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A GLOBAL COMPACTNESS RESULT FOR SINGULAR ELLIPTIC PROBLEMS INVOLVING CRITICAL SOBOLEV EXPONENT

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Abstract. Let $\Omega \subset R^N$ be a bounded domain such that $0 \in \Omega, N \geq 3, 2^* = \frac{2N}{N-2}, \lambda \in R, \epsilon \in R$. Let $\{u_n\} \subset H_0^1(\Omega)$ be a (P.S.) sequence of the functional $E_{\lambda,\epsilon}(u) = \frac{1}{2}\int_\Omega(|\nabla u|^2 - \frac{\lambda u^2}{|x|^2} - \epsilon u^2) - \frac{1}{2^*}\int_\Omega |u|^{2^*}$. We study the limit behaviour of u_n and obtain a global compactness result.

1. Introduction

Let Ω be a bounded domain in R^N such that $0 \in \Omega, N \geq 3, 2^* = \frac{2N}{N-2}, \lambda \in R, \epsilon \in R$.

In recent years, much attention has been paid to the existence of nontrivial solutions to the following problem:

$$(P_{\lambda,\epsilon}) \quad \begin{cases} -\Delta u = \frac{\lambda}{|x|^2}u + |u|^{2^*-2}u + \epsilon u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $0 \leq \lambda < \bar{\lambda} = (\frac{N-2}{2})^2, \epsilon \in R$. For instance, in [11], by using a variational approach, E. Jannelli proved that if $\lambda \leq \bar{\lambda} - 1$, then problem $(P_{\lambda,\epsilon})$ has at least one solution $u \in H_0^1(\Omega)$ when $0 < \epsilon < \epsilon_1(\lambda)$. If $\bar{\lambda} - 1 < \lambda < \bar{\lambda}$, then problem $(P_{\lambda,\epsilon})$ has at least one solution $u \in H_0^1(\Omega)$ when $\epsilon_*(\lambda) < \epsilon < \epsilon_1(\lambda)$, where $\epsilon_1(\lambda)$, $\epsilon_*(\lambda)$ are some positive constants depending on λ.

Let

$$E_{\lambda,\epsilon}(u) = \frac{1}{2}\int_\Omega(|\nabla u|^2 - \frac{\lambda u^2}{|x|^2} - \epsilon u^2) - \frac{1}{2^*}\int_\Omega |u|^{2^*}, \text{ for } u \in H_0^1(\Omega). \tag{1.1}$$

The crucial step in [11] was to overcome the lack of compactness for $E_{\lambda,\epsilon}(u)$. Indeed, the invariance of H_0^1-norm, $L^{2^*}-$norm and $\int_\Omega \frac{\lambda u^2}{|x|^2}$ with respect to rescaling $u \mapsto u_r = r^{\frac{N-2}{2}}u(r(\cdot))$ and the existence of non-trivial entire solution of the limiting

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problem (see [4], [10], [12] and [14])

$$\begin{cases} -\Delta u = \frac{\lambda}{|x|^2}u + |u|^{2^*-2}u, & x \in R^N, \\ u \to 0, & |x| \to \infty, \end{cases} \tag{1.2}$$

result in the failure of the classical Palais-Smale (P.S. for short) condition for $E_{\lambda,\epsilon}$ on $H_0^1(\Omega)$. However a local (P.S.) condition can be established. Indeed, let $|u|_p^p = \int_\Omega |u|^p$ for $p \in (1,\infty)$ and

$$S_\lambda := \inf\{\int_{R^N}(|\nabla u|^2 - \frac{\lambda u^2}{|x|^2}) \mid u \in H_0^1(R^N), |u|_{2^*} = 1\}. \tag{1.3}$$

Suppose $\{u_m\} \subset H_0^1(\Omega)$ is a sequence such that $E_{\lambda,\epsilon}(u_m) \le c < \frac{1}{N}S_\lambda^{\frac{N}{2}}, DE_{\lambda,\epsilon}(u_m) \to 0$ strongly in $H^{-1}(\Omega) = (H_0^1(\Omega))^*$; then $\{u_m\}$ contains a strongly convergent subsequence. Using this local (P.S.) condition, E.Jannelli was able to obtain the existence of one nontrivial solution in [11]. For earlier work, see [3].

As indicated above it is always very important to understand the convergence of the (P.S.) sequence in using variational methods. When $\lambda = 0$, M.Struwe [14] gives a complete description of all energy levels c of $E_{\lambda,\epsilon}$ on which the $(P.S.)_c$ sequence is not compact, and he obtains the well known global compactness theorem, with which J.M.Coron [6] and W.-Y. Ding [7] obtained the positive critical points of $E_{0,0}$ with some non-starshaped domains. In [16] S.Yan generalizes this global compactness result to the case of p-Laplacian successfully. Very recently, Adimurthi and M.Struwe in [1] have studied the convergence of (P.S.) sequences of the energy functional associated with a semilinear elliptic problem on a bounded domain in R^2 with critical nonlinearity $f(s)$ growing like $exp(4\pi s^2)$ as $s \to \infty$.

