ANDRÉ-QUILLEN HOMOLOGY VIA FUNCTOR HOMOLOGY

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Abstract. We obtain André-Quillen homology for commutative algebras using relative homological algebra in the category of functors on finite pointed sets.

1. Introduction

Let $Γ$ be the small category of finite pointed sets. For any $n ≥ 0$, let $[n]$ be the set $\{0, 1, ..., n\}$ with basepoint 0. We assume that the objects of $Γ$ are the sets $[n]$.

A left $Γ$-module is a covariant functor $Γ \to \text{Vect}$ to the category of vector spaces over a field $K$. For a left $Γ$-module $F$, we put $π_0(F) := \text{Coker}(d_0 - d_1 + d_2 : F([2]) \to F([1]))$, where $d_1$ is induced by the folding map $[2] \to [1]$, $1, 2 \mapsto 1$ while $d_0$ and $d_2$ are induced by the projection maps $[2] \to [1]$ given respectively by $1 \mapsto 1, 2 \mapsto 0$ and $1 \mapsto 0, 2 \mapsto 1$. The category $Γ\text{-mod}$ of left $Γ$-modules is an abelian category with enough projective and injective objects. Therefore one can form the left derived functors of the functor $π_0 : Γ\text{-mod} \to \text{Vect}$, which we will denote by $π_*$. Thanks to [5] and [6] we know that $π_*F$ is isomorphic to the homotopy of the spectrum corresponding to the $Γ$-space $F$ according to Segal (see [9] and [11]).

Let $A$ be a commutative algebra over a ground field $K$ and let $M$ be an $A$-module. There exists a functor $\mathcal{L}(A, M) : Γ \to \text{Vect}$, which assigns $M ⊗ A^{⊗n}$ to $[n]$ (see [3] or section 3). Here all tensor products are taken over $K$. It was proved in [7] that $π_*(\mathcal{L}(A, M))$ is isomorphic to a brave new algebra version of André-Quillen homology $H^Γ_*(A, M)$ constructed by Alan Robinson and Sarah Whitehouse [10]. The main result of this paper shows that a similar isomorphism also exists for André-Quillen homology if one takes an appropriate relative derived functor of the same functor $π_0 : Γ\text{-mod} \to \text{Vect}$.

2. A class of proper exact sequences

Thanks to the Yoneda lemma, $Γ^n$, $n ≥ 0$, are projective generators of the category $Γ\text{-mod}$. Here

$Γ^n := K[\text{Hom}_Γ([n], -)]$.

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and $K[S]$ denotes the free vector space spanned by a set $S$. For left $\Gamma$-modules $F$ and $T$ one defines the pointwise tensor product $F \otimes T$ to be the left $\Gamma$-module given by $(F \otimes T)([n]) = F([n]) \otimes T([n])$. Since $\Gamma^n \otimes \Gamma^m \cong \Gamma^{n+m}$ one sees that the tensor product of two projective left $\Gamma$-modules is still projective. We also have $\Gamma^n \cong (\Gamma^1)^{\otimes n}$.

A partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a sequence of natural numbers $\lambda_1 \geq \cdots \geq \lambda_k \geq 1$. The sum of partition is given by $s(\lambda) := \lambda_1 + \cdots + \lambda_k$, while the group $\Sigma(\lambda)$ is a product of the corresponding symmetric groups

$$\Sigma(\lambda) := \Sigma_{\lambda_1} \times \cdots \times \Sigma_{\lambda_k},$$

which is identified with the Young subgroup of $\Sigma_{s(\lambda)}$. Let us observe that $\Sigma_n = \text{Aut}_F([n])$ and therefore $\Sigma_n$ acts on $\Gamma^n \cong (\Gamma^1)^{\otimes n}$. For a partition $\lambda$ with $s(\lambda) = n$ we let $\Gamma(\lambda)$ be the coinvariants of $\Gamma^n$ under the action of $\Sigma(\lambda) \subset \Sigma_n$.

