

THE RATIONAL LS-CATEGORY OF k -TRIVIAL FIBRATIONS

MAXENCE CUVILLIEZ AND BARRY JESSUP

(Communicated by Ralph Cohen)

ABSTRACT. We provide new upper and lower bounds for the rational LS-category of a rational fibration $\xi : F \rightarrow E \rightarrow K(\mathbf{Q}, 2n)$ of simply connected spaces that depend on a measure of the triviality of ξ which is strictly finer than the vanishing of the higher holonomy actions. In particular, we prove that if ξ is k -trivial for some $k \geq 0$ and $H^*(F)$ enjoys Poincaré duality, then

$$\text{cat}_0 E \geq \text{cat}_0 F + k.$$

§1. INTRODUCTION

The Lusternik-Schnirelmann category of a space S , denoted $\text{cat } S$, is the least number of open sets, less one, which cover S and are contractible in S . It is a subtle measure of the complexity of S which is difficult to compute except where it agrees with other well-known homotopy invariants, such as $\dim S$ or the cup length in the cohomology ring. If S is simply connected and has the homotopy type of a CW complex of finite type, the *rational category of S* , $\text{cat}_0 S := \text{cat} S_{\mathbf{Q}}$, introduced by Berstein, is a lower bound for $\text{cat } S$ that is more amenable to computation because Felix and Halperin [FH] provided a complete algebraic description of $\text{cat}_0 S$ in terms of a Sullivan minimal model of S .

We now know that $\text{cat}_0(S_1 \times S_2) = \text{cat}_0 S_1 + \text{cat}_0 S_2^1$ [FHL], so a natural question is to find conditions on a non-trivial fibration $\xi : F \rightarrow E \rightarrow B$ which permit estimates of $\text{cat}_0 E$ in terms of data associated to the “twisting”. Some recent upper bounds in this spirit are provided in [CFJP], [JS] and [C2]. Lower bounds include the Mapping Theorem [FH], which in particular guarantees that $\text{cat}_0 E \geq \text{cat}_0 F$ when $F \hookrightarrow E$ induces an injection in rational homotopy. Extending this, various estimates of the form $\text{cat}_0 E \geq \text{cat}_0 F + k$ for $k > 0$ have been obtained using additional hypotheses on either the holonomy action and B [J], on the homotopy Lie algebra of E [GJ], or on the higher holonomy actions of ξ itself [C1], [C2].

In this paper we attempt to unify and extend some of these ideas by introducing a new measure of the triviality of ξ which is strictly finer than the vanishing of the higher holonomy operations.

Received by the editors October 10, 2000 and, in revised form, February 21, 2002.

2000 *Mathematics Subject Classification*. Primary 53C29, 55M30, 55P62, 55R05.

Key words and phrases. Lusternik-Schnirelmann category, holonomy, minimal model.

This research was partially supported by L’Université Catholique de Louvain-la-Neuve and by the National Science and Engineering Research Council of Canada. The second author thanks colleagues at UCL for their unstinting hospitality during a recent visit.

¹A fact *not* true for cat . See [I].

To describe this new measure of triviality, let $G_k E \rightarrow E$ denote the k^{th} Ganea fibration over E , and let $F \rtimes_E G_k E$ be the join of F and $G_k E$ over E , with induced map $F \xrightarrow{j_k} F \rtimes_E G_k E$. (Recall that the join $X \rtimes_Y Z$ of $X \rightarrow Y \leftarrow Z$ may be obtained by a pull-back followed by a push-out.) Then, we make the following

Definition. The fibration ξ is *trivial of order k* (or *k -trivial*) if j_k has a homotopy retraction, i.e., there exists a map r in the diagram

$$\begin{array}{ccc}
 F & \xrightarrow{j_k} & F \rtimes_E G_k E \cdots \xrightarrow{r} F \\
 & \searrow & \nearrow \\
 & & \text{id}_F
 \end{array}$$

making it commute up to homotopy.

Since the map j_k factorizes through j_{k-1} , k -trivial fibrations are also $(k - 1)$ -trivial. Moreover, a trivial fibration in the usual sense is trivial of arbitrarily high order.

In rational homotopy, 0-triviality of the fibration is exactly the (single) hypothesis of the Mapping Theorem. When $k = 1$, it is precisely the first hypothesis of [GJ, Thm. 1] in that particular case.

The vanishing of the k^{th} (higher) holonomy action [C2] implies, but is not equivalent to, the k -triviality of the fibration. To see this, we recall that the k^{th} holonomy action of $F \hookrightarrow E \rightarrow B$ vanishes iff

$$i : F \rightarrow G_{k+1}(E, F)$$

has a homotopy retraction (where $G_k(E, F)$ is the homotopy pullback of $E \rightarrow B$ and $G_k B \rightarrow B$), and that there is a natural map $F \rtimes_E G_k E \rightarrow G_{k+1}(E, F)$ which is compatible with i and j_k .

