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MEANS ON SOLENOIDS

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Abstract. It is determined which solenoids admit 2-argument continuous means.

A *mean* on a topological space X is a continuous mapping $m : X \times X \to X$ satisfying $m(x, y) = m(y, x)$ and $m(x, x) = x$, for every $x, y \in X$.

There is quite a large literature concerning means, and the main problem is which spaces admit means (see, e.g., [1], [2], [3], [4], [5], [6], [7] where further references are given).

Let $n_1, n_2, \dots$ be a sequence of integers greater than 1. A *solenoid* $\Sigma(n_1, n_2, \dots)$ is the inverse limit $\varprojlim(S_k, f_k)$, where $S_k = \{z \in \mathbb{C} : |z| = 1\}$ and $f_k : S_{k+1} \to S_k$ is given by $f_k(z) = z^{n_k}$. Each solenoid is an abelian topological group under the coordinatewise multiplication $(z_1, z_2, \dots)(z_1', z_2', \dots) = (z_1 z_1', z_2 z_2', \dots)$ with the neutral element $\mathbf{1} = (1, 1, \dots)$. Topologically, solenoids are indecomposable continua whose proper nondegenerate subcontinua are arcs.

Much is known about the existence of homomorphic means on topological abelian groups (see, e.g. [4]); in particular, a mean

$$m : \Sigma(n_1, n_2, \dots) \times \Sigma(n_1, n_2, \dots) \to \Sigma(n_1, n_2, \dots)$$

which is a group homomorphism exists (and is unique) if and only if

(1) there are infinitely many even numbers in the sequence $n_1, n_2, \dots$.

For example, if all numbers in the sequence are of the form $n_i = 2k_i$, $i \in \mathbb{N}$, then the formula

$$m((z_1, z_2, \dots), (z_1', z_2', \dots)) = ((z_2 z_2')^{k_1}, (z_3 z_3')^{k_2}, \dots) \tag{2}$$

defines a homomorphic mean on $\Sigma(n_1, n_2, \dots)$ (see [4]).

The question of whether there exist (non-homomorphic) means on solenoids for odd numbers in the sequence has been asked by J. J. Charatonik at seminars and by A. Illanes at the Spring Topology and Dynamical Systems Conference in 2001.

In this note we completely answer the question by showing that the same criterion (1) is necessary and sufficient for the existence of a mean on $\Sigma(n_1, n_2, \dots)$.

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A simple proof is based on a method of [4] and the following proposition due to W. Scheffer [8, Corollary 2, p. 566].

Proposition 1. *If A and B are continua which are topological groups with neutral elements e_A and e_B, respectively, B is abelian and $f : A \to B$ is a continuous mapping such that $f(e_A) = e_B$, then there exists a (unique) continuous homomorphism $h : A \to B$ which is homotopic to f.*

Theorem 2. *A solenoid $\Sigma(n_1, n_2, \dots)$ admits a mean if and only if condition* (1) *is satisfied.*

Proof. If there are infinitely many even numbers in the sequence $n_1, n_2, \dots$, then $\Sigma(n_1, n_2, \dots)$ is homeomorphic to $\Sigma(n_1', n_2', \dots)$, where all numbers $n_1', n_2', \dots$ are even. So we can assume that numbers $n_1, n_2, \dots$ are even and define a mean by formula (2).

Conversely, let m be a mean on $\Sigma(n_1, n_2, \dots)$ and suppose there exists a number k such that if $i \geq k$, then n_i is odd. Then $\Sigma(n_1, n_2, \dots)$ is homeomorphic to $\Sigma(n_k, n_{k+1}, \dots)$, hence we can assume that all $n_1, n_2, \dots$ are odd. Since $m(\mathbf{1}, \mathbf{1}) = \mathbf{1}$, there exists a continuous homomorphism

$$h : \Sigma(n_1, n_2, \dots) \times \Sigma(n_1, n_2, \dots) \to \Sigma(n_1, n_2, \dots)$$

which is homotopic to m by Proposition 1. We use the symbol $\sim$ to indicate that the values of homotopic mappings lie in the same arc component of $\Sigma(n_1, n_2, \dots)$.

For every $\mathbf{z} \in \Sigma(n_1, n_2, \dots)$, put

$$\mu(\mathbf{z}) = m(\mathbf{z}, \mathbf{1}) = m(\mathbf{1}, \mathbf{z}).$$

We have

$$\mu(\mathbf{z}\mathbf{z}') \sim h(\mathbf{z}\mathbf{z}', \mathbf{1}) = h(\mathbf{z}, \mathbf{1})h(\mathbf{z}', \mathbf{1}) \sim \mu(\mathbf{z})\mu(\mathbf{z}')$$

which implies that

$$\mathbf{z} \sim h(\mathbf{z}, \mathbf{z}) = h(\mathbf{z}, \mathbf{1})h(\mathbf{1}, \mathbf{z}) \sim \mu(\mathbf{z})^2 \sim \mu(\mathbf{z}^2). \tag{3}$$

Now, if $\mathbf{z} = (-1, -1, \dots) \in \Sigma(n_1, n_2, \dots)$, then, by (3), $\mathbf{z} \sim \mu(\mathbf{z}^2) = \mu(\mathbf{1}) = \mathbf{1}$; hence both points $\mathbf{z}$ and $\mathbf{1}$ lie in the same arc component of $\Sigma(n_1, n_2, \dots)$ but this is not the case. □

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