As can easily be seen, problem $(P_{\lambda,\epsilon})$ is the general case of problem $(P_{0,\epsilon})$. So it is very interesting and important to see whether a global compactness result similar to that of problem $(P_{0,\epsilon})$ still exists or not. In this paper we investigate this problem and obtain an affirmative answer.

Our proof is based on rescaling arguments. Such methods have been repeatedly used to extract convergent subsequence from families of solutions or minimizing sequence to nonlinear variational problems; see [13], [14], and [16]).

In the following, let $D^{1,2}(R^N)$ be the completion of $C_0^\infty(R^N)$ with respect to the inner product $(u,v) = \int_{R^N}\nabla u \cdot \nabla v, S_0 = \inf\{\int_{R^N}|\nabla u|^2 \mid u \in H_0^1(R^N), |u|_{2^*} = 1\}$. Let $B(x,r)$ denote the ball centered at x with radius r. For simplicity, we use the same notation $\tilde{\Omega}_m$ and $\tilde{v}_m$ in different situations.

Remark 1.1. (i) By Proposition 8.1 in [4], the solutions of (1.2) are in one-to-one correspondence to solutions of the following type of problem:

$$-div(|x|^{-2a}\nabla w) = |x|^{b2^*}w^{2^*-1}, w \ge 0 \text{ in } R^N,$$

where a and b are constants depending on λ. From Theorem B in [5], any solution of the above equations is radially symmetric with respect to the origin, that is, $u(x)$ depends on $|x|$ only. So according to the theory of ordinary differential equations, the solution of (1.2) is unique (up to a dilation). Using the results in [4], we can deduce that when $0 \le \lambda < \bar{\lambda}$, the solutions of (1.3) are of the following:

$$u(x) = \varepsilon^{\frac{N-2}{2}}U(\varepsilon x), \quad \varepsilon > 0,$$

$$U(x) = C(\lambda,\bar{\lambda})|x|^{-\sqrt{\lambda}+\sqrt{\bar{\lambda}-\lambda}}(1 + |x|^{\frac{4\sqrt{\bar{\lambda}-\lambda}}{N-2}}),$$

where $C(\bar{\lambda}, \lambda)$ is a constant depending only on λ and $\bar{\lambda}$. So S_λ can be attained in R^N when $0 \le \lambda < \bar{\lambda}$.

(ii) $S_\lambda < S_0$ when $0 < \lambda < \bar{\lambda}$. While $\lambda < 0$, we can prove that $S_\lambda = S_0$ and S_λ cannot be attained, even though in R^N.

2. Global compactness

To state the main result, it is convenient to introduce the "problem at infinity". The first one is

$$(P_0^\infty) \qquad -\Delta v = |v|^{2^*-2}v, \qquad v \in D^{1,2}(R^N).$$

Moreover, for $0 \le \lambda < \bar{\lambda}$, the problem "at infinity" is given by

$$(P_\lambda^\infty) \qquad -\Delta v = \frac{\lambda v}{|x|^2} + |v|^{2^*-2}v, \qquad v \in D^{1,2}(R^N).$$

To unify the notations we shall refer to the solutions of problems at infinity as critical points of the following family of functionals:

$$F_\lambda^\infty(u) = \frac{1}{2}\int_{R^N}(|\nabla u|^2 - \frac{\lambda u^2}{|x|^2}) - \frac{1}{2^*}\int_{R^N}|u|^{2^*}$$

and in the cases of (P_0^∞) and (P_λ^∞), the respective energy functionals are $F_0^\infty(u)$ and $F_\lambda^\infty(u)$.

Theorem 2.1. *Let $N \ge 3$, $0 \le \lambda < \bar{\lambda}$, $\epsilon \in R$, and suppose that $\{u_m\} \subset H_0^1(\Omega)$ satisfies $E_{\lambda,\epsilon}(u_m) \le c, DE_{\lambda,\epsilon}(u_m) \to 0$ strongly in $H^{-1}(\Omega)$ as $m \to \infty$. Then there exist a critical point u^0 of $E_{\lambda,\epsilon}$, k sequences of positive numbers $\{r_m^j\}_m$ $(1 \le j \le k)$, l sequences of positive numbers $\{k_m^j\}_m$ $(1 \le j \le l)$ and l sequences of points $\{x_m^j\}_m$ $(1 \le j \le l)$ in Ω which converge to $x_0^j \in \bar{\Omega}$, such that up to a subsequence,*

$$\text{(i)} \quad u_m = u^0 + \sum_{j=1}^{k}(r_m^j)^{\frac{N-2}{2}}v_1^j(r_m^j x) + \sum_{j=1}^{l}(k_m^j)^{\frac{N-2}{2}}v_0^j(k_m^j(x - x_m^j)) + \omega_m,$$

where $\|\omega_m\|_{H_0^1(\Omega)} \to 0, r_m^j \to \infty, k_m^j \to \infty$, v_0^j solves (P_0^∞), and v_1^j solves (P_λ^∞).

$$\text{(ii)} \quad E_{\lambda,\epsilon}(u_m) \to E_{\lambda,\epsilon}(u^0) + \sum_{j=1}^{l}F_0^\infty(v_0^j) + \sum_{j=1}^{k}F_\lambda^\infty(v_1^j).$$

To prove this theorem, let us first recall some known results and establish some preliminary lemmas. The following lemma can be found in [9].

