A WEAK–TYPE ORTHOGONALITY PRINCIPLE

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ABSTRACT. We prove a weak type estimate for operators of the form $f \to \sum_{s \in \mathbf{S}} \langle f, \varphi s \rangle \varphi s$ for certain collections of Schwartz functions $\{\varphi s\}_{s \in \mathbf{S}}$. This extends some of the orthogonality issues involved in the study of the bilinear Hilbert transform by Lacey and Thiele.

1. INTRODUCTION AND PRINCIPAL INEQUALITIES

We are interested in the relationships between three different concepts. First and foremost is that of the phase space, by which we generally mean the Euclidean space formed from the cross product of the spatial variable with the dual frequency variable. Next, we want to associate subsets of that space with functions, the subset describing the location of the function in natural ways. And finally, we want to understand the extent to which orthogonality of the functions can be quantified by geometric conditions on the corresponding sets in the phase plane.

These concerns are not currently very much in the forefront of harmonic analysis, but rather the means towards an end. We treat them as a subject in their own right because the inequalities that we obtain are of an optimal nature and they refine the basic orthogonality issues in the proof of the bilinear Hilbert transform inequalities [4], and complement investigations into "best basis" signal or image processing [7], including the directional issues that arise in the context of brushlets [5].

We state the principal results, and then turn to complementary issues and discussion. For a set $W \subset \mathbb{R}^d$ of finite volume we define $\lambda W := \{c(W) + \lambda(x - c(W)) \mid x \in W\}$ where $c(W)$ is the center of mass of W. We say that W is *symmetric* if

$$-(W - c(W)) = W - c(W).$$

We call the product $s = W_s \times \Omega_s$ of a symmetric convex set W_s and a second set a *tile*. Here, we need not assume that the second set lies in $\mathbb{R}^d$; it could lie in some other set altogether.

We use tiles to study the connection between geometry of the phase plane and orthogonality, which is the intention of the following definition.

1.1. Definition. Let $\mathbf{S}$ be a set of tiles. The functions $\Phi = \{\varphi_s \mid s \in \mathbf{S}\}$ are said to be *adapted to* $\mathbf{S}$ if the functions φ_s are Schwartz functions, and for all $s \in \mathbf{S}$,

$$(1.2) \qquad\qquad\qquad \|\varphi_s\|_2 = 1,$$

$$(1.3) \qquad\qquad \langle\varphi_s, \varphi_{s'}\rangle = 0 \quad \text{if } s' \in \mathbf{S}, \ \Omega_s \cap \Omega_{s'} = \emptyset,$$

$$(1.4) \qquad\qquad |\varphi_s(x)| \leq \frac{C_0}{\sqrt{|W_s|}}(1 + \sigma(x, W_s))^{-2d-5}, \quad x \in \mathbb{R}^d,$$

where we define $\sigma(x, W)$ below.

$$\sigma(x, W) := \inf\{a > 0 \mid x \in aW\}.$$

We call a collection of sets $\mathcal{G}$ a *grid* if for all $G, G' \in \mathcal{G}$, we have $G \cap G' = \emptyset, G$, or G'.

1.5. Theorem. *Let* $\mathbf{S}$ *be a set of tiles such that*

$$(1.6) \qquad\qquad\qquad \{\Omega_s \mid s \in \mathbf{S}\} \quad \text{is a grid,}$$

$$(1.7) \qquad\qquad\qquad \{s \in \mathbf{S}\} \ \text{are pairwise disjoint,}$$

$$(1.8) \qquad s, s' \in \mathbf{S}, \ \Omega'_s \supset \Omega_s \quad \text{implies} \quad W'_s - c(W'_s) \subset W_s - c(W_s).$$

Let $\{\varphi_s \mid s \in \mathbf{S}\}$ *be adapted to* $\mathbf{S}$*, and for* $\lambda > 0$ *and* $f \in L^2(\mathbb{R}^d)$ *let*

$$(1.9) \qquad\qquad \mathbf{S}_\lambda := \left\{ s \in \mathbf{S} \mid \frac{|\langle f, \varphi_s \rangle|}{\sqrt{|W_s|}} \geq \lambda \right\}.$$

Then we have the inequality

$$\sum_{s \in \mathbf{S}_\lambda} |W_s| \leq (2 + KC_0^2)\lambda^{-2}\|f\|_2^2.$$

Here and throughout K *denotes a constant depending only on dimension* d*. This inequality can be rephrased as*

$$\left\| \left\{ \frac{|\langle f, \varphi_s \rangle|}{\sqrt{|W_s|}} \mathbb{I}_{W_s} \mid s \in \mathbf{S} \right\} \right\|_{L^{2,\infty}(\mathbb{R}^d \times \mathbf{S})} \leq \sqrt{2 + KC_0^2}\|f\|_2.$$

