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$\overline{\partial}$-ENERGY INTEGRAL AND HARMONIC MAPPINGS

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ABSTRACT. In this paper, we discuss harmonic mappings on the unit disk with respect to any metric by using the $\overline{\partial}$-energy integral that was first introduced by Li in 1997 to treat quasiconformal harmonic mappings on the Poincaré disk, instead of the total energy integral. Some basic properties of harmonic mappings are given. Moreover, we give a new proof of the uniqueness theorem of Marković and Mateljević, which is more explicit and natural.

§1. INTRODUCTION

Let Δ be the unit disk on the complex plane $\mathbb{C}$. Suppose that $\rho(w)|dw|^2$ is a given smooth metric on Δ. As usual, we say that a mapping $f : \Delta \to \Delta$ is a harmonic mapping with respect to the metric $\rho(w)|dw|^2$ if $f \in C^2$ satisfies the Euler-Lagrange equation

$$f_{z\overline{z}} + \partial_w(\log \rho) \circ f \; f_z f_{\overline{z}} = 0.$$

Moreover, if the Jacobian $J(f) > 0$, the Euler-Lagrange equation can be rewritten as

$$\partial_{\overline{z}}[\, \rho(f)\, f_z \overline{f}_z \,] = 0;$$

namely, $\rho(f)\; f_z\overline{f}_z$ is holomorphic on Δ. Here $\rho(f)\; f_z\overline{f}_z dz^2$, as a quadratic differential, is called the Hopf differential of f.

Usually, we use the energy integral of a mapping f, denoted by

$$E[f] := \iint_\Delta \rho(f)[|f_z|^2 + |f_{\overline{z}}|^2]dxdy,$$

to discuss the properties of harmonic mappings. Actually, a harmonic mapping f that has finite energy is a critical point of the energy functional. However, $E[f] < \infty$ at least implies that Δ has a finite area with respect to the given metric $\rho(w)|dw|^2$. If a metric $\rho(w)|dw|^2$ on Δ is complete, there is no mapping f such that $E[f] < \infty$. So when we use the energy integral, the target metric $\rho(w)|dw|^2$ is actually restricted within narrow limits.

In the present paper, instead of $E[f]$ we shall use another integral:

$$D[f] := \iint_\Delta \rho(f)|f_{\overline{z}}|^2 dxdy,$$

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known as the $\overline{\partial}$-energy integral. It is clear that for any smooth metric ρ, there is a mapping f of Δ such that $D[f] < \infty$. This means that any metric is admissible for the $\overline{\partial}$-energy integral.

It is shown in this paper that a minimizer of $D[f]$ is a harmonic mapping and a harmonic mapping whose Hopf differential has finite L^1-norm always minimizes the functional $D[f]$. Moreover, a uniqueness theorem in [MM] which is an improvement of the result in [We] is proved (see §3).

To state our main result more precisely, we need some notation. For general quasiconformal mappings theory, we refer to [Ah] and [LV]. A mapping f of Δ is called locally quasiconformal on a domain Δ if for any subdomain G with $\overline{G} \subset \Delta$, $f|_G$ is quasiconformal. Let h be an orientation preserving homeomorphism of $\partial\Delta$ onto itself. Set

$$\begin{aligned} Q[h] := \{f : f \text{ is homeomorphic on } \overline{\Delta} \text{ with } f|_{\partial\Delta} = h \text{ and} \\ f|_{\Delta} \ \text{ is locally quasiconformal}\}. \end{aligned}$$

A mapping $f_0 \in Q[h]$ is said to be a minimizer of $D[f]$, if $D[f_0] \leq D[f]$ for any $f \in Q[h]$.