Lemma 2.2 (Hardy inequality)**.** *If $u \in H_0^1(\Omega)$, then $\frac{u}{|x|^2} \in L^2(\Omega)$ and*

$$\int_\Omega \frac{u^2}{|x|^2} \le \frac{1}{\bar{\lambda}}\int_\Omega |\nabla u|^2,$$

moreover $\bar{\lambda}$ is optimal.

Lemma 2.3. *Let $\{u_m\} \subset H_0^1(\Omega), u_m \to u$ weakly in $H_0^1(\Omega)$. Then*

(i) $\int_\Omega |u_m|^{2^*} = \int_\Omega |u_m - u|^{2^*} + \int_\Omega |u|^{2^*} + o(1)$,

(ii) $\int_\Omega |\nabla u_m|^2 = \int_\Omega |\nabla(u_m - u)|^2 + \int_\Omega |\nabla u|^2 + o(1)$,

(iii) $\int_\Omega \frac{|u_m|^2}{|x|^2} = \int_\Omega \frac{|u_m - u|^2}{|x|^2} + \int_\Omega \frac{|u|^2}{|x|^2} + o(1)$.

Proof. (i) and (ii) are the well known results of Brezis-Lieb in [2]. (iii) can be easily proved by Vitali's theorem and we omit the detail.

Lemma 2.4. *Let $\{v_m\} \subset H_0^1(\Omega)$ be a (P.S.) sequence for $E_{\lambda,0}$ at level β, that is, $E_{\lambda,0}(v_m) \to \beta$ as $m \to \infty$, and assume that v_m converges weakly but not strongly to zero in $H_0^1(\Omega)$. Then either*

(i) *there exists a sequence of positive numbers $\{r_m\}$, such that up to a subsequence,*

$$w_m(x) = v_m(x) - r_m^{\frac{N-2}{2}} V_1(r_m x) + o(1) \ \ (x \in \Omega)$$

is a (P.S.) sequence for $E_{\lambda,0}$ in $H_0^1(\Omega)$ at level $\beta - F_\lambda^\infty(V_1)$; moreover, $w_m \to 0$ weakly in $H_0^1(\Omega)$,

or

(ii) *there exists a sequence of positive numbers $\{k_m\}$ and a sequence of points $\{y_m\} \subset \Omega$ satisfying $y_m \to x_0 \in \bar{\Omega}$, such that up to a subsequence,*

$$w_m(x) = v_m(x) - k_m^{\frac{N-2}{2}} V_0(k_m(x - y_m)) + o(1) \ \ (x \in \Omega)$$

is a (P.S.) sequence for $E_{\lambda,0}$ in $H_0^1(\Omega)$ at level $\beta - F_0^\infty(V_0)$. Moreover, $w_m \to 0$ weakly in $H_0^1(\Omega)$, $k_m dist(x_m, \partial\Omega) \to \infty$, where V_0 and V_1 solve (P_0^∞) and (P_λ^∞) respectively, $o(1) \to 0$ in $D^{1,2}(R^N)$ as $m \to \infty$.

Proof. If $E_{\lambda,0}(v_m) \to \beta < \frac{1}{N} S_\lambda^{\frac{N}{2}}$, then sequence $\{v_m\}$ is strongly relatively compact and hence $v_m \to 0, \beta = 0$. Therefore we may assume that $E_{\lambda,0}(v_m) \to \beta \geq \frac{1}{N} S_\lambda^{\frac{N}{2}}$. Moreover, from $DE_{\lambda,0}(v_m) \to 0$, we also have

$$\frac{1}{N} \int_\Omega (|\nabla v_m|^2 - \frac{\lambda v_m^2}{|x^2|}) = E_{\lambda,0}(v_m) - \frac{1}{2} \langle DE_{\lambda,0}(v_m), v_m \rangle \to \beta \geq \frac{1}{N} S_\lambda^{\frac{N}{2}}. \tag{2.1}$$

By Hardy inequality, we have for m large enough

$$S_\lambda^{\frac{N}{2}} \leq N\beta \leq \int_\Omega |\nabla v_m|^2 \leq \frac{N\beta}{1 - \frac{\lambda}{\bar{\lambda}}} \tag{2.2}$$

and hence there are positive constants c_i $(i = 1, 2)$ such that

$$c_1 \leq \int_\Omega |\nabla v_m|^2 \leq c_2, \quad \forall\, m. \tag{2.3}$$

Let $\delta > 0$ small (will be determined later) such that

$$\limsup_{m \to \infty} \int_\Omega |\nabla v_m|^2 > \delta. \tag{2.4}$$

Up to a subsequence, choose minimal $\frac{1}{r_m} > 0$ such that $\int_{B(0, \frac{1}{r_m})} |\nabla v_m|^2 = \delta$. Define $\tilde{v}_m := r_m^{\frac{2-N}{2}} v_m(\frac{x}{r_m})$; then $\int_{B(0,1)} |\nabla \tilde{v}_m|^2 = \delta$.

