BICYCLIC UNITS OF $\mathbb{Z}S_n$

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Abstract. We prove that the group generated by the bicyclic units of $\mathbb{Z}S_n$ has torsion for $n \geq 4$. This answers a question of Sehgal (1993).

Let $G$ be a finite group. For every $x \in G$ of order $k$ let $\hat{x} = \sum_{i=0}^{k-1} x^i \in \mathbb{Z}G$. The bicyclic units of $\mathbb{Z}G$ are the units of the form

$$b(x, y) = 1 + \hat{x}y(1-x)$$

for $x, y \in G$. The following appears in [5] as Problem 19:

Problem: Is the group $\langle b(x, y) : x, y \in G \rangle$, generated by the bicyclic units of $\mathbb{Z}G$, torsionfree?

As a consequence of [5, Theorem 31.3] it is easy to prove that the problem has a positive answer for several groups, including dihedral groups.

The units of the form $b'(x, y) = 1 + (1-x)y\hat{x}$ are also called bicyclic units and in fact the problem was stated in [5] for the group generated by the $b'(x, y)$'s. It is obvious that both versions are equivalent. We have chosen the $b(x, y)$'s for computational reasons.

In this paper we show that the problem has a negative answer proving the following theorem.

Theorem 1. For every positive integer $n$ let $S_n$ be the symmetric group on $n$ letters and $B_n$ the group generated by the bicyclic units of the symmetric group ring $\mathbb{Z}S_n$. Then

$$B_n \cap S_n = \begin{cases} 1 & \text{if } n \leq 3, \\ \langle (1 2)(3 4), (1 3)(2 4) \rangle & \text{if } n = 4, \\ A_n \text{ or } S_n & \text{if } n \geq 5. \end{cases}$$

Since $S_2$ is abelian and $B_4$ is free [3] Theorem 1 is clear of $n \leq 3$. We consider $S_n$ embedded in $S_{n+1}$ in the obvious way so that $B_n \subseteq B_{n+1}$. If $g, x, y \in G$, then $g^{-1}b(x, y)g = b(g^{-1}xg, g^{-1}yg)$. Therefore $B_n$ is normalized by $S_n$ and hence $B_n \cap S_n$ is a normal subgroup of $S_n$. Thus to prove Theorem 1 it is enough to prove

$$\langle (1 2)(3 4), (1 3)(2 4) \rangle = B_4 \cap S_4.$$ 

In the remainder of the paper we prove this equality and in the way we obtain a full description of $B_4$ in terms of some groups of integral matrices.

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Consider the following four elements of $S_4$:

$$a = (12)(34), \quad b = (13)(24), \quad c = (123), \quad d = (12).$$

Recall that $S_4 = \langle a, b \rangle \rtimes \langle c, d \rangle$ and $\langle c, d \rangle = S_3$. Let $\tau : S_4 \to S_3$ be the projection given by the previous decomposition, that is, $\tau$ is the identity in $\langle c, d \rangle$ and $\text{Ker} \tau = \langle a, b \rangle$. Extend $\tau$ by linearity to a homomorphism of rational algebras $\mathbb{Q}S_4 \to \mathbb{Q}S_3$, also denoted by $\tau$. $S_4$ has two inequivalent representations of degree 3. We take from [1] $\rho_1$ and $\rho_2$ given by

$$\rho_1(a) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \rho_1(b) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\rho_1(c) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \rho_1(d) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and

$$\rho_2(g) = \begin{cases} \rho_1(g), & \text{if } g \in A_4, \\ -\rho_1(g), & \text{if } g \not\in A_4. \end{cases}$$

(The representation $\rho_1$ and $\rho_2$ are denoted $\rho$ and $\rho'$ in [1]. Note that there is an error in the definition of $\rho$ in [1] where $\rho(a)$ and $\rho(b)$ should be interchanged.)

Extend $\rho_1$ and $\rho_2$ to homomorphisms of rational algebras $\mathbb{Q}S_4 \to M_3(\mathbb{Q})$ and let $\rho : \mathbb{Q}S_4 \to M_3(\mathbb{Q})^2$ be the direct sum of $\rho_1$ and $\rho_2$. It is well known that $\tau \oplus \rho : \mathbb{Q}S_4 \to \mathbb{Q}S_3 \oplus M_3(\mathbb{Q})^2$ is an isomorphism.