Theorem 1. *Let h be an orientation preserving homeomorphism of $\partial\Delta$, and let $\rho(w)|dw|^2$ be an arbitrarily given smooth metric on Δ. If $f_0 \in Q[h]$ is a minimizer of $D[f]$, then f_0 is a harmonic map of Δ onto Δ. Conversely, if $f_0 \in Q[h]$ is a harmonic mapping and satisfies the condition*

$$\iint_\Delta \rho(f_0)|f_{0z}||\overline{f_0}_z|dxdy < \infty\,, \tag{1.1}$$

then f_0 is the unique minimizer of $D[f]$.

Remarks. When $\rho(w)|dw|^2$ is the Poincaré metric on Δ, Li Zhong [Li] first introduced the functional $D[f]$ to treat mappings harmonic with respect to $\rho(w)$. It is shown in [Li] that there is a subspace T^* of the universal Teichmüller space such that for any $h \in T^*$, $Q[h]$ contains a harmonic mapping f. Here the universal Teichmüller space is viewed as a set of normalized quasi-symmetric homeomorphisms of $\partial\Delta$.

The main idea of this paper comes from the paper of Li [Li]. The key tool of our discussion is the Main Inequality of Reich-Strebel (see [RS1] or [RS2]). The importance of the Main Inequality in the study of the extremal problem of quasiconformal mappings is well known. It is also useful in the discussion on harmonic mappings (for example, see [Re], [RS], [Li], [We], or [MM]). In this paper, all proofs of our main result are based on the Main Inequality or a new version of the Main Inequality in [MM].

§2. $\overline{\partial}$-energy functional and proof of Theorem 1

First, we state the Main Inequality of Reich and Strebel as follows:

Main Inequality. *Let $F : \Delta \to \Delta$ be a quasiconformal mapping of the unit disc Δ onto itself with the Beltrami coefficient $\mu(z)$. Suppose the boundary corresponding to F is the identity, namely $F|_{\partial\Delta} = id$. Then we have, for any holomorphic function*

ϕ on Δ with finite L^1-norm,

$$\iint_\Delta |\phi| dxdy \le \iint_\Delta \frac{\left|1-\mu\frac{\phi}{|\phi|}\right|^2}{1-|\mu|^2}|\phi| dxdy,$$

or equivalently,

$$\iint_\Delta \frac{|\mu(z)|^2|\phi(z)|}{1-|\mu(z)|^2} - Re\frac{\mu(z)\phi(z)}{1-|\mu(z)|^2} dxdy \ge 0. \tag{2.1}$$

It is shown in [MM] that when F is locally quasiconformal, the Main Inequality still holds.

Suppose that $\rho(w)|dw|^2$ is a given smooth metric on Δ. Let $h : \partial\Delta \to \partial\Delta$ be an orientation preserving homeomorphism. If h has a homeomorphic extension f to the closed disk such that $f|_\Delta$ is locally quasiconformal and $D[f] < \infty$, then we say that h is admissible for $\rho(w)|dw|^2$. For example, if h is a boundary corresponding to a quasiconformal mapping of Δ whose Beltrami coefficient vanishes in a neighborhood of the boundary, then h is admissible for any smooth metric $\rho(w)|dw|^2$.

In general case, we have the following:

Lemma 2.1. *Let $h : \partial\Delta \to \partial\Delta$ be an orientation preserving homeomorphism and let $\rho(w)|dw|^2$ be a given smooth metric on Δ. Then h is admissible for $\rho(w)|dw|^2$ if and only if there is a locally quasiconformal extension $g : \Delta \to \Delta$ of h^{-1} such that*

$$\iint_\Delta \frac{|\mu_g(w)|^2}{1-|\mu_g(w)|^2}\rho(w) dudv < \infty,$$

where μ_g is the Beltrami coefficient of g.

The proof of this lemma is a simple computation omitted here (see [Li]).

Now we want to show some lemmas on the relationship between harmonic mappings and the $\overline{\partial}$-energy functional.