In this inequality, $\mathbb{I}_A$ *denotes the indicator function for the set* A*,* $L^{2,\infty}$ *denotes the weak* L^2 *space, and we assign* $\mathbb{R}^d \times \mathbf{S}$ *the product measure of Lebesque measure times counting measure on* $\mathbf{S}$*.*

Note that in the last inequality if the weak L^2 norm could be replaced by the L^2 norm, we would have a Bessel inequality. However, the weak L^2 space cannot be replaced by any smaller Lorentz space. We demonstrate this with an example at the conclusion of the paper. However, in the context of computation, one cannot distinguish between L^2 and weak–L^2.

It is worth noting that the concluding Lemmas of [4] and [3] study exactly the question of orthogonality for more restrictive class of functions φ_s; therein the tiles are rectangles in the phase plane. Also, the analysis shows that orthogonality is linked to the boundedness of the the related maximal function.

There is a second form of this Theorem in which a multiscale object plays the role of a single tile. We call that object a *cluster*.

1.10. Definition. Let $\{\varphi_s \mid s \in \mathbf{C}\}$ be adapted to a set of tiles $\mathbf{C}$. We call $\{\varphi_s \mid s \in \mathbf{C}\}$ a *cluster with shadow I* if the following four conditions are met: (a) $\{s \in \mathbf{C}\}$ are pairwise disjoint, (b) for all $s \in \mathbf{C}$, $W_s \subset I$,

(c) $$s \neq s' \in \mathbf{C}, \ \Omega_s \cap \Omega'_s \neq \emptyset \quad \text{implies} \quad \langle \varphi_s, \varphi'_s \rangle = 0,$$

(d) $$\sum_{s \in \mathbf{C}} |W_s|^{-1} \left[\int_{I^c} (1 + \sigma(x, W_s))^{-d-1} \, dx \right]^2 \leq C_1 |I|.$$

1.11. Theorem. *Let $\mathbf{S}$ be a collection of tiles which is a disjoint union of subcollections $\{\mathbf{C}_I \mid I \in \mathbf{I}\}$. Let $\{\varphi_s \mid s \in \mathbf{S}\}$ be adapted to $\mathbf{S}$, so that for each $I \in \mathbf{I}$, $\{\varphi_s \mid s \in \mathbf{C}_I\}$ is a cluster. Suppose that (1.6) and (1.8) hold. Finally, suppose that the clusters $\mathbf{C}_I$ satisfy this condition.*

(1.12) $$s \neq s' \in \mathbf{S}, \ s \in \mathbf{C}_I, \ \Omega_s \subset \Omega'_s \quad \text{implies} \quad s' \in \mathbf{C}_I \text{ or } W'_s \cap I = \emptyset.$$

Define

$$SQ(I, f)^2 := \frac{1}{|I|} \sum_{s \in \mathbf{C}_I} |\langle f, \varphi_s \rangle|^2, \quad I \in \mathbf{I}.$$

Then, under the assumptions $SQ(I, f) \geq \lambda$ for all $I \in \mathbf{I}$ and $|\langle f, \varphi_s \rangle| \leq 2\lambda \sqrt{|W_s|}$ for all $s \in \mathbf{S}$, we have

$$\sum_{I \in \mathbf{I}} |I| \leq \lambda^{-2} K (1 + C_1^{1/2} C_0^2) \|f\|_2^2.$$

The non–linear form of the hypotheses of this last Theorem preclude a natural formulation of a weak–type inequality.

These Theorems can also be used to study two complicated operators of harmonic analysis, namely Carleson's operator controlling the maximum partial Fourier sums [1] and the bilinear Hilbert transform [4]. A result clearly related to these Theorems can be found in a neglected paper of Prestini [6]. But the first forms of these Theorems, again for special tiles, is in [2].