Let us point out that, thanks to (2.3) and (2.4), the sequence $\{r_m\}$ is bounded away from zero.

Denote $\tilde{\Omega}_m := \{x \in R^N \mid \frac{x}{r_m} \in \Omega\}$; then $\tilde{v}_m \in H_0^1(\tilde{\Omega}_m) \subset D^{1,2}(R^N)$. Moreover, $\|\tilde{v}_m\|^2_{D^{1,2}(R^N)} = \|v_m\|^2_{D^{1,2}(R^N)} \to N\beta < \infty$. Up to a subsequence there exists $V_1 \in D^{1,2}(R^N)$ satisfying $\tilde{v}_m \to V_1$ weakly in $D^{1,2}(R^N), \tilde{v}_m \to V_1$ a.e. in R^N. We have either $V_1 \not\equiv 0$ or $V_1 \equiv 0$.

(I): Assume $V_1 \not\equiv 0$. Since $v_m \to 0$ weakly in $H_0^1(\Omega)$, we have $r_m \to \infty, \tilde{\Omega}_m \to R^N$. For this case we claim that V_1 satisfies (P_λ^∞) and the sequence $w_m(x) := \tilde{v}_m(x) - r_m^{\frac{N-2}{2}} V_1(r_m x)$ is a (P.S.) sequence for $E_{\lambda,0}$ at level $\beta - F_\lambda^\infty(V_1)$.

Indeed, let us fix a ball $B(0,r)$ and a test function $\psi \in C_0^\infty(B(0,r))$ and remark that for sufficiently large $m, B(0,r) \subset \tilde{\Omega}_m$. So

$$\begin{aligned}
&\langle DF_\lambda^\infty(V_1), \psi\rangle \\
&= \int_{B(0,r)} \nabla V_1 \cdot \nabla\psi - \int_{B(0,r)} \frac{\lambda V_1 \psi}{|x|^2} - \int_{B(0,r)} |V_1|^{2^*-2} V_1 \psi \\
&= \int_{\tilde{\Omega}_m} \nabla \tilde{v}_m \cdot \nabla\psi - \int_{\tilde{\Omega}_m} \frac{\lambda \tilde{v}_m \psi}{|x|^2} - \int_{\tilde{\Omega}_m} |\tilde{v}_m|^{2^*-2} \tilde{v}_m \psi + o(1) \\
&= \int_{\Omega} \nabla v_m \cdot \nabla\bar{\psi}_m - \int_{\Omega} \frac{\lambda v_m \bar{\psi}_m}{|x|^2} - \int_{\Omega} |v_m|^{2^*-2} v_m \bar{\psi}_m + o(1) \\
&= \langle DE_{\lambda,0}(v_m), \bar{\psi}_m\rangle + o(1) \qquad \text{(by the (P.S.) condition)}
\end{aligned}$$

where $\bar{\psi}_m(x) = r_m^{\frac{N-2}{2}} \psi(r_m x)$. Thus V_1 solves (P_λ^∞).

Let $\varphi \in C_0^\infty(R^N)$ satisfying $0 \le \varphi \le 1, |\nabla\varphi| \le 2$ in R^N,$\varphi \equiv 1$, in $B(0,1), \varphi \equiv 0$ outside $B(0,2)$, and let

$$w_m(x) = v_m(x) - r_m^{\frac{N-2}{2}} V_1(r_m x)\varphi(\bar{r}_m x) \in H_0^1(\Omega)$$

where the sequence $\{\bar{r}_m\}$ is chosen such that $\tilde{r}_m := \frac{r_m}{\bar{r}_m} \to \infty$ while $\bar{r}_m dist(0, \partial\tilde{\Omega}_m) \to \infty$ as $m \to \infty$, that is, $\tilde{w}_m(x) = \tilde{v}_m(x) - V_1(x)\varphi(\frac{x}{\tilde{r}_m})$.