For a vector space $V$ we let $S^*(V)$, $\Lambda^*(V)$ and $D^*(V)$ be respectively the symmetric, exterior and divided power algebra generated by $V$. Let us recall that $S^n(V) = (V^{\otimes n})/\Sigma_n$ is the space of coinvariants of $V^{\otimes n}$ under the action of the symmetric group $\Sigma_n$, while $D^n(V) = (V^{\otimes n})_{\Sigma_n}$ is the space of invariants. Moreover for a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ we put

$$S^\lambda := S^{\lambda_1} \otimes \cdots \otimes S^{\lambda_k}.$$

We similarly define $\Lambda^\lambda$ and $D^\lambda$. It follows from the definition that

$$\Gamma(\lambda) \cong S^\lambda \circ \Gamma^1.$$

In particular $\Gamma(1, \ldots, 1) \cong \Gamma^n$ and $\Gamma(n) \cong S^n \circ \Gamma^1$.

Let

$$0 \rightarrow T_1 \rightarrow T \rightarrow T_2 \rightarrow 0$$

be an exact sequence of left $\Gamma$-modules. It is called a $\mathcal{U}$-exact sequence if for any partition $\lambda$ with $s(\lambda) = n$ the induced map

$$T([n])^{\Sigma(\lambda)} \rightarrow T_2([n])^{\Sigma(\lambda)}$$

is surjective. Here and elsewhere, $M^G$ denotes the subspace of $G$-fixed elements of a $G$-module $M$. For a $\mathcal{U}$-exact sequence $0 \rightarrow T_1 \rightarrow T \rightarrow T_2 \rightarrow 0$ the sequence

$$0 \rightarrow T_1([n])^{\Sigma(\lambda)} \rightarrow T([n])^{\Sigma(\lambda)} \rightarrow T_2([n])^{\Sigma(\lambda)} \rightarrow 0$$

is also exact. Following to Section XII.4 of [4] we introduce the relative notions. An epimorphism $f : F \rightarrow T$ is called a $\mathcal{U}$-epimorphism if

$$0 \rightarrow \text{Ker}(f) \rightarrow F \rightarrow T \rightarrow 0$$

is a $\mathcal{U}$-exact sequence. Similarly, a monomorphism $f : F \rightarrow T$ is called a $\mathcal{U}$-monomorphism if

$$0 \rightarrow F \rightarrow T \rightarrow \text{Coker}(f) \rightarrow 0$$

is a $\mathcal{U}$-exact sequence. A morphism $f : F \rightarrow T$ is called a $\mathcal{U}$-morphism if $F \rightarrow \text{Im}(f)$ is a $\mathcal{U}$-epimorphism and $\text{Im}(f) \rightarrow T$ is a $\mathcal{U}$-monomorphism. A left $\Gamma$-module $Z$ is called $\mathcal{U}$-projective if for any $\mathcal{U}$-epimorphism $f : F \rightarrow T$ and any morphism $g : Z \rightarrow T$ there exists a morphism $h : Z \rightarrow F$ such that $g = fh$.

**Lemma 2.1.**

i) If a short exact sequence is isomorphic to a $\mathcal{U}$-exact sequence, then it is also a $\mathcal{U}$-exact sequence.

ii) A split short exact sequence is $\mathcal{U}$-exact.

iii) A composition of two $\mathcal{U}$-epimorphisms is still a $\mathcal{U}$-epimorphism.
iv) If $f$ and $g$ are two composable epimorphisms and $fg$ is a $\gamma$-epimorphism, then $f$ is also a $\gamma$-epimorphism.

v) A composition of two $\gamma$-monomorphisms is still a $\gamma$-monomorphism.

vi) If $f$ and $g$ are two composable monomorphisms and $fg$ is a $\gamma$-monomorphism, then $g$ is also a $\gamma$-monomorphism.

Proof. The properties i)-iv) are clear. Let $f : B \to C$ and $g : A \to B$ be monomorphisms. One can form the following diagram:

Assume $f$ and $g$ are are $\gamma$-monomorphisms; then for any partition $\lambda$ with $s(\lambda) = n$ one has a commutative diagram:

