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$\cos \pi\lambda$ AGAIN

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Abstract. It is shown that if, for an entire function,

$$\liminf_{r\to\infty} \log M(r)/r^{\lambda} = 0$$

where $0 < \lambda < 1$, then

$$\limsup_{r\to\infty} (\log m(r) - \cos \pi\lambda \log M(r))/\log r = \infty.$$

In the proof, the zeros of the function are redistributed to minimize the large values of $\log m(r) - \cos \pi\lambda \log M(r)$.

1. Introduction

For an entire function $f(z)$, let $M(r)$ and $m(r)$ be the maximum and minimum modulus:

$$M(r) = M(r, f) = \max_{|z|=r} |f(z)|, \quad m(r) = m(r, f) = \min_{|z|=r} |f(z)|.$$

This note contains a proof of the $\cos\pi\lambda$ theorem [5, Theorem II] in the following form:

Theorem 1. *If*

$$\liminf_{r\to\infty} \frac{\log M(r)}{r^{\lambda}} = 0, \tag{1}$$

where $0 < \lambda < 1$*, then*

$$\limsup_{r\to\infty} \frac{\log m(r) - \cos \pi\lambda \log M(r)}{\log r} = \infty. \tag{2}$$

This result has been overtaken by later work of Kjellberg [6], Baernstein [1] and others, but the proof given here is new and elementary and throws light on the way in which the growth of the maximum modulus is smoothed when the minimum modulus is pressed down. A key part is played by the following lemma:

Lemma 2 ([4])**.** *Suppose that* $K : [-1, 0) \cup (0, 1] \to \mathbf{R}$ *is twice differentiable and concave, decreasing on* $[-1, 0)$ *and increasing on* $(0, 1]$*, and such that* $\lim_{t\to 0} |K'(t)|$

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$= \infty$ and $\lim_{t\to 0} K(t)$ exists (possibly $-\infty$). Also suppose that $J : [-1,1] \to \mathbf{R}$ is concave, and, given a partition T of $[0,1] : 0 = t_1 \le t_2 \le \ldots \le t_N = 1$, define

$$\phi(t) = J(t) + \sum_{m=1}^{N} K(t - t_m).$$

Let $\phi_m = \max_{t_m \le t \le t_{m+1}} \phi(t)$, $1 \le m \le N-1$, and let

$$\Phi = \max_{1\le m\le N-1} \phi_m, \quad \Psi = \min_{1\le m\le N-1} \phi_m.$$

The problems of minimizing Φ or maximizing Ψ have the same solution. In the extremal configuration, which is unique, $0 = t_1 < t_2 < \ldots < t_N = 1$ and $\phi_1 = \phi_2 = \ldots = \phi_{N-1}$.

The graph of ϕ consists of concave fingers between distinct successive points of T, each one having a single maximum or 'peak'. The upshot of the lemma is that the largest peak is minimised in the same configuration in which the smallest peak is maximised.

We also need an estimate of $\cos \pi\lambda$ type for a particular function.

Lemma 3. *Given positive numbers σ, λ and ϵ, with σ a non-negative integer and $0 < \lambda < 1$, let*

$$G(z) = z^\sigma \prod_{m=1}^{\infty} \left(1 - \frac{z}{\rho_m}\right),$$

where $\rho_m = (m/\epsilon)^{1/\lambda}$, $m = 1, 2, \ldots$. If $r = (\rho_m + \rho_{m+1})/2$, $m = 1, 2, \ldots$, then

$$\log|G(r)| - \cos\pi\lambda \log G(-r) \ge (\sigma(1 - \cos\pi\lambda) - 4)\log r + 4\lambda^{-1}\log(1/\epsilon) + C,$$

where C depends only on λ.