2. Proofs of the Theorems

We will need a precise estimate of $\langle \varphi_s, \varphi'_s \rangle$, which is the purpose of

2.1. Lemma. *Let s and s' be two tiles with $W'_s - c(W'_s) \subset W_s - c(W_s)$ but $W'_s \cap W_s = \emptyset$. Let $\{\varphi_s, \varphi'_s\}$ be adapted to $\{s, s'\}$. Then there is a constant K so that*

(2.2) $$|\langle \varphi_s, \varphi'_s \rangle| \leq K C_0^2 \sqrt{\frac{|W'_s|}{|W_s|}} \inf_{x \in W'_s} (1 + \sigma(x, W_s))^{-d-5}.$$

Proof. Heuristically, the Lemma follows from the estimate

$$|\langle \varphi_s, \varphi'_s \rangle| \simeq \int_{W'_s} |\varphi_s(x) \varphi'_s(x)| \, dx \leq \|\varphi_s\|_{L^\infty(W'_s)} \|\varphi'_s\|_1$$

and (1.4). Of course the first step must be made precise.
    Observe that

$$|\varphi_s(x)| \leq K \frac{C_0}{\sqrt{|W_s|}} \int_0^\infty \mathbb{I}_{aW_s}(x) \frac{da}{(1 + a)^{2d+6}},$$

which is a consequence of (1.4). We can estimate

$$|\langle \varphi_s, \varphi_s' \rangle| \leq K \frac{C_0^2}{\sqrt{|W_s||W_s'|}} \int_0^\infty \int_0^\infty \langle \mathbb{I}_{aW_s}, \mathbb{I}_{\alpha W_s'} \rangle \frac{da}{(1+a)^{2d+6}} \frac{d\alpha}{(1+\alpha)^{2d+6}}$$

$$\leq K C_0^2 \sqrt{\frac{|W_s'|}{|W_s|}} \int_0^\infty \int_0^\infty \mathbb{I}_{\{aW_s \cap \alpha W_s' \neq \emptyset\}} \frac{da}{(1+a)^{2d+6}} \frac{\alpha^d d\alpha}{(1+\alpha)^{2d+6}}.$$

The point to exploit is that $aW_s \cap \alpha W_s' = \emptyset$ for $a, \alpha \leq \tilde{\sigma} := \sup\{a \mid aW_s \cap aW_{s'} \neq \emptyset\}$.

We bound the double integral above by breaking the region of integration into four pieces. Take $R^0 = [0, \tilde{\sigma})$, $R^1 = [\tilde{\sigma}, \infty)$ and define regions in the (a, α) plane by

$$R_{\varepsilon_1 \varepsilon_2} = R^{\varepsilon_1} \times R^{\varepsilon_2}, \qquad \varepsilon_i \in \{0, 1\}.$$

The integral over R_{00} is zero by the choice of $\tilde{\sigma}$. The integral over R_{01} is

$$\int_0^{\tilde{\sigma}} \int_{\tilde{\sigma}}^\infty \frac{da}{(1+a)^{2d+6}} \frac{\alpha^d d\alpha}{(1+\alpha)^{2d+6}} \leq K(1+\tilde{\sigma})^{-d-5}.$$

We have a similar estimate for the integral over R_{10}. Finally, the estimate over R_{11} is

$$\int_{\tilde{\sigma}}^\infty \int_{\tilde{\sigma}}^\infty \frac{da}{(1+a)^{2d+6}} \frac{\alpha^d d\alpha}{(1+\alpha)^{2d+6}} \leq K(1+\tilde{\sigma})^{-d-5}.$$

These estimates supply us with

$$|\langle \varphi_s, \varphi_s' \rangle| \leq K C_0^2 \sqrt{\frac{|W_s'|}{|W_s|}} (1+\tilde{\sigma})^{-d-5},$$

which is quite close to our claim.

To finish, observe that $\tilde{\sigma} \geq 1$, as W_s and W_s' are disjoint. We shall also see that $\sigma(x, W_s) \leq 1 + 2\tilde{\sigma}$ for all $x \in W_s'$. These two points finish the proof of the Lemma.

Indeed, we can assume that W_s and W_s' are closed sets. Let $\tilde{x}$ be a point in $\tilde{\sigma}W_s \cap \tilde{\sigma}W_s'$. Note that for any point $x \in W_s'$,

$$x - c(W_s) = (x - c(W_s')) + (c(W_s') - \tilde{x}) + (\tilde{x} - c(W_s)).$$

Of the three terms on the right, the first is in $W_s' - c(W_s') \subset W_s - c(W_s)$, by assumption, the second is in $\tilde{\sigma}(W_s' - c(W_s')) \subset \tilde{\sigma}(W_s - c(W_s))$ by symmetry of W_s' and assumption, and the third is in $\tilde{\sigma}(W_s - c(W_s))$. The convexity of W_s then implies that $x \in (2\tilde{\sigma} + 1)W_s$. Hence, $\sigma(x, W_s) \leq 1 + 2\tilde{\sigma}$ for all $x \in W_s'$ as was to be shown. □

