} \quad \Delta_g g_j = \lambda_1(g) g_j.$$

Now, for any $j \leq 2n$,

$$\lambda_1(g) f_j = \Delta_g f_j = (e^{-2\alpha}) \left[\Delta_{g_\Gamma} f_j - (n-2)\langle \nabla\alpha, \nabla f_j\rangle\right]$$

and

$$\lambda_1(g) g_j = \Delta_g g_j = (e^{-2\alpha}) \left[\Delta_{g_\Gamma} g_j - (n-2)\langle \nabla\alpha, \nabla g_j\rangle\right].$$

Multiplying the first equality by f_j and the second one by g_j, we obtain after summing:

$$\lambda_1(g) = (e^{-2\alpha}) 4\pi^2 |\tau_j|^2.$$

It follows that α is constant and, since $V(g) = V(g_\Gamma)$, g is equal to g_Γ. Moreover, $|\tau_1|^2 = ... = |\tau_n|^2 = c(\Gamma^\star)/n$. □

References

[1] Bando, S., Urakawa, H.: Generic properties of the eigenvalue of Laplacian for compact Riemannian manifolds. *Tôhoku Math. J.* 35, 155-172 (1983). MR **84h:**58146

[2] Berger, M.: Sur les premières valeurs propres des variétés riemanniennes. *Compositio. Math.* 26, 129-149 (1973). MR **47:**5461

[3] Colbois, B., Dodziuk, J.: Riemannian metrics with large λ_1. *Proc. Amer. Math. Soc.* 122 (3), 905-906 (1994). MR **95a:**58130

[4] Conway, J.H, Sloane, N.J.A.: Sphere packings, lattices and groups, Grundlehren der Mathematischen Wissenschaften, 290, *Springer Verlag*, 1999. MR **2000b:**11077

[5] Eells, J., Lemaire, L.: A report on harmonic maps. *Bull. London Math. Soc.* 10, 1-68 (1978). MR **82b:**58033

[6] El Soufi, A., Ilias, S.: Immersions minimales, première valeur propre du laplacien et volume conforme. *Math. Ann.* 275, 257-267 (1986). MR **87j:**53088

[7] El Soufi, A., Ilias, S.: Riemannian manifolds admitting isometric immersions by their first eigenfunctions, *Pacific J. Math.* 195, 91–99 (2000). MR **2001f:**53128

[8] El Soufi, A., Jeune, A.: Indice de Morse des applications p-harmoniques. *Ann. Inst. Henri Poincaré*, 2, 229-250 (1996). MR **97d:**58057

[9] El Soufi, A., Ilias, S., Ros, A.: Sur la première valeur propre des tores. *Séminaire de th'eorie spectrale et géométrie*, Grenoble, 17-23 (1996-1997). MR **99e:**58192

[10] Li, P., Yau, S.T.: A new conformal invariant and its applications etc. *Invent. Math.* 69, 269-291 (1982). MR **84f:**53049

[11] Nadirashvili N.: Berger's isoperimetric problem and minimal immersions of surfaces, *GAFA* 6, 877-897 (1996). MR **98f:**53061

[12] Rockafellar, R.T.: Convex Analysis. *Princeton University Press*, Princeton 1970. MR **43:**445

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