PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 131, Number 7, Pages 2247–2253
S 0002-9939(02)06754-0
Article electronically published on October 15, 2002

IMBEDDINGS OF FREE ACTIONS ON HANDLEBODIES

DARRYL MCCULLOUGH

(Communicated by Ronald A. Fintushel)

ABSTRACT. Fix a free, orientation-preserving action of a finite group G on a 3-dimensional handlebody V. Whenever G acts freely preserving orientation on a connected 3-manifold X, there is a G-equivariant imbedding of V into X. There are choices of X closed and Seifert-fibered for which the image of V is a handlebody of a Heegaard splitting of X. Provided that the genus of V is at least 2, there are similar choices with X closed and hyperbolic.

INTRODUCTION

Any finite group acts (smoothly and) freely preserving orientation on some 3-dimensional handlebody. For a given group and genus, there might be many inequivalent actions. For example, when p is an odd prime, $\mathbb{Z}/p \times \mathbb{Z}/p$ has $(p-1)/2$ inequivalent actions on the handlebody of genus p^2+1 (this is a special case of Theorem 4.1 of [6], other theorems in that paper give additional kinds of examples). Consequently, the following imbedding property of free actions may appear surprising at first glance:

Theorem 1. *Let G be a finite group acting freely and preserving orientation on two handlebodies V_1 and V_2, not necessarily of the same genus. Then there is a G-equivariant imbedding of V_1 into V_2.*

In fact, this result is almost a triviality, as is the following theorem of which it is a special case:

Theorem 2. *Let G be a finite group acting freely and preserving orientation on a handlebody V and on a connected 3-manifold X. Then there is a G-equivariant imbedding of V into X.*

By a result of D. Cooper and D. D. Long [1], any finite group acts freely on some hyperbolic rational homology 3-sphere. So Theorem 2 shows that a free G-action on a handlebody always has an extension to an action on such a 3-manifold. Also, by a result of S. Kojima [4], for any finite G there is a closed hyperbolic 3-manifold whose full isometry group is G, and Kojima's construction actually produces a free action. So there is an extension to a free action on a closed hyperbolic 3-manifold whose full isometry group is G.

Received by the editors October 9, 2001 and, in revised form, February 14, 2002.
2000 *Mathematics Subject Classification.* Primary 57M60; Secondary 57M50.
Key words and phrases. 3-manifold, handlebody, group action, free, free action, imbed, imbedding, equivariant, invariant, hyperbolic, Seifert, Heegaard, Heegaard splitting, Whitehead link.
The author was supported in part by NSF grant DMS-0102463.

One might ask for a more natural kind of extension, to a free G-action on a closed 3-manifold M, for which V is one of the handlebodies in a G-invariant Heegaard splitting of M. Simply by forming the double of V and taking an identical action on the second copy of V, one obtains such an extension with M a connected sum of $S^2 \times S^1$'s. A better question is whether V is an invariant Heegaard handlebody for a free action on an irreducible 3-manifold. Our main result answers this affirmatively.

Theorem 3. *Let G be a finite group acting freely and preserving orientation on a handlebody V. Then the action is the restriction of a free G-action on a closed irreducible 3-manifold M, which has a G-invariant Heegaard splitting with V as one of the handlebodies. One may choose M to be Seifert-fibered. Provided that V has genus greater than 1, one may choose M to be hyperbolic. In both cases, there are infinitely many choices of M.*

We remark that any orientation-preserving action of a finite group on a closed 3-manifold, free or not, has an invariant Heegaard splitting. For the quotient is a closed orientable 3-orbifold with 1-dimensional (possibly empty) singular set. One may triangulate the quotient so that the singular set is a subcomplex of the 1-skeleton. Then, the preimage of a regular neighborhood of the 1-skeleton is invariant and is one of the handlebodies in a Heegaard splitting.

In the remaining sections of this paper, we prove theorems 2 and 3. In [6], a number of results about free G-actions on handlebodies are obtained using more algebraic methods.