Lemma 2.2. *Let h be an orientation preserving homeomorphism of $\partial\Delta$ onto itself and let $\rho(w)|dw|^2$ be a given smooth metric on Δ. Suppose that $f_0 \in Q[h]$ is harmonic on Δ and the Hopf differential of f_0 has a finite norm:*

$$\iint_\Delta \rho(f_0)|f_{0z}\overline{f_0}_z| dxdy < \infty. \tag{2.2}$$

Then for any $f \in Q[h]$, we have

$$D[f] \ge D[f_0] + \iint_\Delta \frac{|\mu_F|^2}{1-|\mu_F|^2}\rho(f_0)[|f_{0z}| - |f_{0\overline{z}}|]^2 dxdy, \tag{2.3}$$

and $D[f] = D[f_0]$ holds if and only if $f = f_0$, where $F = f^{-1} \circ f_0$ and μ_F is the Beltrami coefficient of F.

Proof. Suppose $f : \Delta \to \Delta$ is an arbitrarily given element of $Q[h]$. Without any loss of generality, we assume that $D[f] < \infty$. Let $F = f^{-1} \circ f_0$. A simple computation shows that

$$\begin{aligned} D[f] &= \iint_\Delta \rho(f_0 \circ F^{-1})|\partial_{\overline{z}}(f_0 \circ F^{-1})|^2 dxdy \\ &= \iint_\Delta \rho(f_0 \circ F^{-1}) \left| f_{0\zeta} \circ F^{-1}\partial_{\overline{z}}F^{-1} + f_{0\overline{\zeta}} \circ F^{-1}\partial_{\overline{z}}\overline{F^{-1}}\right|^2 dxdy. \end{aligned}$$

By making a transformation $z \mapsto \zeta = F(z)$ in the above integral, we get

$$D[f] = \iint_{\Delta} \rho(f_0) \frac{|f_{0\zeta}\mu_F - f_{0\overline{\zeta}}|^2}{1-|\mu_F|^2} d\xi d\eta \ (\zeta = \xi + i\eta),$$

which can be rewritten as

$$\begin{aligned} D[f] = D[f_0] &+ \iint_{\Delta} \frac{|\mu_F|^2}{1-|\mu_F|^2}\rho(f_0)\left(|f_{0z}|^2 + |f_{0\overline{z}}|^2\right)dxdy \\ &- 2Re\iint_{\Delta} \frac{\mu_F}{1-|\mu_F|^2}\rho(f_0)\overline{f_0}_z dxdy \\ &= D[f_0] + \iint_{\Delta} \frac{|\mu_F|^2}{1-|\mu_F^2}\rho(f_0)[|f_{0z}| - |f_{0\overline{z}}|]^2 dxdy \\ &+ 2\iint_{\Delta} \frac{|\mu_F|^2}{1-|\mu_F|^2}\rho(f_0)|f_{0z}\overline{f_0}_z| - Re\frac{\mu_F}{1-|\mu_F|^2}\rho(f_0)f_{0z}\overline{f_0}_z dxdy. \end{aligned} \tag{2.4}$$

Let $\phi = \rho(f_0)f_{0z}\overline{f_0}_z$. Then ϕ is holomorphic on Δ and has finite norm by condition (2.2). Note that F is a locally quasiconformal mapping and $F|_{\partial\Delta} = id$. Now we apply the new version of the Main Inequality of Reich-Strebel [MM] to the mapping F. Then we find the last term in right-hand side of (2.4) is nonnegative. Hence we get (2.3). If $D[f] = D[f_0]$ holds, then $\mu_F(z) = 0$ a.e. in Δ, which implies $f \equiv f_0$. □

Since the Hopf differential of a harmonic map does not always have finite L^1-norm, we do not know whether or not a harmonic map is always a minimizer of $D[f]$. However, the following lemma tell us that a minimizer of $D[f]$ on $Q[h]$ is always harmonic.