Set $\varphi_m(x) = \varphi(\frac{x}{\tilde{r}_m})$. Note that $|\nabla V_1| \in L^2(R^N)$, so as $m \to \infty$,

$$\begin{aligned}
&\int_{R^N} |\nabla V_1(\varphi_m - 1)|^2 \\
&\le 2\int_{R^N} |\nabla V_1|^2(\varphi_m - 1)^2 + 2\int_{R^N} |V_1|^2 |\nabla(\varphi_m - 1)^2| \\
&\le 2\int_{R^N \setminus B(0,r_m)} |\nabla V_1|^2 + 8\tilde{r}_m^{-2} \int_{B(0,2\tilde{r}_m)\setminus B(0,\tilde{r}_m)} |V_1|^2 \\
&\le 2\int_{R^N \setminus B(0,r_m)} |\nabla V_1|^2 + 8(\int_{B(0,2\tilde{r}_m)\setminus B(0,\tilde{r}_m)} |V_1|^{2^*})^{\frac{2}{2^*}} \\
&= o(1).
\end{aligned}$$

Thus we have $\tilde{w}_m = \tilde{v}_m - V_1 + o(1)$, where $o(1) \to 0$ in $D^{1,2}(R^N)$. So by Lemma 2.3 and the invariance of dilation, we have for m large

$$E_{\lambda,0}(w_m) = E_{\lambda,0}(\tilde{v}_m) - F_\lambda^\infty(V_1) + o(1) = \beta - F_\lambda^\infty(V_1) + o(1),$$

$$\|DE_{\lambda,0}(w_m)\|_{H^{-1}(\Omega)} = o(1).$$

Also, it is obvious that $w_m \to 0$ weakly in $H_0^1(\Omega)$.

(II): Assume $V_1 \equiv 0$. Let $h \in C_0^\infty(B(0,1))$; we have

$$\begin{aligned}
&\int_{R^N} |\nabla(\tilde{v}_m h)|^2 \\
&= \int_{R^N} \nabla\tilde{v}_m \cdot \nabla(h^2\tilde{v}_m) + \int_{R^N} \tilde{v}_m^2 |\nabla h|^2 \\
&= \int_{R^N} \nabla\tilde{v}_m \cdot \nabla(h^2\tilde{v}_m) + o(1) \\
&= \langle DE_{\lambda,0}(\tilde{v}_m), h^2\tilde{v}_m\rangle + \int_{R^N} \frac{\lambda h^2 \tilde{v}_m}{|x|^2} + \int_{R^N} |\tilde{v}_m|^{2^*} h^2 + o(1) \\
&\le \frac{4\lambda}{(N-2)^2} \int_{R^N} |\nabla(\tilde{v}_m h)|^2 + S_0^{-1}(\int_{B(0,1)} |\tilde{v}_m|^{2^*})^{\frac{N}{2}} \int_{R^N} |\nabla(\tilde{v}_m h)|^2 + o(1).
\end{aligned}$$

Choose δ suitably small. From the above inequality and the fact $0 \le \lambda < \frac{(N-2)^2}{4}$, we can find an $a \in (0,1)$ such that

$$\int_{B(0,a)} |\nabla\tilde{v}_m|^2 \to 0. \tag{2.5}$$

For simplicity, substitute $\tilde{v}_m$ by z_m and $\tilde{\Omega}_m$ by Ω (because of dilation invariance), and without loss of generality, we still suppose that z_m satisfies (2.3) and (2.4). Denote

$$Q_m(r) = \sup_{x\in\Omega} \int_{B(x,r)} |\nabla z_m|^2$$

the concentration function of z_m, choose $x_m \in \bar{\Omega}$ and scale

$$z_m \mapsto \tilde{z}_m(x) := R_m^{\frac{2-N}{2}} z_m(\frac{x}{R_m} + x_m) \tag{2.6}$$

such that

$$\tilde{Q}_m(1) = \sup_{\frac{x}{R_m}+x_m, x\in R^N} \int_{B(x,1)} |\nabla\tilde{z}_m|^2 = \int_{B(0,1)} |\nabla\tilde{z}_m|^2 = \frac{1}{2L} S_\lambda^{\frac{N}{2}}, \tag{2.7}$$

where L denotes the least number of balls with radius 1 in R^N that are needed to cover a ball of radius 2.

Note that $\{R_m\}$ is obviously bounded away from zero. Setting $\tilde{\Omega}_m := \{x \in R^N \mid \frac{x}{R_m} + x_m \in \Omega\}$, we may regard $\tilde{z}_m \in D^{1,2}(R^N)$, moreover, $\{\tilde{z}_m\}$ is bounded uniformly in $D^{1,2}(R^N)$. Thus up to a subsequence,

$$\tilde{z}_m \to V_0 \ \text{ weakly in } D^{1,2}(R^N). \tag{2.8}$$

We are going to prove that the convergence actually holds in the strong $H^1_{loc}(R^N)$ sense. To do this, following the analogous argument in [14], for any $x \in R^N$, we can find $\rho \in [1,2]$ such that the solution ϕ_m of the Dirichlet problems

$$\begin{cases} -\Delta\phi = 0, & x \in B(x,3) \setminus B(x,\rho), \\ \phi|_{\partial B(x,\rho)} = \tilde{z}_m - V_0, & \phi|_{\partial B(x,3)} = 0, \end{cases}$$

satisfies

$$\phi_m \to 0 \text{ in } H^1(B(x,3) \setminus B(x,\rho)). \tag{2.9}$$

Let

$$\varphi_m = \begin{cases} \tilde{z}_m - V_0, & x \in B(x,\rho), \\ \phi_m, & x \in B(x,3) \setminus B(x,\rho). \end{cases} \tag{2.10}$$