1. Proof of Theorem 2

Recall that two G-actions on spaces X and Y are *equivalent* if there is a homeomorphism $j\colon X \to Y$ such that $h(x) = j^{-1}(h(j(x)))$ for all $x \in X$ and all $h \in G$. If G acts properly discontinuously and freely on a path-connected space X, then the quotient map $X \to X/G$ is a regular covering map, so by the theory of covering spaces the action determines an extension

$$1 \longrightarrow \pi_1(X) \longrightarrow \pi_1(X/G) \xrightarrow{\phi} G \longrightarrow 1 \ .$$

Since we have not specified basepoints, the homomorphism ϕ is well-defined only up to an inner automorphism of G.

Suppose now that G is finite and acts freely and preserving orientation on a handlebody V. The quotient manifold V/G is orientable and irreducible with nonempty boundary, so $\pi_1(V/G)$ is torsionfree. A torsionfree finite extension of a finitely generated free group is free (by [3] any finitely generated virtually free group is the fundamental group of a graph of groups with finite vertex groups, and if the group is torsionfree, the vertex groups must be trivial). So $\pi_1(V/G)$ is free, and Theorem 5.2 of [2] shows that V/G is a handlebody. In this context, we obtain a simple algebraic criterion for equivalence.

Lemma 4. *Suppose that G acts freely and preserving orientation on handlebodies V_1 and V_2, with quotient handlebodies W_1 and W_2, determining homomorphisms $\phi_i\colon \pi_1(W_i) \to G$. The actions are equivalent if and only if there is an isomorphism $\Psi\colon \pi_1(W_1) \to \pi_1(W_2)$ for which $\phi_2 \circ \Psi = \phi_1$.*

Proof. An equivalence $j\colon V_1 \to V_2$ between the actions induces a homeomorphism $\overline{j}\colon W_1 \to W_2$ for which $\phi_2 \circ \overline{j}_\# = \phi_1$. Conversely, suppose Ψ exists. Since both W_1

and W_2 are orientable, there is a homeomorphism $f\colon W_1 \to W_2$. Using well-known constructions of homeomorphisms of W_2 (as, for example, in [5]), all of Nielsen's [7] generators of the automorphism group of the free group $\pi_1(W_2)$ can be induced by homeomorphisms, so f may be selected to induce Ψ. The condition that $\phi_2 \circ \Psi = \phi_1$ then shows that f lifts to a homeomorphism of covering spaces $j\colon V_1 \to V_2$, and moreover, ensures that $h(x) = j^{-1}(h(j(x)))$. $\square$

To illustrate Lemma 4, while establishing a fact that we shall need in the proof of Theorem 3 below, we will check that any free G-action on a solid torus $V = D^2 \times S^1$ is equivalent to a cyclic action that just rotates in the S^1 factor. The Euler characteristic shows that the quotient W must have genus 1, and the extension $1 \longrightarrow \pi_1(V) \longrightarrow \pi_1(W) \xrightarrow{\phi} G \longrightarrow 1$ shows that G is cyclic of order n. Fix a generator γ of $\pi_1(W)$. Since $V \to W$ is an n-fold covering, we may choose coordinates on V so that the covering transformation t of V determined by γ is the identity in the D^2 factor and rotates by an angle of $2\pi/n$ in the S^1 factor. Identify G abstractly with the group of covering transformations $\{1, t, t^2, \ldots, t^{n-1}\}$. The action of G by covering transformations corresponds to the extension in which ϕ takes γ to t. The G-action corresponding to the surjective homomorphism $\psi_k\colon \pi_1(W) \to G$ that takes γ to t^k corresponds to the action of G in which t rotates through $2\pi\ell/n$, where $k\ell \equiv 1 \mod n$, since the k^{th} power of t must be the covering transformation corresponding to γ. Any extension $1 \to \pi_1(V) \to \pi_1(W) \xrightarrow{\phi} G \to 1$ is equivalent to one of the extensions $1 \to \pi_1(V) \to \pi_1(W) \xrightarrow{\psi_k} G \to 1$, in the sense of Lemma 4, so the corresponding action is equivalent to one that just rotates in the S^1 factor.