According to a preliminary result of Kjellberg [5, p. 192], a function $f(z)$ satisfying (1) is uniformly approximated on any compact set by a sequence of partial products formed from its zeros:

$$f_k(z) = A_0 z^\sigma \prod_{m=1}^{N_k} \left(1 - \frac{z}{z_m}\right). \tag{3}$$

Here A_0 and σ are constants, σ being a non-negative integer, and z_1, z_2, ... are the non-zero zeros of f arranged in order of increasing modulus. For our purposes, there is no loss of generality in assuming that $A_0 = 1$. Evidently Theorem 1 will be proved if we can show the following:

Lemma 4. *Given any positive number κ, there are arbitrarily large numbers R_1 and R_2 such that $R_2 > R_1$ and, for some $r = r(k) \in [R_1, R_2]$,*

$$\log m(r, f_k) - \cos\pi\lambda \log M(r, f_k) \ge \kappa \log r, \tag{4}$$

for all large k.

To prove Lemma 4, it may be assumed, using a standard argument [2, p. 40], that $z_1, z_2 \ldots$ are real and positive, with $0 < z_1 \le z_2 \le \ldots$, and then $M(r, f_k) = f_k(-r)$ and $m(r, f_k) = |f_k(r)|$. Writing $n_k(t)$ for the counting function of the non-zero zeros of f_k, we have

$$\log|f_k(z)| = \sigma \log r + \int_0^\infty \log\left|1 - \frac{z}{t}\right| dn_k(t) = \sigma \log r + \int_0^\infty \frac{z n_k(t)}{t(z-t)} dt. \tag{5}$$

In the context of Lemma 4, the distribution of the first few zeros of f is immaterial. The contribution of any factor in the representation (3) to $\log|f_k(r)| - \cos\pi\lambda \log f_k(-r)$ is, asymptotically as $r \to \infty$, $(1-\cos\pi\lambda)\log r$, which does not change if the zero is moved slightly. Some of the initial zeros may thus be pushed to 0 if it is convenient to do so (as it turns out to be). Notice too that to prove Lemma 4 it is enough to establish

$$\log|f_k(r)| - \cos\pi\lambda \log f_k(-r) \geq 0, \tag{6}$$

rather than (4). Suppose we could do that, and we are given an entire function f satisfying (1). Since, for any positive integer q, $g(z) = f(z)/\prod_{m=1}^{q}(1-z/z_m)$ also satisfies (1), (6) would hold for g_k at $r = r(k)$, and so

$$\log|f_k(r)| - \cos\pi\lambda \log f_k(-r) \geq (q(1-\cos\pi\lambda) + o(1))\log r.$$

Since q is arbitrary, Lemma 4 would follow.

2. Proof of Lemma 4

If $f(z)$ satisfies (1) and $A_0 = 1$ in (3), then from Jensen's theorem,

$$\int_0^r \frac{n(t)}{t}dt = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta - \sigma\log r \leq \log M(r), \quad r \geq 1,$$

where $n(t)$ is the counting function of the non-zero zeros of f. Since

$$\int_0^{2r} \frac{n(t)}{t}dt \geq \int_r^{2r} \frac{n(t)}{t}dt \geq n(r)\log 2,$$

we deduce that

$$\liminf_{r\to\infty} \frac{n(r)}{r^\lambda} = 0. \tag{7}$$

It is thus possible to find R_1 arbitrarily large such that

$$\frac{n(t)}{t^\lambda} \geq \frac{n(R_1)}{{R_1}^\lambda} \equiv \epsilon, \quad z_1 < t \leq R_1. \tag{8}$$

Notice that, since $n(R_1)$ is an integer, so is $\epsilon {R_1}^\lambda$. With ϵ as in (8) and ϵ' a positive number satisfying

$$0 < \epsilon' < c\epsilon, \tag{9}$$

where $c = c(\lambda)$ is a constant to be determined by later constraints, we choose R_2 arbitrarily large such that $R_2 > R_1$, $\log f(-4R_2) < \epsilon' R_2^\lambda$, which is possible from (1), and ϵR_2^λ is an integer. (This last condition can be ensured since, if $\log f(-4R_2) < 2^{-\lambda}\epsilon' R_2^\lambda$ for some R_2, then $\log f(-4R_2') < \epsilon' R_2'^\lambda$ for any R_2' such that $R_2/2 \leq R_2' \leq R_2$, and some such R_2' may be chosen for which $\epsilon R_2'^\lambda$ is an integer.) For all large k then, $\log f_k(-4R_2) < \epsilon' R_2^\lambda$, and thus, from (5),

$$\int_{2R_2}^\infty \frac{4R_2 n_k(t)}{t(t+4R_2)}dt < \epsilon' R_2^\lambda,$$

from which it follows that

$$\int_{2R_2}^\infty \frac{n_k(t)}{t^2}dt < \frac{3}{4}\epsilon' R_2^{\lambda-1}. \tag{10}$$