For an arbitrary finite group $G$, $V(ZG)$ denotes the group of units of $ZG$ of augmentation 1. The homomorphisms $\tau$ and $\rho$ induce group homomorphisms $\tau : V(ZS_4) \to V(ZS_3)$ and $\rho : V(ZS_4) \to \text{GL}_3(\mathbb{Z})^2$. Clearly $\tau(B_4) = B_3$. Since $B_3$ is free in $[1]$, one has that

$$B_4 = (B_4 \cap K) \rtimes B_3,$$

where $K = \{ \alpha \in V(ZS_4) : \tau(\alpha) = 1 \}$. Moreover, $\rho$ is an isomorphism between $K$ and $\rho(K)$ (because $\tau \oplus \rho$ is an isomorphism) and the last has been described in [1].

Since we need this description we are going to recall it.

Let $\hat{E}(n)$ denote the principal congruence group of level $n$, of $\text{SL}_3(\mathbb{Z})$ ($n \in \mathbb{Z}$); that is,

$$\hat{E}(n) = \{ A \in \text{SL}_3(\mathbb{Z}) : A \equiv 1 \mod n \}.$$

Let

$$X = \{(x_{ij}) \in \hat{E}(2) : x_{12} + x_{23} + x_{31} \equiv x_{13} + x_{21} + x_{32} \mod 4 \}$$

and

$$X_1 = \{(x_{ij}) \in \hat{E}(2) : x_{12} + x_{23} + x_{31} \equiv x_{13} + x_{21} + x_{32} \equiv 0 \mod 4 \}.$$

Let $G = \langle Q, R, Q^i, R^i, \hat{E}(8) \rangle$ where

$$Q = \begin{pmatrix} 1 & 0 & 4 \\ 4 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \quad R = \begin{pmatrix} 5 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 4 & 5 \end{pmatrix},$$
and $A^t$ denotes the transpose of a matrix $A$. Finally let

$$T = \begin{pmatrix} 17 & 0 & -4 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}.$$ 

(Note that the matrices $Q$ and $R$ are different from the corresponding matrices in [I]. This does not affect the definition of $G$ because our $Q$ and $R$ are congruent to the $Q$ and $R$ in [I] modulo 8.)

Now we are ready to give the description of $\rho(K)$ in terms of integral matrices.

**Theorem 2** ([I]).

$\rho(K) = \{(A, T^sAG) : A \in X, G \in G, s = 0, if A \in X_1 \ and \ s = 1, otherwise\}.$

For a permutation $\sigma \in S_n$ and a matrix $A \in M_n(R)$ let $A^\sigma$ denote the matrix obtained by permuting the rows and columns of $A$ by $\sigma$, that is, $A^\sigma = P_{\sigma}^{-1}AP_{\sigma}$ where $P_{\sigma}$ is the permutation matrix defined by

$$P_{\sigma}(i,j) = \begin{cases} 1, & \text{if } j = \sigma(i), \\ 0, & \text{otherwise}. \end{cases}$$

For every $x, y \in S_4$ let

$$\kappa_{x,y} = b(x,y) \cdot \tau(b(x,y))^{-1} \in B_4 \cap K$$

and let $K_0$ be the group generated by all the $\kappa_{x,y}$’s.

**Remark 3.** Let $H$ be a group of units of $\mathbb{Z}S_4$ normalized by $S_3 = \langle c, d \rangle$. Then $\rho_i(H)$ is normalized by $\rho_i(S_3)$ ($i = 1, 2$). This implies that $A^\sigma \in \rho_i(H)$ for every $A \in \rho_i(H)$ and $\sigma \in S_3$.

Some groups normalized by $S_3$ are $B_4$, $K$ and $\text{Ker } \rho_i$ ($i = 1, 2$). Another example is $K_0$ because $\tau$ acts as the identity in $S_3$. \hfill \Box

For every $1 \leq i \neq j \leq 3$ and $n$ an integer let $e_{ij}(n)$ be the $3 \times 3$ matrix having $n$ in the $(i, j)$ entry and zeroes elsewhere. Set $E_{ij}(n) = I + e_{ij}(n)$. Let $E(n) = \langle E_{ij}(n) : 1 \leq i \neq j \leq 3 \rangle$.

**Lemma 4.**

1. $SL_3(\mathbb{Z}) = E(1)$.
2. $\hat{E}(n)$ is the normal subgroup of $SL_3(\mathbb{Z})$ generated by $E_{12}(n)$.
3. $E(n) = \{(a_{ij}) \in SL_3(\mathbb{Z}) : n|a_{ij} \ and \ a_{ii} \equiv 1 \ mod \ n^2\}$, in particular $\hat{E}(n^2) \subseteq E(n)$.
4. $\hat{E}(n) = \langle A_n, A_n^c, E(n) \rangle$ (recall that $c = (1 \ 2 \ 3)$), where

$$A_n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+n & n \\ 0 & -n & 1-n \end{pmatrix}.$$

5. $X = \langle A_2^\sigma, B^\sigma, \hat{E}(4) : \sigma \in S_3 \rangle$ where $B = E_{23}(2) \cdot E_{12}(2)$.