Now we prove Theorem 2. Let W be the quotient handlebody of the action on V, let $Y = X/G$, and let $\phi\colon \pi_1(W) \to G$ and $\psi\colon \pi_1(Y) \to G$ be the homomorphisms determined by the actions.

There is an imbedding $k\colon W \to Y$ so that $\psi \circ k_\# = \phi$. For we can regard W as a regular neighborhood of a 1-point union K of circles, so that $\pi_1(K) = \pi_1(W)$, and construct a map k_0 of K into Y for which $\psi \circ (k_0)_\# = \phi$. Since K is 1-dimensional, k_0 is homotopic to an imbedding, and since W and Y are orientable, this imbedding extends to an imbedding k of W into Y. Since $\psi \circ k_\# = \phi$, the preimage of $k(W)$ in X is connected, and by Lemma 4 the restricted G-action on it is equivalent to the original action on V.

Theorem 2 extends to the case when some elements of G reverse the orientation. The equivariant imbedding exists if and only if the subgroups of elements of G that reverse orientation on V and on X are identical. The proof is affected only at the step when the imbedding of K into Y is extended to an imbedding of W into Y. The equality of the orientation-reversing subgroups is precisely the condition needed for the extension to exist.

2. Proof of Theorem 3

If the genus of V is 0, then G is trivial and we take $M = S^3$. Suppose that the genus of V is 1. As shown in the paragraph after the proof of Lemma 4, G is cyclic and in some product coordinates on V the action is a rotation in the S^1 factor. Regarding V as a trivially fibered solid torus in the Hopf fibering of S^3, the action extends to a free action on S^3 with V an invariant Heegaard splitting (it also extends to free actions on infinitely many lens spaces containing ∂V as a fibered Heegaard torus). So we may assume that the genus of V is greater than 1.

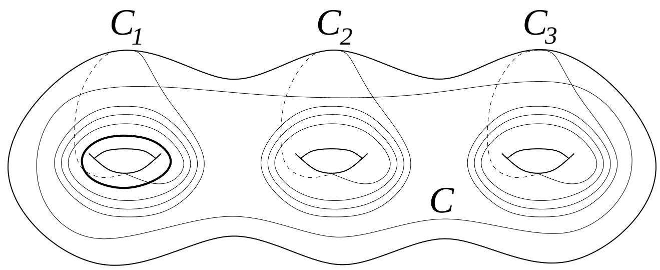

FIGURE 1. The quotient handlebody W.

The quotient handlebody $W = V/G$ has genus at least 2, since V and consequently W have negative Euler characteristic.

We first construct the Seifert-fibered extension. As in section 1, there is a homomorphism $\phi\colon \pi_1(W) \to G$ that determines the action. Let g be the genus of W, and let n be any positive integer divisible by the orders of all the elements of G. We consider a collection of simple closed curves $C_1, \ldots, C_g$ in the boundary ∂W, as shown in Figure 1 for the case when $g = 3$ and $n = 4$. Each C_i winds n times around one of the handles of W. Let C_i' be the image of C_i under the n^{th} power of a Dehn twist of ∂W about the curve C. The union of the C_i does not separate ∂W, so neither does the union of the C_i'. So we can obtain a closed 3-manifold Y with W as a Heegaard handlebody by attaching 2-handles along the C_i' and filling in the resulting 2-sphere boundary component with a 3-ball.

Let $x_1, \ldots, x_g$ be a standard set of generators of $\pi_1(W)$, where x_i is represented by a loop that goes once around the i^{th} handle. In $\pi_1(W)$, C_i represents x_i^n (up to conjugacy), and C_i' represents $x_i^n\,(x_1 \cdots x_g)^{-n}$. Since every element of G has order dividing n, it follows that ϕ carries each C_i' to the trivial element of G, so induces a homomorphism $\psi\colon \pi_1(Y) \to G$. If $k\colon W \to Y$ is the inclusion, then $\psi \circ k_\# = \phi$. The covering space M of Y has a free G-action and an invariant Heegaard splitting, one of whose handlebodies is the covering space of W corresponding to the kernel of ϕ, that is, V.