The diagram chasing shows that $h$ is an epimorphism and therefore $fg$ is a $\gamma$-monomorphism and v) is proved. Assume now that $fg$ is a $\gamma$-monomorphism.
Then we have the following commutative diagram:

\[
\begin{array}{ccccccc}
0 & \rightarrow & A([n])^{\Sigma(\lambda)} & \rightarrow & B([n])^{\Sigma(\lambda)} & \rightarrow & X([n])^{\Sigma(\lambda)} & \rightarrow & 0 \\
& & \downarrow{l} & & \downarrow{l} & & \downarrow{l} & & \\
0 & \rightarrow & A([n])^{\Sigma(\lambda)} & \rightarrow & C([n])^{\Sigma(\lambda)} & \rightarrow & Z([n])^{\Sigma(\lambda)} & \rightarrow & 0 \\
& & \downarrow{1_\lambda} & & \downarrow{1_\lambda} & & \downarrow{1_\lambda} & & \\
0 & \rightarrow & A([n])^{\Sigma(\lambda)} & \rightarrow & C([n])^{\Sigma(\lambda)} & \rightarrow & Y([n])^{\Sigma(\lambda)} & \rightarrow & 0 \\
& & \downarrow{1_Y} & & \downarrow{1_Y} & & \downarrow{1_Y} & & \\
& & & & & \rightarrow & & & & \\
\end{array}
\]

The diagram chasing shows that \( l \) is an epimorphism and therefore \( f \) is a \( \mathcal{Y} \)-monomorphism and therefore we get vi).

As an immediate corollary we obtain that the class of all \( \mathcal{Y} \)-exact sequences is proper in the sense of Mac Lane [4]. We now show that there are enough \( \mathcal{Y} \)-projective objects.

**Lemma 2.2.**

i) For any partition \( \lambda \) the left \( \Gamma \)-module \( \Gamma(\lambda) \) is a \( \mathcal{Y} \)-projective object.

ii) A morphism \( f : F \rightarrow T \) of left \( \Gamma \)-modules is a \( \mathcal{Y} \)-epimorphism iff for any partition \( \lambda \) the induced morphism

\[
\text{Hom}_{\Gamma\text{-mod}}(\Gamma(\lambda), F) \rightarrow \text{Hom}_{\Gamma\text{-mod}}(\Gamma(\lambda), T)
\]

is an epimorphism.

iii) For any left \( \Gamma \)-module \( F \) there is a \( \mathcal{Y} \)-projective object \( Z \) and a \( \mathcal{Y} \)-epimorphism \( f : Z \rightarrow F \).

iv) Any projective \( \mathcal{Y} \)-module is a direct summand of the sum of objects of the form \( \Gamma(\lambda) \).

v) The tensor product of two \( \mathcal{Y} \)-projective left \( \Gamma \)-modules is still \( \mathcal{Y} \)-projective.

**Proof.** Let \( \lambda \) be a partition with \( s(\lambda) = n \). By definition \( \Gamma(\lambda) = H_0(\Sigma(\lambda), \Gamma^n) \).

Hence for any left \( \Gamma \)-module \( F \) one has

\[
\text{Hom}_{\Gamma\text{-mod}}(\Gamma(\lambda), F) \cong H^0(\Sigma(\lambda), \text{Hom}_{\Gamma\text{-mod}}(\Gamma^n, F) \cong F(n)^{\Sigma(\lambda)}.
\]

The assertions i) and ii) are immediate consequences of this isomorphism. To prove iii) we set

\[X(\lambda) := \text{Hom}_{\Gamma\text{-mod}}(\Gamma(\lambda), F)\]

Moreover, for each \( x \in X(\lambda) \) we let \( f_x : \Gamma(\lambda) \rightarrow F \) be the corresponding morphism. Take \( Z = \bigoplus_{x \in X(\lambda)} \Gamma(\lambda) \). Then the collection \( f_x, x \in X(\lambda) \), yields the morphism \( f : Z \rightarrow F \). We have to show that it is a \( \mathcal{Y} \)-epimorphism. Let \( g : \Gamma(\lambda) \rightarrow F \) be a morphism of left \( \Gamma \)-modules. By ii) we need to lift \( g \) to \( Z \). By our construction \( g \in X(\lambda) \) and therefore the inclusion \( \Gamma(\lambda) \rightarrow Z \) corresponding to the summand \( g \in X(\lambda) \) is an expected lifting and iii) is proved. The proof of iii) shows that one can assume \( P \) to be a sum of \( \Gamma^\lambda \) and iv) follows. To prove the last statement one observes that, for any partitions \( \lambda \) and \( \mu \), one has

\[
\Gamma(\lambda) \otimes \Gamma(\mu) \cong (\Gamma^{s(\lambda)} \otimes \Gamma^{s(\mu)})^{\Sigma(\lambda) \times \Sigma(\mu)} = (\Gamma^{s(\lambda) + s(\mu)})^{\Sigma(\lambda) \times \Sigma(\mu)}
\]

and therefore \( \Gamma(\lambda) \otimes \Gamma(\mu) \) is \( \mathcal{Y} \)-projective. \( \square \)
3. Definition of André-Quillen homology and the functor $\mathcal{L}$