Moreover, from Jensen's theorem,

$$n_k(2R_2)\log 2 \le \int_{2R_2}^{4R_2} \frac{n_k(t)}{t}dt \le \log f_k(-4R_2) < \epsilon' R_2^\lambda,$$

so that, with c suitably chosen in (9),

$$n_k(2R_2) < \epsilon R_2^\lambda. \tag{11}$$

From (8) and the fact that, for all large k, $n_k(t) = n(t)$ for $0 < t \le R_1$,

$$[\epsilon t^\lambda] \le n_k(t), \quad z_1 < t \le R_1, \tag{12}$$

and thus, taking account of (11),

$$\nu_k(t) = \begin{cases} [\epsilon t^\lambda], & t < R_1, \\ n_k(t), & R_1 \le t < R_2, \\ [\epsilon t^\lambda], & t \ge R_2 \end{cases} \tag{13}$$

is the counting function of the non-zero zeros of an entire function $F_k(z)$ which has, apart from σ zeros at 0 (as for f), real positive zeros (which we label $s_1, s_2, \dots$). From (8), (11) and the fact that ϵR_2^λ is an integer, F_k has zeros at R_1 and R_2.

For $r \in [R_1, R_2]$ we have, from (5),

$$\begin{aligned} &\log|f_k(r)| - \cos\pi\lambda \log f_k(-r) - \left\{\log|F_k(r)| - \cos\pi\lambda \log F_k(-r)\right\} \\ &\qquad = \int_0^\infty \frac{r(At+Br)}{t(r^2-t^2)}(n_k(t) - \nu_k(t))dt, \end{aligned} \tag{14}$$

where

$$A = (1+\cos\pi\lambda) \quad \text{and} \quad B = (1-\cos\pi\lambda). \tag{15}$$

Moreover, from (13), (12), (11) and (10),

$$\begin{aligned} \int_0^\infty \frac{r(At+Br)}{t(r^2-t^2)}(n_k(t)-\nu_k(t))dt &\ge \int_{2R_2}^\infty \frac{r(At+Br)}{t(r^2-t^2)}(n_k(t) - [\epsilon t^\lambda])dt \\ &\ge -\frac{8}{3}r\int_{2R_2}^\infty \frac{n_k(t)}{t^2}dt + \frac{2^{\lambda-2}A}{1-\lambda}\epsilon r R_2^{\lambda-1} \\ &> 0, \end{aligned}$$

with a suitable choice of c in (9). It follows that, for all $r \in [R_1, R_2]$,

$$\log|f_k(r)| - \cos\pi\lambda \log f_k(-r) > \log|F_k(r)| - \cos\pi\lambda \log F_k(-r). \tag{16}$$

The plan is to arrange the zeros of F_k in $[R_1, R_2]$ to make

$$\max_{r\in[R_1,R_2]}(\log|F_k(r)| - \cos\pi\lambda \log F_k(-r))$$

as small as possible, the only restriction being that, in any arrangement, zeros are retained at R_1 and R_2.

With Lemma 2 in view, write $r = e^{at+b}$, where $a = \log(R_2/R_1)$ and $b = \log R_1$, and $s_n = e^{at_n+b}$, $n = 1, 2, \dots$. Let

$$K(t) = \log|1 - e^{at}| - \cos\pi\lambda \log(1 + e^{at}),$$

and, supposing that $R_1 = s_p$ (p being the smallest such integer) and $R_2 = s_q$ (q being the largest such integer), let

$$J(t) = \sigma(1-\cos\pi\lambda)(at+b) + \sum_{m<p, m>q} K(t - t_m),$$

so that

$$\log|F_k(r)| - \cos\pi\lambda \log F_k(-r) = J(t) + \sum_{p\le m\le q} K(t-t_m).$$

Direct differentiation shows that K and J satisfy the hypotheses of Lemma 2, and we conclude that the smallest value of the largest peak of

$$\log|F_k(r)| - \cos\pi\lambda \log F_k(-r)$$

in $[R_1, R_2]$ is the same as the largest value of the smallest peak. To prove Theorem 1 we need only produce an arrangement of the zeros of F_k that makes all peaks in $[R_1, R_2]$ positive. According to Lemma 3, this occurs when $\nu(t) \equiv [\epsilon t^\lambda]$, provided σ is sufficiently large, which, as we have noticed, can always be ensured.