We will show that Y is Seifert-fibered, from which it follows that M is Seifert-fibered. Choose imbedded loops in the interior of W: L near and parallel to C, and $L_1, \ldots, L_g$ near and parallel to loops ℓ_i in ∂W with each ℓ_i going once around the i^{th} handle, meeting C_i in one point. The loop ℓ_1 appears in Figure 1. By a standard procedure, as explained for example on pp. 275-278 of [9], whose notation we follow, we may change the attaching curves for the discs by Dehn twists about C and the ℓ_i, at the expense of performing Dehn surgery on L and the L_i. First we twist n times along C, introducing a $1/n$ coefficient on L and moving each C_i' back to C_i. Then, $n-1$ twists along each ℓ_i move C_i to a loop C_i'' in ∂W that looks like C_i except it goes only once around the handle. This creates surgery coefficients of $-1/(n-1)$ on the L_i. We may change the attaching homeomorphism of the Heegaard splitting by any homeomorphism of ∂W that extends over W, without changing Y. In particular, we may perform left-hand twists in the meridinal 2-discs of the 2-handles to move the C_i'' to the ℓ_i. This subtracts 1 from the surgery coefficients of the L_i, and subtracts g from the coefficient of L, yielding the diagram in Figure 2. The ℓ_i are the attaching curves for the discs of a Heegaard description of S^3, so Y is obtained from S^3 by Dehn surgery using the diagram.

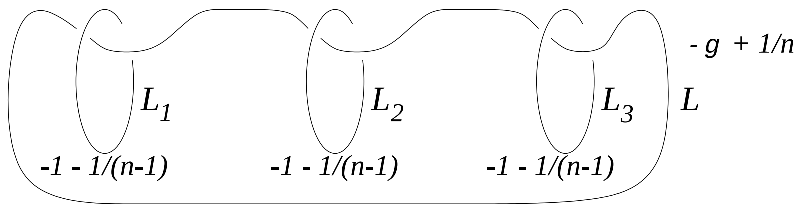

FIGURE 2. Surgery description of Y in the Seifert-fibered case.

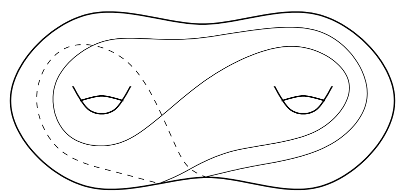

FIGURE 3. The loop C_0.

We can already see a Seifert fibering, but we will work out the exact Seifert invariants. The complement of a regular neighborhood of L is a solid torus T, for which the L_i are fibers of a product fibering. A cross-sectional surface in this fibering contained in a meridian disc of T meets the boundary of a regular neighborhood of each L_i in a meridian circle and meets the boundary of a regular neighborhood of L in the negative of the longitude, while the fiber meets them in longitude circles and a meridian circle respectively.

A surgery coefficient a/b means that a solid torus is filled in so that $am + b\ell$ becomes contractible, where m and ℓ are a meridian-longitude pair for a boundary torus of a regular neighborhood of the link component. A Seifert invariant (α, β) determines a filling in which $\alpha q + \beta t$ becomes contractible, where q is the cross-section and t is the fiber. So the surgery coefficients of our link produce one exceptional fiber with Seifert invariant $(n, 1-n)$ for each L_i, and one with Seifert invariants $(n, gn-1)$ for L. In the notation of [8], the unnormalized Seifert invariants of Y are $\{0; (o_1, 0); (n, 1-n), \ldots, (n, 1-n), (n, gn-1)\}$, where there are $g+1$ exceptional orbits. The normalized invariants are $\{-1; (o_1, 0); (n, 1), \ldots, (n, 1), (n, n-1)\}$.