Proof of Theorem 1.5. It is sufficient to prove a different assertion. For $f \in L^2(\mathbb{R}^d)$, we assume that

$$(2.3) \qquad\qquad 1 \leq \frac{|\langle f, \varphi_s \rangle|}{\sqrt{|W_s|}} \leq 2, \quad s \in \mathbf{S},$$

and prove that

$$(2.4) \qquad\qquad \sum_{s \in \mathbf{S}} |W_s| \leq (1 + K C_0^2) \|f\|_2^2.$$

For $k \geq 0$ define $\mathbf{S}_k := \{s \in \mathbf{S} \mid 2^k \leq |W_s|^{-1/2}|\langle f, \varphi_s \rangle| \leq 2^{k+1}\}$. One sees that the sum over this set of tiles is at most 2^{-2k} times the upper bound in (2.4). The Theorem is then established in the case of $\lambda = 1$, but this is sufficient as $f \in L^2(\mathbb{R}^d)$ is arbitrary.

For the proof of (2.4) we can assume that $\mathbf{S}$ is a finite collection, so that *a priori* the quantity

$$B := \left\| \sum_{s \in \mathbf{S}} \langle f, \varphi_s \rangle \varphi_s \right\|_2$$

is finite. It suffices to estimate B, for by using (2.3) we see that

$$(2.5) \qquad \sum_{s \in \mathbf{S}} |W_s| \leq \sum_{s \in \mathbf{S}} |\langle f, \varphi_s \rangle|^2 \ = \ \left\langle f, \sum_{s \in \mathbf{S}} \langle f, \varphi_s \rangle \varphi_s \right\rangle \leq B \|f\|_2.$$

We expand B^2 into diagonal and off–diagonal terms.

$$(2.6) \qquad B^2 = \left\| \sum_{s \in \mathbf{S}} \langle f, \varphi_s \rangle \varphi_s \right\|_2^2 \leq \sum_{s \in \mathbf{S}} |\langle f, \varphi_s \rangle|^2 + 2 \sum_{s \in \mathbf{S}} |\langle f, \varphi_s \rangle| \mathcal{O}(s),$$

where we define $\mathbf{S}(s) = \{ s' \in \mathbf{S} - \{s\} \mid \Omega_s \subset \Omega'_s, \langle \varphi_s, \varphi'_s \rangle \neq 0 \}$, and

$$(2.7) \qquad \mathcal{O}(s) := \sum_{s' \in \mathbf{S}(s)} |\langle \varphi_s, \varphi'_s \rangle \langle \varphi'_s, f \rangle|.$$

Recall that $\langle \varphi_s, \varphi'_s \rangle \neq 0$ only if $\Omega_s \cap \Omega'_s \neq \emptyset$. But then from the grid structure, we may assume that $\Omega_s \subset \Omega'_s$.

We have already seen that the diagonal term is dominated by $B\|f\|_2$, so that the term $\mathcal{O}(s)$ is our concern. Using (2.2) and the upper bound on $\langle \varphi'_s, f \rangle$ we have

$$\mathcal{O}(s) \leq K C_0^2 |W_s|^{-1/2} \sum_{s' \in \mathbf{S}(s)} \inf_{x \in W'_s} (1 + \sigma(x, W_s))^{-d-1} |W'_s|$$

$$\leq K C_0^2 |W_s|^{-1/2} \int_{(W_s)^c} (1 + \sigma(x, W_s))^{-d-1} \, dx$$

$$\leq K C_0^2 |W_s|^{1/2}.$$

For the middle line above, the sets Ω'_s for $s' \in \mathbf{S}(s)$ contain Ω_s. But the tiles are disjoint, thus the sets W'_s are pairwise disjoint and contained in $(W_s)^c$.

Therefore, the off diagonal term is, by (2.3) and (2.5),

$$\sum_{s \in \mathbf{S}} |\langle f, \varphi_s \rangle| \mathcal{O}(s) \leq K C_0^2 \sum_{s \in \mathbf{S}} |W_s| \leq K C_0^2 B \|f\|_2.$$

Combining this with (2.6) we see that

$$B^2 \leq B\|f\|_2 + K C_0^2 B\|f\|_2,$$

which gives the desired upper bound B. $\square$

Proof of Theorem 1.11. It suffices to consider the case of $\lambda = 1$. The initial steps are just as before. We assume that the collection of tiles is finite and set

$$B := \left\| \sum_{s \in \mathbf{S}} \langle f, \varphi_s \rangle \varphi_s \right\|_2$$

and estimate B. This is sufficient since

$$\sum_{I \in \mathbf{I}} |I| \leq \sum_{s \in \mathbf{S}} |\langle f, \varphi_s \rangle|^2 \leq B \|f\|_2.$$