Recall the definition of relative LS-category in the sense of Fadell and Husseini [FaHu]. If (M, A) is an NDR pair, the relative category of (M, A) , denoted $cat(M, A)$, is the least integer n such that there exists an open covering $(M_j)_{0 \leq j \leq n}$ of M such that $M_0 \hookrightarrow M$ factorizes up to homotopy (relative to A) through A and that the open sets $M_j, j \geq 1$, are contractible in M . Moreover, $cat(M, A) \leq n + 1$ iff the map $i \rtimes g_n : A \rtimes_M G_n M \rightarrow M$ has a homotopy section s satisfying $s \circ i \simeq j_k$, where j_k is the inclusion $A \hookrightarrow A \rtimes_M G_k M$ [M]. Note that we have

$$cat M/A \leq cat(M, A) \leq cat M + 1.$$

Remark 1. Note that when $cat(E, F) \leq k + 1$ and the fibration $\xi : F \hookrightarrow E \rightarrow B$ is k -trivial, then ξ is in fact homotopically trivial, since the composition of the two homotopy retractions guaranteed by these conditions shows that $F \hookrightarrow E$ has a homotopy retraction. This has two interesting consequences:

- If ξ is a k -trivial fibration which is not homotopically trivial, then $cat(E, F) > k + 1$ and, in general, for any k -trivial fibration, we have

$$cat(E, F) > min(cat B - 1, k + 1).$$

Indeed, assume $cat(E, F) \leq k + 1$. Then the fibration ξ is homotopically trivial and

$$cat(E, F) = cat(B \times F, F) \geq cat B.$$

- Since $cat(E, F) \leq cat E + 1$, if ξ is a k -trivial fibration with $cat E \leq k$, then the fibration is homotopically trivial in the usual sense.

With $K(\mathbf{Q}, 2n)$ denoting as usual an Eilenberg-Mac Lane space and $\text{cat}_0(E, F)$ denoting $\text{cat}(E_{\mathbf{Q}}, F_{\mathbf{Q}})$, we can now state

Theorem 1. *Suppose $F \rightarrow E \rightarrow K(\mathbf{Q}, 2n)$ is a rational k -trivial fibration for some $k \geq 0$. Then:*

- $\text{cat}_0 E \leq \text{cat}_0(E, F) + \max(\text{cat}_0 F, k + 1) - k - 1$.
- *If in addition $H^*(F; \mathbf{Q})$ satisfies Poincaré duality, we have $\text{cat}_0 E \geq \text{cat}_0 F + k$.*

In the general case, we have $\text{cat } M \leq \text{cat}(M, A) + \text{cat } A$ and so the first inequality is an improvement on this bound. Similarly, the second point of Remark 1 implies that $\text{cat } E > k$ and the second inequality is an improvement on this bound.

Stated this way, the second inequality relies on the equality $\text{cat}_0 = e_0$ for Poincaré duality spaces [FHL]. Here, e_0 denotes Toomer’s invariant [T], which is the largest p such that in the spectral sequence of Milnor and Moore, $E_{\infty}^{p,*} \neq 0$. Under our assumptions, $H^*(E; \mathbf{Q})$ is also a P.d.a [FHT2, Thms. 4.3 & 3.1], so what we actually prove is that $e_0 E \geq e_0 F + k$. As already mentioned, second point of Theorem 1 is a generalization of [GJ, Thm. 1], and, once we have algebraically characterized the condition of k -triviality, it is proved in the same way.

Theorem 1 is proven using the standard methods of rational homotopy, and we remark that it is a straightforward matter to check for k -triviality when one has a minimal model of E , especially when E is an elliptic space, i.e., when $\pi_* E \otimes \mathbf{Q}$ and $H^*(E; \mathbf{Q})$ are finite dimensional.

This paper is organized as follows. In the next section, we give a characterization of our hypotheses at the level of minimal models, and we state a proposition that will imply the second point of Theorem 1. In section 3, we will prove the two points of Theorem 1. In the final section, we present 3 examples to illustrate our results.

§2. RATIONAL HOMOTOPY AND cat_0

All our spaces will be simply connected with the homotopy type of CW complexes with rational cohomology of finite type. We will work with \mathbf{Q} as ground field and our principal tools are Sullivan models. A detailed description of these and the standard tools of rational homotopy can be found in [FHT1]. For our purposes, we recall the following.