3. Proof of Lemma 3

As in (14),

$$\log|G(r)| - \cos\pi\lambda \log G(-r) = \sigma(1-\cos\pi\lambda)\log r + \int_0^\infty \frac{r(At+Br)}{t(r^2-t^2)}\nu(t)dt,$$

where $\nu(t) = [\epsilon t^\lambda]$. Also, for $r = (\rho_m + \rho_{m+1})/2$,

$$\int_0^\infty \frac{r(At+Br)}{t(r^2-t^2)}[\epsilon t^\lambda]dt \ge I_1 - I_2 - I_3 - I_4, \tag{17}$$

where

$$I_1 = \int_0^\infty \frac{r(At+Br)}{t(r^2-t^2)}\epsilon t^\lambda dt = 0, \tag{18}$$

a standard result from contour integration [3, pp. 140-141],

$$I_2 = \int_0^{\rho_1} \frac{r(At+Br)}{t(r^2-t^2)}\epsilon t^\lambda dt, \tag{19}$$

$$I_3 = \int_{\rho_1}^{\rho_m} \frac{r(At+Br)}{t(r^2-t^2)}dt, \tag{20}$$

$$I_4 = \int_{\rho_m}^{\rho_{m+1}} \frac{r(At+Br)}{t(r^2-t^2)}(\epsilon t^\lambda - [\epsilon t^\lambda])dt. \tag{21}$$

We estimate these in turn. For I_2: since $(At+Br)/(r+t) \le A+B = 2$ and $r/(r-t) \le (\rho_2+\rho_1)/(\rho_2-\rho_1)$, we have $I_2 \le 2\lambda^{-1}(2^{1/\lambda}+1)/(2^{1/\lambda}-1)$.

For I_3: again $(At+Br)/(r+t) \le 2$, so

$$I_3 \le 2\int_{\rho_1}^{\rho_m} \frac{r}{t(r-t)}dt = 2\log\left(\frac{\rho_m}{r-\rho_m}\frac{r-\rho_1}{\rho_1}\right).$$

Also

$$\frac{\rho_m}{r-\rho_m} = \frac{2}{(1+1/m)^{1/\lambda}-1} \le 2m\lambda < 2m$$

and

$$\frac{r-\rho_1}{\rho_1} \le r/\rho_1 \le \rho_{m+1}/\rho_1 = (m+1)^{1/\lambda} \le (2m)^{1/\lambda},$$

and $m < \epsilon r^\lambda$, so $I_3 \le 4\lambda^{-1}\log(2\epsilon r^\lambda)$.

For I_4: write $\alpha(t) = r(At + Br)/t(r + t)$ and $\beta(t) = \epsilon t^\lambda - [\epsilon t^\lambda]$.

$$\begin{aligned} I_4 &= \int_{\rho_m}^{\rho_{m+1}} \frac{\alpha(t) - 1}{r - t}\beta(t)dt + \int_{\rho_m}^{\rho_{m+1}} \frac{\beta(t)}{r - t}dt \\ &= \int_{\rho_m}^{\rho_{m+1}} \frac{r(1 - \cos \pi\lambda) + t}{t(r + t)}\beta(t)dt + \int_{\rho_m}^{\rho_{m+1}} \frac{\epsilon t^\lambda}{r - t}dt \\ &\leq \int_{\rho_m}^{\rho_{m+1}} \frac{r(1 - \cos \pi\lambda) + t}{t(r + t)}\beta(t)dt \\ &\leq 2\int_{\rho_m}^{\rho_{m+1}} t^{-1}dt \leq 2\lambda^{-1} \log 2. \end{aligned} \tag{22}$$

Lemma 3 follows by combining the estimates for I_2, I_3 and I_4.

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