**Proof.**

1. See [4, 1.2.11].
2. See [4, 1.2.26] and [2, Corollary 4.3] or the proof of [4, 4.3.1].
3. We first prove $\hat{E}(n^2) \subseteq E(n)$. By 1 and 2 it is enough to show that $E_{ij}(1)E_{12}(n^2)E_{ij}(1)^{-1}$ belongs to $E(n)$ for every $i \neq j$. This is obvious if $(i, j) \neq (2, 1)$. Finally

$$E_{21}(1)E_{12}(n^2)E_{21}(1)^{-1} = E_{21}(1)[E_{13}(n), E_{32}(n)]E_{21}(1)^{-1} = [E_{21}(1)E_{13}(n)E_{21}(1)^{-1}, E_{21}(1)E_{32}(n)]E_{21}(1)^{-1} = [E_{23}(n)E_{13}(n), E_{31}(-n)E_{32}(n)].$$
Let $E = \{ (a_{ij}) \in SL_3(\mathbb{Z}) : n(a_{ij} \text{ if } i \neq j \text{ and } a_{ii} \equiv 1 \mod n^2) \ $. Plainly $\hat{E}(n^2) \subseteq E(n) \subseteq E$. Now notice that if $A = I + n(a_{ij})$ and $B = I + n(b_{ij})$, then $AB \equiv I + n(a_{ij} + b_{ij}) \mod n^2$. Using this it is easy to see that $E(n)/\hat{E}(n^2) \simeq \mathbb{Z}_n^k \simeq E/\hat{E}(n^2)$, so that $E(n)/\hat{E}(n^2) = E/E(n^2)$ and hence $E(n) = E$.

(4) and (5). A trivial verification shows that $E(n)/\hat{E}(n^2) = \langle A_n, A'_n, E(n) \rangle/\hat{E}(n^2)$ and $X/E(4) = \langle A_2^\sigma, B^\sigma, E(4) : \sigma \in S_3 \rangle/E(4)$.

\begin{remark}
Let $H$ be as in Remark 4. By Lemma 4 to prove that $E(n) \subseteq \rho_i(H)$ it is enough to show that $E_{ij}(n) \subseteq \rho_i(H)$ for some $i \neq j$, and to prove that $\hat{E}(n) \subseteq \rho_i(H)$ it is enough to additionally prove that $A_n \in \rho_i(H)$.
\end{remark}

\begin{lemma}
$\rho_1(K_0) = X$.
\end{lemma}

\begin{proof}
By Theorem 2 $\rho_1(K_0) \subseteq X$. To prove the other inclusion we are going to use Lemma 4 and Remarks 4 and 5 several times without specific mention.

Note that $\rho_1(\kappa_{a_1,ca_2}) = E_{21}(4)$ and $A_2 = \rho_1(\kappa_{d,b,c,d,a})$. Since $A_4 = A_2^2$, $\hat{E}(4) \subseteq \rho_1(K_0)$.

The proof is completed by showing that $B \in \rho_1(K_0)$. Let $C = \rho_1(\kappa_{a_2,b_2,c_2,d_2})$ and $D = \rho_1(\kappa_{a_2,b_2,c_2,d_2})$. Consider $B_1 = C \cdot (D \cdot A_2)^c$. Then $B \in B_1 \hat{E}(4)$ and therefore $B \in \rho_1(K_0)$. This completes the proof.
\end{proof}

\begin{lemma}
$\mathbb{G} = \rho_2(K_0 \cap \ker \rho_1)$.
\end{lemma}

\begin{proof}
Let $N = K_0 \cap \ker \rho_1$. By Theorem 2 $\rho_2(N) \subseteq \rho_2(K \cap \ker \rho_1) \subseteq \mathbb{G}$. We obtain the other embedding by proving $E(8) \subseteq \rho_2(N)$ and $Q, Q^t, R, R^t \in \rho_2(N)$. Again we are going to use Lemma 4 and Remarks 4 and 5 without specific mention.