Sullivan [S] defined a contravariant functor \mathcal{A}_{PL} which associates to each space S a commutative graded differential algebra (hereafter cgda) $\mathcal{A}_{PL}(S)$ which represents the rational homotopy type of S . He also constructed, for each simply connected cgda (A, d) (i.e. satisfying $H^0(A, d) = H^1(A, d) = 0$), another cgda $(\Lambda X, d)$ and a map

$$(\Lambda X, d) \xrightarrow{\cong} (A, d)$$

which induces an isomorphism in cohomology (hereafter called a *quasi-isomorphism*), where ΛX denotes the free commutative-graded algebra on the graded vector space $X = \sum_{n \geq 2} X^n$, which has a well ordered, homogeneous basis $\{x_\alpha\}$ such that, if $X_{<\alpha}$ denotes $\text{span}\{x_\beta \mid \beta < \alpha\}$, we have $dx_\alpha \in \Lambda^{\geq 2}(X_{<\alpha})$. The cgda $(\Lambda X, d)$ is called a (*minimal*) *Sullivan model* of (A, d) or a *Sullivan model* of S if $(A, d) = \mathcal{A}_{PL}(S)$.

He also defined a *geometric realization* functor $|\cdot|$ which converts a (minimal) Sullivan model $(\Lambda X, d)$ (of finite type) into a rational space $|\Lambda X, d|$ so that $(\Lambda X, d)$

is a Sullivan model for $|(\Lambda X, d)|$. These functors define bijections

$$\left\{ \begin{array}{c} \text{rational homotopy} \\ \text{types of spaces} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{minimal Sullivan models} \end{array} \right\},$$

$$\left\{ \begin{array}{c} \text{homotopy classes of} \\ \text{maps between rational spaces} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{homotopy classes of maps} \\ \text{between minimal Sullivan models} \end{array} \right\}.$$

When two cgda's have isomorphic minimal models, we say that they are *quasi-isomorphic cgda's*, even though there may not be a quasi-isomorphism between them.

Let $\Lambda(t, dt)$ be the cgda generated as an algebra by t in degree 0, and dt in degree 1, with $d(t) = dt$ and $d(dt) = 0$, and let $\varepsilon_0, \varepsilon_1 : \Lambda(t, dt) \rightarrow (\mathbf{Q}, 0)$ denote the unique maps satisfying $\varepsilon_i(t) = it$. If $(\Lambda X, d)$ is a minimal model and (A, d_A) is any 0-connected cgda, two maps $\phi_0, \phi_1 : (\Lambda X, d) \rightarrow (A, d_A)$ are *homotopic* if there is a map

$$\Phi : (\Lambda X, d) \rightarrow (A, d_A) \otimes \Lambda(t, dt)$$

with $\phi_i = \varepsilon_i \Phi$, $i = 0, 1$.

If $\phi : (A, d) \rightarrow (B, d)$ is a morphism of 1-connected cgda's, a *Sullivan* or *relative model* of ϕ is a factoring $\phi = \psi i$ in

$$(A, d_A) \xrightarrow{i} (A \otimes \Lambda X, d) \xrightarrow{\psi} (B, d_B)$$

where $i(a) = a \otimes 1$ for $a \in A$, ψ is a quasi-isomorphism and

$$(\Lambda X, \bar{d}) := (A \otimes \Lambda X, d) / (A^+ \otimes \Lambda X, d)$$

is a minimal Sullivan model.

For every Serre fibration $\xi : F \xrightarrow{i} E \xrightarrow{p} B$ of simply connected spaces, there is a commutative diagram of augmented cgda's

$$\begin{array}{ccccc} \mathcal{A}_{PL}(B) & \xrightarrow{\mathcal{A}_{PL}(p)} & \mathcal{A}_{PL}(E) & \xrightarrow{\mathcal{A}_{PL}(i)} & \mathcal{A}_{PL}(F) \\ \simeq \uparrow \phi_B & & \simeq \uparrow & & \simeq \uparrow \\ (\Lambda X, d) & \longrightarrow & (\Lambda X \otimes \Lambda Y, d) & \longrightarrow & (\Lambda Y, \bar{d}) \end{array}$$

in which $(\Lambda X, d)$ and $(\Lambda Y, \bar{d})$ are Sullivan models for B and F respectively, and the bottom row is the Sullivan model of $\mathcal{A}_{PL}(p) \circ \phi_B$. The bottom row of this diagram is called a minimal K-S extension and a minimal model of the fibration. In general, the middle cgda need not be a minimal model of E , but will be precisely when the kernel of the homomorphism $\pi_k(F) \rightarrow \pi_k(E)$ is strictly torsion.