It follows that

$$\varphi_m = \tilde{\varphi}_m + o(1) \in H_0^1(\tilde{\Omega}_m) + D^{1,2}(R^N), \tag{2.11}$$

where $\tilde{\varphi}_m \in H_0^1(\tilde{\Omega}_m)$ and $o(1) \to 0$ in $D^{1,2}(R^N)$ as $m \to \infty$ (see for instance [14]). Hence, scaling back the φ_m's,

$$\bar{\varphi}_m(x) := R_m^{\frac{N-2}{2}} \varphi_m(R_m(x - x_m)),$$

we have, taking into account of (2.9), (2.10) and (2.11),

$$\|\nabla \bar{\varphi}_m\|^2_{L^2(\Omega)} = \|\varphi_m\|^2_{D^{1,2}(R^N)} + o(1) = \|\tilde{z}_m - V_0\|^2_{D^{1,2}(B(x,\rho))} + o(1). \tag{2.12}$$

By scale invariance, (2.11) and the fact $\{z_m\}$ is a (P.S.) sequence for $E_{\lambda,0}$,

$$\langle DE_m(\tilde{z}_m), \varphi_m \rangle = \langle DE_{\lambda,0}(z_m), \bar{\varphi}_m \rangle + o(1) = o(1)$$

where

$$E_m(\tilde{z}_m) := \frac{1}{2} \int_{\tilde{\Omega}_m} (|\nabla \tilde{z}_m|^2 - \frac{\lambda \tilde{z}_m^2}{|x + R_m x_m|^2}) - \frac{1}{2^*} \int_{\tilde{\Omega}_m} |\tilde{z}_m|^{2^*}.$$

Therefore, from the definitions of E_m, φ_m and (2.9), we infer by using arguments similar to those in [14]

$$\begin{aligned}
o(1) &= \int_{\tilde{\Omega}_m \cap B(x,\rho)} \nabla \tilde{z}_m \cdot \nabla(\tilde{z}_m - V_0) - \int_{\tilde{\Omega}_m \cap B(x,\rho)} \frac{\lambda \tilde{z}_m(\tilde{z}_m - V_0)}{|x + R_m x_m|^2} \\
&\quad - \int_{\tilde{\Omega}_m \cap B(x,\rho)} |\tilde{z}_m|^{2^*-2} \tilde{z}_m(\tilde{z}_m - V_0) \\
&= \int_{\tilde{\Omega}_m \cap B(x,\rho)} |\nabla(\tilde{z}_m - V_0)|^2 - \int_{\tilde{\Omega}_m \cap B(x,\rho)} \frac{\lambda(\tilde{z}_m - V_0)^2}{|x + R_m x_m|^2} \\
&\quad - \int_{\tilde{\Omega}_m \cap B(x,\rho)} |\tilde{z}_m - V_0|^{2^*} + o(1) \\
&= \int_{\tilde{\Omega}_m} |\nabla \varphi_m|^2 - \int_{\tilde{\Omega}_m} \frac{\lambda \varphi_m^2}{|x + R_m x_m|^2} - \int_{\tilde{\Omega}_m} |\varphi_m|^{2^*} + o(1).
\end{aligned}$$

Moreover, by scale invariance, Sobolev inequality and Hardy inequality,

$$\begin{aligned}
o(1) &= \int_\Omega |\nabla \bar{\varphi}_m|^2 - \int_\Omega \frac{\lambda \bar{\varphi}_m^2}{|x|^2} - \int_\Omega |\bar{\varphi}_m|^{2^*} \\
&\geq (\int_\Omega |\nabla \bar{\varphi}_m|^2 - \int_\Omega \frac{\lambda \bar{\varphi}_m^2}{|x|^2})(1 - \frac{\|\nabla \bar{\varphi}_m\|^{2^*-2}_{L^2(\Omega)}}{S_\lambda^{\frac{2^*}{2}}}) \\
&\geq c \int_\Omega |\nabla \bar{\varphi}_m|^2 (1 - \frac{\|\tilde{z}_m\|^{2^*-2}_{D^{1,2}(B(x,\rho))}}{S_\lambda^{\frac{2^*}{2}}}),
\end{aligned} \tag{2.13}$$

where (2.12) and the convergence of $\tilde{z}_m$ to V_0 have been used to get the last inequality. Recalling (2.7) we have $\|\nabla \tilde{z}_m\|^2_{L^2(B(x,\rho))} \leq L \|\nabla \tilde{z}_m\|^2_{L^2(B(0,1))} \leq \frac{1}{2} S_\lambda^{\frac{N}{2}}$, so that (2.13) yields $\|\nabla \bar{\varphi}_m\|_{L^2(\Omega)} \to 0$. From (2.12) we obtain $\|\tilde{z}_m - V_0\|_{D^{1,2}(B(x,\rho))} \to 0$. In particular, $\int_{B(0,1)} |\nabla V_0|^2 = \frac{1}{2L} S_\lambda^{\frac{N}{2}} > 0$ and $V_0 \not\equiv 0$. By $z_m \to 0$ weakly, we also have $R_m \to \infty$ as $m \to \infty$.