Now, we will construct the extension of the G-action on V to a hyperbolic 3-manifold. As before, let n be any positive integer divisible by the orders of all the elements of G. Assume for the time being that $g = 2$. We take the same curves C_1 and C_2 as in the Seifert-fibered construction, but for C we take the image of the loop C_0 shown in Figure 3 under the homeomorphism of W which is a right-hand twist in a meridinal disc in each of the two 1-handles.

As before, we take the images of the C_i under the n^{th} power of a Dehn twist about C as the attaching curves, form the closed manifold Y, and use the induced homomorphism ψ to obtain the original G-action as an invariant Heegaard handlebody of a free G-action on a covering space M of Y. Let ℓ_1, ℓ_2, L_1, L_2 and L be as before. We choose the L_i to lie closer to the boundary of W than L. Again, change the attaching curves first by the inverse of the n Dehn twists about C, introducing a surgery coefficient of $1/n$ on L, then by the $n-1$ twists about the ℓ_i, introducing surgery coefficients $-1/(n-1)$ on the L_i. Applying left-hand twists in the meridian

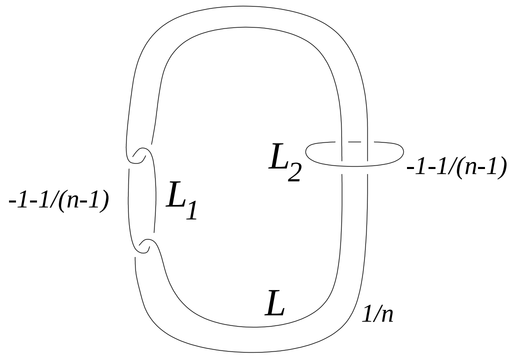

FIGURE 4. Surgery description of Y in the hyperbolic case.

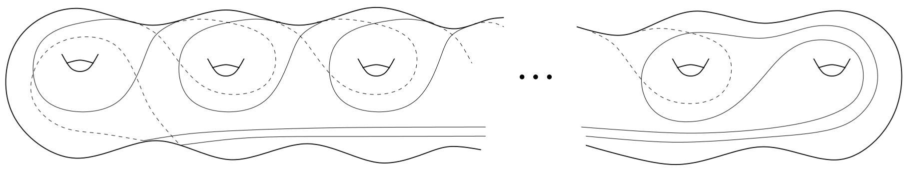

FIGURE 5. The loops C_0 for the general hyperbolic construction.

discs of the two 1-handles of W, we move the attaching curves to the ℓ_i, obtaining the surgery description of Y shown in Figure 4. This time, the coefficient of L is still $1/n$, because L has algebraic intersection 0 with each of the meridian discs of the handles of Y where the left-hand twists were performed. The complement of this link is a 2-fold covering of the complement of the Whitehead link, so is hyperbolic. By [10], Dehn surgery on the link produces a hyperbolic 3-manifold, provided that one avoids finitely many choices for the coefficients of each component. So all but finitely many choices for n yield a hyperbolic 3-manifold for Y, and hence for M.

Finally, to adapt the construction to arbitrary genus, one simply adds more components to C_0 to obtain the $g-1$ circles shown in Figure 5. In the surgery description for Y, the chain $L \cup L_1$ in Figure 4 is replaced by a chain of length $2g-2$, in which $L_1, \ldots, L_{g-1}$ alternate with components from C_0, and the component L_g links the chain as did L_2 in Figure 4. The L_i have surgery coefficients $-1-1/(n-1)$ as before, and the components coming from C_0 have coefficient $1/n$, since each had algebraic intersection 0 with the union of the meridian discs. The link complement is the $(2g-2)$-fold covering of the Whitehead link complement, so is hyperbolic, and the argument is completed as before.

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Department of Mathematics, University of Oklahoma, Norman, Oklahoma 73019
E-mail address: `dmccullough@math.ou.edu`
URL: `www.math.ou.edu/~dmccullo/`