The definition of André-Quillen homology is based on the framework of homotopical algebra \([\mathbb{S}]\) and it is given as follows. We let $C_{*}(V_{*})$ be the chain complex associated to a simplicial vector space $V_{*}$. Let $A$ be a commutative algebra over a ground field $K$ and let $M$ be an $A$-module. A simplicial resolution of $A$ is an augmented simplicial object $P_{*} \to A$ in the category of commutative algebras, which is a weak equivalence (in other words $C_{*}(P_{*}) \to A$ is a weak equivalence). A simplicial resolution is called free if $P_{n}$ is a polynomial algebra over $K$ for all $n \geq 0$.

Any commutative algebra possesses a free simplicial resolution which is unique up to homotopy. Then the André-Quillen homology is defined by

$$D_{*}(A, M) := H_{*}(C_{*}(\Omega^{1}_{\mathcal{P}} \otimes_{P_{*}} M)), \tag{3.1}$$

where $\Omega^{1}$ is the Kähler 1-differential and $P_{*} \to A$ is a free simplicial resolution. In dimension 0 we have $D_{0}(A, M) \cong \Omega^{1}_{A} \otimes_{A} M$.

As we mentioned above the functor $\mathcal{L}(A, M) : \Gamma \to \text{Vect}$ is given on objects by $[n] \mapsto M \otimes A^{\otimes n}$. For a pointed map $f : [n] \to [m]$, the action of $f$ on $\mathcal{L}(A, M)$ is given by

$$f_{*}(a_{0} \otimes \cdots \otimes a_{n}) := b_{0} \otimes \cdots \otimes b_{m}, \tag{3.2}$$

where $b_{j} = \prod_{f(i)=j} a_{i}$, $j = 0, \cdots, n$.

Example 3.1. Let $M = A = K[t]$. In this case one has an isomorphism

$$\mathcal{L}(K[t], K[t]) \cong S^* \Gamma^{1}.$$

To see this isomorphism, one observes that $\Gamma^{1}$ assigns the free vector space on a set $[n]$ to $[n]$ and therefore both functors in question assign the ring $K[t_{0}, \cdots, t_{n}]$ to $[n]$. An important consequence of this isomorphism is the fact that the functor $\mathcal{L}(K[t], K[t])$ is $\mathcal{Y}$-projective.

Lemma 3.2. For any commutative algebra $A$ and any $A$-module $M$, one has a natural isomorphism $\pi_{0}(\mathcal{L}(A, M)) \cong \Omega^{1}_{A} \otimes_{A} M$.

Proof. By the definition we have $\pi_{0}(\mathcal{L}(A, M)) = \text{Coker}(b : M \otimes A^{\otimes 2} \to M \otimes A)$, where $b(m \otimes a \otimes b) = ma \otimes b - m \otimes ab + mb \otimes a$. Since

$$adb \otimes m \mapsto (ma \otimes b) \text{ mod Im}(b)$$

yields the isomorphism $\Omega^{1}_{A} \otimes_{A} M \to \text{Coker}(b)$, the result follows.

Lemma 3.3.

i) Let $A$ be a commutative algebra and let

$$0 \to M_{1} \to M \to M_{2} \to 0$$

be a short exact sequence of $A$-modules. Then

$$0 \to \mathcal{L}(A, M_{1}) \to \mathcal{L}(A, M) \to \mathcal{L}(A, M_{2}) \to 0$$

is a $\mathcal{Y}$-exact sequence.

ii) Let $f : B \to A$ be a surjective homomorphism of commutative algebras. Then for any $A$-module $M$ the induced morphism of left $\Gamma$-modules

$$\mathcal{L}(B, M) \to \mathcal{L}(A, M)$$

is a $\mathcal{Y}$-epimorphism.
Proof. One observes that for any partition \( \lambda \) with \( s(\lambda) = n \) one has
\[
(L(A, M)([n]))^{\Sigma(\lambda)} = (M \otimes A^{\otimes n})^{\Sigma(\lambda)} \cong M \otimes D^\lambda(A).
\]
Since we are over a field the tensor product is exact and we obtain i). By the same reason \( f \) has a linear section, which also yields a linear section of \( D^\lambda(B) \to D^\lambda(A) \), because \( D^\lambda \) is a functor defined on the category of vector spaces. \( \square \)