Now using the result of case (I), we have

$$\tilde{z}_m(x) = R_m^{\frac{2-N}{2}} \tilde{v}_m(\frac{x}{R_m} + x_m) = (R_m r_m)^{\frac{2-N}{2}} v_m(\frac{x}{R_m r_m} + \frac{x_m}{r_m}).$$

Define $k_m = R_m r_m, y_m = \frac{x_m}{r_m}$; then $y_m \to y_0 \in \bar{\Omega}$, $k_m|y_m| = R_m|x_m|$. By (2.5) we have $|x_m| > a > 0$, so $k_m|y_m| \to +\infty$. Also, by the fact that $\{r_m\}$ is bounded away from zero, $k_m \to +\infty (m \to \infty)$. Denote $\tilde{\Omega}_m = \{x \in R^N \mid \frac{x}{k_m} + y_m \in \Omega\}, \tilde{v}_m = k_m^{\frac{2-N}{2}} v_m(\frac{x}{k_m} + y_m)$.

Now we distinguish two cases:

(1) $k_m dist(y_m, \partial\Omega) \le c < \infty$, uniformly. Then after an orthogonal transformation,

$$\tilde{\Omega}_m \to \tilde{\Omega}_\infty = R^N_+ = \{x = (x_1, \dots, x_N), x_1 > 0\},$$

(2) $k_m dist(y_m, \partial\Omega) \to \infty$, in this case $\tilde{\Omega}_m \to \tilde{\Omega}_\infty = R^N$.

In each case we have for large m and any given $\varphi \in C_0^\infty(\tilde{\Omega}_\infty)$, $\int_{\tilde{\Omega}_m} \frac{\tilde{v}_m \varphi}{|x+k_m y_m|^2} = o(1)$. Hence there holds, as $m \to \infty$,

$$\begin{aligned}
&\int_{\tilde{\Omega}_\infty} (\nabla V_0 \cdot \nabla\varphi - |V_0|^{2^*-2} V_0 \varphi) \\
&= \int_{\tilde{\Omega}_m} (\nabla \tilde{v}_m \cdot \nabla\varphi - \frac{\lambda \tilde{v}_m \varphi}{|x + k_m y_m|^2} - |\tilde{v}_m|^{2^*-2} \tilde{v}_m \varphi) + o(1) \\
&= \int_\Omega (\nabla v_m \cdot \nabla\varphi - \frac{\lambda v_m \varphi}{|x|^2} - |v_m|^{2^*-2} v_m \varphi) + o(1) \\
&= o(1).
\end{aligned} \tag{2.14}$$

If $\tilde{\Omega}_\infty = R^N_+$, (2.14) implies that $V_0 \in H_0^1(\tilde{\Omega}_\infty)$ is a weakly solution of the equation

$$-\Delta u = u^{2*-1}, \quad u > 0, \quad x \in R^N_+; \qquad u = 0,\ x \in \partial R^N_+.$$

Thus, $V_0 \equiv 0$, which is impossible. So case (2) is true, that is, V_0 solves (P_0^∞).

Let

$$w_m(x) = v_m(x) - k_m^{\frac{N-2}{2}} V_0(k_m(x - y_m))\varphi(\bar{k}_m(x - y_m)) \in H_0^1(\Omega),$$

where $\bar{k}_m, \varphi$ are defined similarly to case (I). We can also get $\tilde{w}_m = \tilde{v}_m - V_0 + o(1)$, where $o(1) \to 0$ in $D^{1,2}(R^N)$. Using the fact that $k_m|y_m| \to +\infty$, the invariance of scaling and Lemma 2.3 we obtain that $w_m(x) = v_m(x) - k_m^{\frac{N-2}{2}} V_0(k_m(x - x_m)) + o(1)$ is still a (P.S.) sequence for $E_{\lambda,0}$ at level $\beta - F_0^\infty(V_0)$ and of course it converges weakly to zero.

This concludes the proof of Lemma 2.4.

Proof of Theorem 2.1. By $E_{\lambda,\epsilon}(u_m) \le c, DE_{\lambda,\epsilon}(u_m) \to 0$ strongly in $H^{-1}(\Omega)$, we have $u_m \to u^0$ weakly in $H_0^1(\Omega)$ and u^0 solves problem $(P_{\lambda,\epsilon})$. Moreover, setting $v_m = u_m - u^0$, we then have $v_m \to 0$ strongly in $L^2(\Omega)$ and by Lemma 2.3 we get

$$\begin{aligned}
\int_\Omega |u_m|^{2^*} &= \int_\Omega |u_m - u|^{2^*} + \int_\Omega |u|^{2^*} + o(1), \\
\int_\Omega |\nabla u_m|^2 &= \int_\Omega |\nabla(u_m - u)|^2 + \int_\Omega |\nabla u|^2 + o(1), \\
\int_\Omega \frac{|u_m|^2}{|x|^2} &= \int_\Omega \frac{|u_m - u|^2}{|x|^2} + \int_\Omega \frac{|u|^2}{|x|^2} + o(1).
\end{aligned}$$