Lemma 2.3. *Let h be an orientation preserving homeomorphism of $\partial\Delta$ and let $\rho(w)|dw|^2$ be a given smooth metric on Δ. Suppose that $f_0 \in Q[h]$ minimizes $D[f]$ for all $f \in Q[h]$. Then f_0 is a harmonic map.*

Proof. The proof of this lemma is essentially the same as the proof of Lemma 1.1 in [RS], provided the energy integral there is replaced by our $\overline{\partial}$-energy integral $D[f]$. Following the idea of E. Reich and K. Strebel, we write the proof as follows.

Let ψ be a C^∞-function whose support is a compact subset of Δ. We look at $F_\epsilon(z) = z + \epsilon\psi(z)$. It is easy to check that f_ϵ is a quasiconformal mapping and keeps the points of $\partial\Delta$ fixed for any complex number ϵ with the condition

$$0 \le |\epsilon| < \left(\sup_{z\in\Delta}\{|\psi_z| + |\psi_{\overline{z}}|\}\right)^{-1}. \tag{2.5}$$

By a simple computation, we get

$$\begin{aligned} D[f_0 \circ F_\epsilon^{-1}] = D[f_0] &+ \iint_{\Delta} \frac{|\mu_\epsilon|^2}{1-|\mu_\epsilon|^2}\rho(f_0)\left(|f_{0z}|^2 + |f_{0\overline{z}}|^2\right)dxdy \\ &- 2Re\iint_{\Delta} \frac{\mu_\epsilon}{1-|\mu_\epsilon|^2}\phi(z)dxdy, \end{aligned} \tag{2.6}$$

where $\phi(z) = \rho(f_0(z))f_{0z}(z)\overline{f_0}_z(z)$ and μ_ϵ is the complex dilatation of F_ϵ. It is clear that

$$\mu_\epsilon = \frac{\epsilon\psi_{\overline{z}}}{1+\epsilon\psi_z}.$$

By the assumption that f_0 minimizes $D(f)$, we have $D(f_0 \circ F_\epsilon^{-1}) \geq D(f_0)$ and hence from (2.6),

$$2Re \iint_\Delta \frac{\mu_\epsilon}{1-|\mu_\epsilon|^2}\phi(z)dxdy \leq \iint_\Delta \frac{|\mu_\epsilon|^2}{1-|\mu_\epsilon|^2}\rho(f_0)\left(|f_{0z}|^2+|f_{0\overline{z}}|^2\right)dxdy.$$

Letting ϵ tend to zero along the straight line $\{t\exp(i\theta) : t \in \mathbb{R}\}$, we get

$$Re\{\exp(i\theta)\iint_\Delta \psi_{\overline{z}}(z)\phi(z)dxdy\} \leq 0. \tag{2.7}$$

Since θ is an arbitrary real number, (2.7) implies

$$\iint_\Delta \psi_{\overline{z}}(z)\phi(z)dxdy = 0 \quad \text{for any } \psi \in C_0^\infty(\Delta).$$

It follows from Weyl's Lemma that $\phi(z)$ is holomorphic in Δ. □

Proof of Theorem 1. Let f_0 be a harmonic map contained in $Q[h]$ that satisfies (1.1). Then we have

$$D[f_0] = \iint_\Delta \rho(f_0)|f_{0\overline{z}}|^2dxdy \leq \iint_\Delta \rho(f_0)|f_{0z}||f_{0\overline{z}}|dxdy < \infty.$$

Let f be an arbitrarily given element of $Q[h]$. If $D[f] = \infty$, then $D[f] > D[f_0]$. Now suppose that $D[f] < \infty$. By Lemma 2.2, we get

$$D[f] \geq D[f_0] + \iint_\Delta \frac{|\mu_F|^2\rho(f_0)[|f_{0z}|-|f_{0\overline{z}}|]^2}{1-|\mu_F|^2}dxdy \geq D[f_0].$$

So f_0 is a minimizer of $D[f]$ on $Q[h]$. The uniqueness of the minimizer of $D[f]$ also derives from Lemma 2.2. Therefore, the second part of Theorem 1 is proved.