4. Relative derived functors

By Lemma 2.2 the class of \( \mathcal{Y} \)-exact sequences has enough projective objects. Thanks to [1] this allows us to construct the relative derived functors. Let us recall that an augmented chain complex \( X_* \to F \) is called a \( \mathcal{Y} \)-resolution of \( F \) if it is exact (that is, \( H_i(X_*) = 0 \) for \( i > 0 \) and \( H_0(X_*) \cong F \)) and all boundary maps \( X_{n+1} \to X_n \) are \( \mathcal{Y} \)-morphisms, \( n \geq 0 \). It follows from Lemma 2.2 that \( X_* \to F \) is a \( \mathcal{Y} \)-resolution iff for any partition \( \lambda \) the augmented complex
\[
\text{Hom}_{\Gamma-\text{mod}}(\Gamma(\lambda), X_*) \to \text{Hom}_{\Gamma-\text{mod}}(\Gamma(\lambda), F)
\]
is exact. A \( \mathcal{Y} \)-resolution \( Z_* \to F \) is called a \( \mathcal{Y} \)-projective resolution if for all \( n \geq 0 \) the left \( \Gamma \)-module \( Z_n \) is a \( \mathcal{Y} \)-projective object. We define \( \pi^\mathcal{Y}_n(F) \) using relative derived functors of the functor \( \pi_0 : \Gamma\text{-mod} \to \text{Vect} \). In other words we put
\[
\pi^\mathcal{Y}_n(F) := H_n(\pi_0(Z_*)), \quad n \geq 0,
\]
where \( Z_* \to F \) is a \( \mathcal{Y} \)-projective resolution. By [1] this gives the well-defined functors \( \pi^\mathcal{Y}_n : \Gamma\text{-mod} \to \text{Vect} \), \( n \geq 0 \).

Lemma 4.1. If \( K \) is a field of characteristic zero, then \( \pi_*(F) \cong \pi^\mathcal{Y}_*(F) \).

Proof. In this case all exact sequences are \( \mathcal{Y} \)-exact, because for any finite group \( G \), the functor \( M \mapsto M^G \) is exact. \( \square \)

Lemma 4.2. For left \( \Gamma \)-modules \( F, T \) one has an isomorphism
\[
\pi^\mathcal{Y}_n(F \otimes T) \cong \pi^\mathcal{Y}_n(F) \otimes T([0]) \oplus F([0]) \otimes \pi^\mathcal{Y}_n(T).
\]

Proof. The result in dimension 0 is known (see Lemma 4.2 of [5]). Let \( Z_* \to F \) and \( R_* \to T \) be \( \mathcal{Y} \)-projective resolutions. By Lemma 2.2 \( Z_* \otimes R_* \to F \otimes T \) is also a \( \mathcal{Y} \)-projective resolution. Thus
\[
\pi^\mathcal{Y}_n(F \otimes T) = H_*(\pi_0(Z_* \otimes R_*)) \cong H_*(\pi^\mathcal{Y}_0(Z_* \otimes R_*([0]) \oplus Z_*([0]) \otimes \pi^\mathcal{Y}_0(R_*))) \\
\cong \pi^\mathcal{Y}_n(F) \otimes T([0]) \oplus F([0]) \otimes \pi^\mathcal{Y}_n(T),
\]
where the last isomorphism follows from the Eilenberg-Zilber theorem and Künneth theorem. \( \square \)

Lemma 4.3. Let \( \epsilon : X_* \to A \) be a simplicial resolution in the category of commutative algebras and let \( M \) be an \( A \)-module. Then the associated chain complex of the simplicial \( \Gamma \)-module \( C_*(L(X_*, M)) \to L(A, M) \) is a \( \mathcal{Y} \)-resolution.

Proof. Since \( \epsilon \) is a weak equivalence of simplicial algebras it is a homotopy equivalence in the category of simplicial vector spaces. Thus \( M \otimes D^\lambda(X_*) \to M \otimes D^\lambda(A_*) \) is also a homotopy equivalence, for any partition \( \lambda \). It follows that
\[
L(X_*, M)([n]^{\Sigma(\lambda)}) \to L(A, M)([n]^{\Sigma(\lambda)}
\]
is a homotopy equivalence of simplicial vector spaces. \( \square \)
The following is our main result.