Hence,

$$\begin{aligned}
&E_{\lambda,\epsilon}(u_m) = E_{\lambda,\epsilon}(u^0) + E_{\lambda,0}(v_m) + o(1),\\
&DE_{\lambda,\epsilon}(u_m) = DE_{\lambda,\epsilon}(u^0) + DE_{\lambda,0}(v_m) + o(1),\\
&DE_{\lambda,0}(v_m) = o(1).
\end{aligned}$$

By applying Lemma 2.4 to $\{v_m\}$ recursively, the iteration must stop after finite steps; moreover the last (P.S.) sequence must strongly converge to zero. □

Applying Theorem 2.1, we can prove the following:

Corollary 2.5. *Suppose* $\frac{1}{N}S_\lambda^{\frac{N}{2}} < c < \frac{2}{N}S_\lambda^{\frac{N}{2}}$ *and* $c \neq \frac{1}{N}S_0^{\frac{N}{2}}$*, then functional* $E_{\lambda,0}$ *satisfies the* $(P.S.)_c$ *condition.*

Proof. Suppose $\{u_m\}$ is a $(P.S.)_c$ sequence for $E_{\lambda,0}(u)$. Notice that for any $u \in D^{1,2}(R^N)\backslash\{0\}$ satisfying (P_λ^∞) $(0 \le \lambda < \bar{\lambda})$, $F_\lambda^\infty(u) \ge \frac{1}{N}S_\lambda^{\frac{N}{2}}$. Moreover, if u changes sign, then $F_\lambda^\infty(u) \ge \frac{2}{N}S_\lambda^{\frac{N}{2}}$. By Theorem 2.1,

$$c + o(1) = E_{\lambda,0}(u_m) \to E_{\lambda,0}(u^0) + \sum_{j=1}^{l} F_0^\infty(v_0^j) + \sum_{j=1}^{k} F_\lambda^\infty(v_1^j).$$

Suppose that $u^0 \equiv 0$; then $c = \sum_{j=1}^{l} F_0^\infty(v_0^j) + \sum_{j=1}^{k} F_\lambda^\infty(v_1^j)$. If there exist some l or k such that v_0^l or v_1^k changes sign, then by Remark 1.1 (ii) we get $c \ge \frac{2}{N}S_\lambda^{\frac{N}{2}}$, which is a contradiction, so we can assume $v_0^i \ge 0, v_1^j \ge 0$ $(1 \le i \le l, 1 \le j \le k)$. By the uniqueness of solutions of problem (P_0^∞) and (P_λ^∞) (see Remark 1.1 (i)), we have $c = \frac{l}{N}S_0^{\frac{N}{2}} + \frac{k}{N}S_\lambda^{\frac{N}{2}}$, which is impossible. So $u^0 \not\equiv 0$, and we can infer $E_{\lambda,0}(u^0) \ge \frac{1}{N}S_\lambda^{\frac{N}{2}}$, which follows that $l = 0, k = 0$. Hence $u_m \to u^0$ in $H_0^1(\Omega)$.

Remark 2.6. (i) Let

$$\begin{aligned}
\sigma &= \{E_{\lambda,\epsilon}(u) \mid u \in H_0^1(\Omega) \text{ solves } (P_\lambda)\},\\
\sigma_0 &= \{F_0^\infty(u) \mid u \in H_0^1(R^N) \text{ solves } (P_0^\infty)\},\\
\sigma_\lambda &= \{F_\lambda^\infty(u) \mid u \in H_0^1(R^N) \text{ solves } (P_\lambda^\infty)\},
\end{aligned}$$

be the "spectral" of $(P_\lambda), (P_0^\infty)$ and (P_λ^∞), respectively. Then in particular, Theorem 2.1 implies that any sequence $\{u_m\}$ satisfying $E_{\lambda,\epsilon}(u_m) \to \beta, DE_{\lambda,\epsilon}(u_m) \to 0$ in $H^{-1}(\Omega)(m \to \infty)$ is strongly relatively compact in $H_0^1(\Omega)$, provided

$$\beta \bar{\in} \{\bar{\beta} + \sum_{i=1}^{k} \beta_0^i + \sum_{i=1}^{l} \beta_\lambda^i \mid \bar{\beta} \in \sigma \setminus \beta, \beta_0^i \in \sigma_0 \setminus 0, \beta_\lambda^i \in \sigma_\lambda \setminus 0\}.$$

(ii) Assume $u_\epsilon \in H_0^1(\Omega)$ is the ground state solution of $(P_{\lambda,\epsilon})$, $0 < \lambda < \bar{\lambda} - 1$. Then $|\nabla u_\epsilon|^2 - \frac{\lambda u_\epsilon^2}{|x|^2} \to S_\lambda \delta_0$ in the sense of measure as $\epsilon \to 0$, where δ_x denotes the Dirac mass at x.

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