Note that $N$ is normalized by $S_3$ and $\rho(\kappa_{b,c,d} - \kappa_{b,c,d}) = (1, E_{12}(8))$, so that $E(8) \subseteq \rho_2(N)$. Let $b = (\kappa_{b,c,d} \cdot \kappa_{a,b,c,d})^{-1} \in N$ and $B = \rho_2(b)$. Then
\[
B \equiv \begin{pmatrix}
41 & 48 & 0 \\
25 & 16 & 1 \\
56 & 16 & 1
\end{pmatrix} \mod 64
\]
and hence $A_8 \in (B^3)^c \cdot dE(8)$. Thus $A_8 \in \rho_2(N)$ and we conclude $\hat{E}(8) \subseteq \rho_2(N)$.

Consider the following elements of $\rho_2(N)$:
\[
Q_1 = \rho_2(\kappa_{c,d,a,b,c,d,a}^{-1}), \\
Q_2 = \rho_2(\kappa_{d,b,a,b,c,d}^{-1}), \\
Q_3 = \rho_2(\kappa_{b,c,d,a,b,c,d}^{-1}).
\]

Then $R \equiv Q_1 \cdot Q_2 \mod 8$ and $R^t \equiv Q_2 \cdot Q_3 \mod 8$ and hence $R, R^t \in \rho_2(N)$. Since $Q = R^{-1}$, we have that $Q, Q^t \in \rho_2(N)$.
\end{proof}

\begin{proposition}
$B_4 = K \times B_3$.
\end{proposition}

\begin{proof}
By 2, it is enough to show that $K \subseteq B_4$. Since $K_0 \subseteq B_4 \cap K \subseteq K$ and the restriction of $\rho$ to $K$ is injective, it is enough to prove that $\rho(K) \subseteq \rho(K_0)$. By Theorem 2 any element of $\rho(K)$ is of the form $(A, T^sAG)$ with $A \in X, G \in \mathbb{G}$ and $s = 0$ if $A \in X_1$ and $s = 1$ otherwise. By Lemma 4 $A \in \rho_1(K_0)$. Thus, by Theorem 2 we have that $(A, T^sAG_1) \in \rho(K_0)$ for some $G_1 \in \mathbb{G}$. By Lemma 4 $(1, G_1)$ and $(1, G_1)$ belong to $\rho(K_0)$. Then
\[
(A, T^sAG) = (A, T^sAG_1) \cdot (1, G_1)^{-1} \cdot (1, G) \in \rho(K_0).
\]
\end{proof}
Proposition 8 contains the announced description of $B_4$. Indeed, $B_3$ is isomorphic to the congruence subgroup of level 3 of $\text{SL}_2(\mathbb{Z})$, which is free of rank 3 [3]. Moreover we have already mentioned that $\rho$ is an isomorphism between $K$ and $\rho(K)$ and the last has been described in Theorem 2.

Proof of (1). By Proposition 8, $\langle a, b \rangle \subseteq K \cap S_4 \subseteq B_4 \cap S_4$. Since the last is a normal subgroup of $S_4$, then $B_4 \cap S_4$ is either $\langle a, b \rangle$, $A_4$ or $S_4$. We prove $B_4 \cap S_4 = \langle a, b \rangle$ by proving that $B_4$ has only 2-torsion (that is, every torsion element of $B_4$ has order $\leq 2$).

By Proposition 8, $B_4 = (K \cap B_4) \rtimes B_3$. Let $b$ be a torsion element of $B_4$. Then $b = gh$ with $g \in K \cap B_4$ and $h \in B_3$. However, $h$ is a torsion element of $B_3$ and hence $h = 1$, because $B_3$ is torsionfree. Therefore $b = g$ is a torsion element of $K$. Since $K \simeq \rho(K) \subseteq \hat{E}(2)$, the order of $b = g$ is $\leq 2$. □

Final remark

After this paper was accepted, we received a note from Martin Hertweck with the following two remarks.

First $B_n \cap S_n \subseteq A_n$ because $B_n$ is embedded in the kernel of the sign representation of $S_n$. Therefore, by Theorem 1, $B_n \cap S_n = A_n$ if $n \geq 5$. This improves Theorem 1.

Second he has expressed the group element $b \in S_4$ as a product of seven bicyclics:

$$b = b(b, c^2) \cdot b(ab, bc) \cdot b(d, abc^2) \cdot b(d, bc^2) \cdot b(abd, c) \cdot b(bc^2d, abc).$$

Motivated by this we have performed an exhaustive search, using Mathematica, looking for a shorter product of bicyclic units of $\mathbb{Z}S_4$ of finite order and this search has produced the following expression of $a$ as a product of four bicyclics:

$$a = b(cd, c) \cdot b(bc^2d, abc) \cdot b(c^2d, abc^2) \cdot b(cd, ac^2).$$

REFERENCES


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