**Theorem 4.4.** For any commutative ring $A$ and any $A$-module $M$, there is a canonical isomorphism

$$D_i(A, M) \cong \pi_i^\vee(\mathcal{L}(A, M)), \ i \geq 0,$$

between the André-Quillen homology and relative derived functors of $\pi_0$ applied on the functor $\mathcal{L}(A, M)$.

**Proof.** Thanks to Lemma 3.2 the result is true for $i = 0$. First consider the case when $M = A = K[t]$. In this case the André-Quillen homology vanishes in positive dimensions by definition. On the other hand $\mathcal{L}(K[t], K[t])$ is $\mathcal{Y}$-projective thanks to Example 5.1 and therefore $\pi_i^\vee(\mathcal{L}(A, M))$ vanishes for all $i > 0$. One can use Lemma 4.3 to conclude that $\pi_i^\vee(\mathcal{L}(A, M))$ vanishes for all $i > 0$ provided $A$ is a polynomial algebra. For the next step, we prove that the result is true if $A$ is a polynomial algebra and $M$ is any $A$-module. We have to prove that $\pi_i^\vee(\mathcal{L}(A, M))$ also vanishes for $i > 0$. We already proved this fact if $M = A$. By additivity the functor $\pi_i^\vee(\mathcal{L}(A, -))$ vanishes on free $A$-modules. By Lemma 4.3 the functor $\pi_i^\vee(\mathcal{L}(A, -))$ assigns the long exact sequence to a short exact sequence of $A$-modules. Therefore we can consider such an exact sequence associated to a short exact sequence of $A$-modules

$$0 \to N \to F \to M \to 0$$

with free $F$. Since the result is true if $i = 0$, one obtains by induction on $i$ that $\pi_i^\vee(\mathcal{L}(A, M)) = 0$ provided $i > 0$. Now consider the general case. Let $P_* \to A$ be a free simplicial resolution in the category of commutative algebras. Then we have

$$\Omega^1_{P_*} \otimes_{P_*} M \cong \pi_0^\vee(\mathcal{L}(P_*, M))$$

Thanks to Lemma 4.3 $C_*(\mathcal{L}(P_*, M)) \to \mathcal{L}(A, M)$ is a $\mathcal{Y}$-resolution consisting of $\pi^n$-acyclic objects and the result follows.

The main theorem together with the main result of [7] yields:

**Corollary 4.5.** If $\text{Char}(K) = 0$, then for any commutative algebra $A$ and any $A$-module $M$ one has a natural isomorphism

$$D_*(A, M) \cong H_*(A, M).$$

This fact was also proved in [10] based on the combinatorical and homotopical analysis of the space of fully grown trees.

**Remarks.** i) We let $t : \Gamma^\text{op} \to \text{Vect}$ be the functor which assigns the vector space of all maps $f : [n] \to K$, $f(0) = 0$ to $[n]$. Then $t \otimes_{\Gamma} F \cong \pi_0(F)$ (see Proposition 2.2 of [5]). Hence $\pi^n_\vee$ can also be defined as the relative derived functors of the functor $t \otimes_{\Gamma} (-) : \Gamma\text{-mod} \to \text{Vect}$. More generally one can take any functor $T : \Gamma^\text{op} \to \text{Vect}$ and define $\text{Tor}_n^\vee(T, F)$ as the value of the relative derived functors (with respect to $\mathcal{Y}$-exact sequences) of the functor $T \otimes_{\Gamma} (-) : \Gamma\text{-mod} \to \text{Vect}$. Then our result claims that

$$D_*(A, M) \cong \text{Tor}_*^\vee(t, \mathcal{L}(A, M)).$$

Based on Proposition 1.15 of [5] the argument given above shows that

$$D_*(n)(A, M) \cong \text{Tor}_n^\vee(\Lambda^n \circ t, \mathcal{L}(A, M)).$$
where  \( D^{(n)}_*(A, M) \) are defined using Kähler \( n \)-differentials:

\[
D^{(n)}_*(A, M) := H_*(C_*(\Omega^n_{P_\ast} \otimes P_* M))
\]

and for \( n = 1 \) one recovers the main theorem.

ii) All results remains true if \( K \) is any commutative ring and \( A \) and \( M \) are projective as \( K \)-modules.

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**References**


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