Conversely, if f_0 is a minimizer of $D[f]$, then f_0 must be harmonic by Lemma 2.3.

Theorem 1 is proved.

§3. Application of the main result

Theorem A ([MM]). *Let f and g be two harmonic mappings of Δ onto Δ with respect to a given smooth metric $\rho(w)|dw|^2$. Suppose that they are locally quasiconformal on Δ and satisfy the following conditions:*

$$\iint_\Delta \rho(f)|f_z||f_{\overline{z}}|dxdy < \infty \quad \text{and} \quad \iint_\Delta \rho(g)|g_z||g_{\overline{z}}|dxdy < \infty. \tag{3.1}$$

If f and g can be extended to $\overline{\Delta}$ as homeomorphisms with $f|_{\partial\Delta} = g|_{\partial\Delta}$, then $f \equiv g$ on Δ.

Proof. Let f and g be two harmonic mappings of Δ onto Δ contained in $Q[h]$ satisfying condition (3.1).

By Theorem 1, both of them are minimizers for the $\overline{\partial}$-functional on the set $Q[h]$, where $h = f|_{\partial\Delta}$. So we have $D[f] = D[g]$. The uniqueness of the minimizer of $D[f]$ implies that $f = g$.

The proof of Theorem A is completed. □

Remarks. In the most known results related to the uniqueness theorem of harmonic mappings, the curvature of the given metric is assumed to be negative or non-positive curvature (see [J]). Due to Peter Li and Luen-Fai Tam ([LT]), a uniqueness theorem of quasiconformal harmonic mappings w.r.t. the Poincaré metric on Δ was proved. In [We], H. B. Wei proved a uniqueness theorem on harmonic mappings without any assumption on the curvature. However, his theorem is based on the assumption that f and g have finite total energy:

$$\iint_\Delta \rho(f)[|f_z|^2 + |f_{\overline{z}}|^2]dxdy < \infty \text{ and } \iint_\Delta \rho(g)[|g_z|^2 + |g_{\overline{z}}|^2] < \infty. \tag{3.2}$$

It is clear that condition (3.2) is stronger than (3.1). The condition, as mentioned above, restricts the given metric $\rho(w)|dw|^2$ within narrow limits. In a recent work [MM] by V. Marković and M. Mateljević, an improvement, i.e. Theorem A, is given. However, our approach to the uniqueness theorem is simpler and more explicit than that in [MM].

§4. Remark and some open problems

By making use of our approach and the Main Inequality, one can easily prove the following uniqueness theorem on harmonic mappings on Riemann surfaces:

Theorem B. *Let S_0 and S be two Riemann surfaces with the same genus that is larger than one and let $\rho(w)|dw|^2$ be an arbitrarily given smooth metric on S. Suppose that $f : S_0 \to S$ and $g : S_0 \to S$ are two quasiconformal harmonic maps and f is homotopic to g. Then $f \equiv g$.*

This uniqueness theorem was known only in the case when $\rho(w)|dw|^2$ is with nonpositive curvature on S (see [S] and [SY]). Now it holds for any metric. In Theorem 1 and Theorem A, we suppose that the Hopf differential of harmonic mappings under consideration have finite L^1-norm. We do not know what happens if we replace this condition by finite $\overline{\partial}$-energy. More precisely, the problems are as follows:

Problem 1. *Is a quasiconformal harmonic mapping f_0 of Δ onto itself with $D[f_0] < \infty$ always a minimizer of $D[f]$ in $Q[h]$? Here $h = f_0|_{\partial\Delta}$.*

Problem 2. *Suppose f and g are two quasiconformal harmonic mappings of Δ onto itself with $f|_{\partial\Delta} = g|_{\partial\Delta}$ and suppose $D[f] < \infty$ and $D[g] < \infty$. Under these conditions, can one conclude that $f \equiv g$?*

At present, we know that the answer to these two problems is affirmative if the Hopf differentials satisfy certain conditions [Ya].

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