Then by Cauchy–Schwartz,

$$B^2 \le \sum_{s\in\mathbf{S}}|\langle f,\varphi_s\rangle|^2 + 2\sum_{s\in\mathbf{S}}|\langle f,\varphi_s\rangle|\mathcal{O}(s),$$

$$\le B\|f\|_2 + [B\|f\|_2]^{1/2}\left[\sum_{s\in\mathbf{S}}|\mathcal{O}(s)|^2\right]^{1/2}.$$

Here as before, $\mathbf{S}(s)$ and $\mathcal{O}(s)$ are as in (2.7). Then by (1.12) and Definition 1.10 (c) the sets $\{W'_s \mid s' \in \mathbf{S}(s)\}$ are contained in I^c if $s \in \mathbf{C}_I$. But they are also pairwise disjoint. Indeed, consider $s', s'' \in \mathbf{S}(s)$. If they fall into the same cluster, they are disjoint by assumption on clusters. Suppose they are in different clusters. Assuming as we may that $\Omega'_s \subset \Omega''_s$, we see that (1.12) implies $W'_s \cap W''_s = \emptyset$.

Recalling Lemma 2.1 and the fact that we have the upper bound $|\langle f,\varphi'_s\rangle| \le 2\sqrt{|W_s|}'$ by assumption, we see that

$$\mathcal{O}(s) \le K C_0^2 |W_s|^{-1/2} \sum_{s'\in\mathbf{S}(s)} \inf_{x\in W'_s}(1+\sigma(x,W_s))^{-d-1}|W'_s|$$

$$\le K C_0^2 |W_s|^{-1/2} \int_{I^c} (1+\sigma(x,W_s))^{-d-1}\,dx$$

if $s \in \mathbf{C}_I$. Hence by the definition of a cluster

$$\sum_{s\in\mathbf{C}_I} \mathcal{O}(s)^2 \le K C_0^4 \sum_{s\in\mathbf{C}_I} |W_s|^{-1}\left[\int_{I^c}(1+\sigma(x,W_s))^{-d-1}\,dx\right]^2$$

$$\le K C_1 C_0^4 |I|,$$

and so $B^2 \le B\|f\|_2 + K C_1^{1/2} C_0^2 B\|f\|_2$, which proves the Theorem. $\qquad\square$

3. Counterexample

We demonstrate the optimality of the $L^{2,\infty}$ norm in our Theorem. It suffices to consider the first Theorem on $\mathbb{R}$, and this we will do with the collection of disjoint rectangles

$$\mathbf{S} := \{[2^j, 2^{j+1}) \times [(n-1/2)2^{-j}, (n+1/2)2^{-j}) \mid j, n \in \mathbb{Z}\}.$$

Let φ denote a Schwartz function with $\varphi(x) > 0$ for all x, L^2 norm one, and $\hat{\varphi}$ supported on $[-1/2, 1/2]$. For $s = W_s \times \Omega_s \in \mathbf{S}$ define

$$\varphi_s(x) := e^{2\pi i c(\Omega_s)x}|W_s|^{-1/2}\varphi\Big(\frac{x-c(W_s)}{|W_s|}\Big).$$

Clearly, $\{\varphi_s \mid s \in \mathbf{S}\}$ is adapted to $\mathbf{S}$.

Take $f = \mathbb{I}_{[-1,0)}$. For integers $j \ge 0$ and $|n| < 2^{j-1}$, let $s = [2^j, 2^{j+1}) \times [(n-1/2)2^{-j}, (n+1/2)2^{-j})$. We have

$$2^{-j/2}\langle f,\varphi_s\rangle = 2^{-j}\int_{-1}^{0} e^{2\pi i n 2^{-j}x}\varphi\Big(\frac{x-32^{j-1}}{2^j}\Big)\,dx$$

$$= e^{3\pi i n}\int_{-\frac{3}{2}-2^{-j}}^{-\frac{3}{2}} e^{2\pi i n x}\varphi(x)\,dx.$$

Since $\varphi(-3/2) > 0$, we see that $|2^{-j/2}\langle f, \varphi_s\rangle| \geq c2^{-j}$. This estimate is uniform in n and j as we have specified. Thus, for all $0 < \lambda < 1$,

$$\lambda^2 \sum_{s \in \mathbf{S}} \mathbb{I}\{|\langle f, \varphi_s\rangle| \geq \lambda\sqrt{|W_s|}\}|W_s| \geq c'.$$

Thus, for any finite t,

$$\left\| \left\{ \frac{|\langle f, \varphi_s\rangle|}{\sqrt{|W_s|}} \mathbb{I}_{W_s} \mid s \in \mathbf{S} \right\} \right\|_{L^{2,t}(\mathbb{R}^d \times \mathbf{S})} = \infty.$$

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