Now let S be a space and $(\Lambda X, d)$ a minimal model of S . The projection $\Lambda X \rightarrow \Lambda X / \Lambda^{>m} X$ induces a differential D in $\Lambda X / \Lambda^{>m} X$ which makes it a map of differential algebras. Let

$$(\Lambda X, d) \rightarrow (\Lambda X \otimes \Lambda V, d) \xrightarrow{\cong} (\Lambda X / \Lambda^{>m} X, D)$$

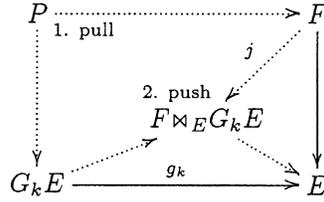
be a relative model for this projection. The *rational category* of $(\Lambda X, d)$, denoted $\text{cat}_0 S$, is the least m such that there is a map $r : (\Lambda X \otimes \Lambda V, d) \rightarrow (\Lambda X, d)$ of cgda's which satisfies $r(x) = x$ for all $x \in X$. Félix and Halperin [FH] proved that $\text{cat}_0 S = \text{cat} S_{\mathbf{Q}}$.

Toomer’s invariant, denoted e_0S , may also be defined as the least m for which there is a map $r : (\Lambda X \otimes \Lambda V, d) \rightarrow (\Lambda X, d)$ of graded differential *vector spaces* which satisfies $r(x) = x$ for all $x \in X$. Clearly, $e_0S \leq \text{cat}_0S$ and it is straightforward that

$$e_0S = \sup\{k \mid \exists \alpha \in \Lambda^{\geq k} X \text{ with } 0 \neq [\alpha] \in H^*(S; \mathbf{Q})\}.$$

Moreover, if S is a Poincaré duality space, the top class is the ‘longest’ class, that is, we may assume there exists $\alpha \in \Lambda^{\geq e_0S} X$ such that $[\alpha] \neq 0$ and $H^{>|\alpha|}(S; \mathbf{Q}) = 0$.

We are now in a position to interpret the k -triviality of the fibration on the level of minimal models. Let E and F be simply connected rational spaces and $\xi : F \rightarrow E \rightarrow K(Q, 2n)$ a rational fibration. Suppose $k \geq 0$, that $g_k : G_k E \rightarrow E$ is the k^{th} Ganea fibration of E and let $F \rtimes_E G_k E$ be the join of g_k and the inclusion $i : F \rightarrow E$. That is, if P denotes the homotopy pullback of g_k and i , then $F \rtimes_E G_k E$ is the homotopy pushout of the projection $P \rightarrow F$ and $P \rightarrow G_k E$. Let $i \rtimes g_k : F \rtimes_E G_k E \rightarrow E$ be the induced map as in the diagram



where we have denoted by j the inclusion $F \rightarrow F \rtimes_E G_k E$.

Proposition 2. *The fibration ξ is k -trivial iff there exists a minimal model*

$$\Lambda(a; 0) \rightarrow \Lambda(a, X; d) \rightarrow \Lambda(X; \bar{d})$$

of ξ with

$$d : X \rightarrow \Lambda X \oplus \Lambda^+ a. \Lambda^{>k}(a, X).$$

We remark [C2] that the vanishing of the k^{th} -higher holonomy is equivalent, on this level, to the existence of a model of ξ as above with

$$d : X \rightarrow \Lambda X \oplus \Lambda^{>k+1} a. \Lambda(a, X),$$

and an example exhibiting that this is stronger than the above is presented in the last section.

The proof of Proposition 2 will follow directly from Lemmas 3–6. However, before proceeding with them, we indicate why 1-triviality of the fibration is equivalent to the hypotheses of [GJ, Theorem 1].

If $(\Lambda X, d)$ is the minimal model of S , then as graded vector spaces, $L_S^n := \text{Hom}(X^n, \mathbf{Q}) \cong \pi_n(S) \otimes \mathbf{Q}$. Then, $L_S^n \cong \pi_n(\Omega S) \otimes \mathbf{Q}$, and the homotopy Lie algebra of $S_{\mathbf{Q}}$ is encoded in the minimal model $(\Lambda X, d)$ as follows.

The differential d can be written as a sum of derivations $d = d_2 + d_3 + \dots$ where $d_i : X \rightarrow \Lambda^i X$. The fact that $d^2 = 0$ implies the same for d_2 . The dual of $d_2 : X \rightarrow \Lambda^2 X$ induces a bilinear and antisymmetric $[\cdot, \cdot] : L_S \otimes L_S \rightarrow L_S$, which represents the Samelson product in $\pi(\Omega S) \otimes \mathbf{Q}$. The Jacobi identity for $[\cdot, \cdot]$ is equivalent to $d_2^2 = 0$. One calls $(L_S, [\cdot, \cdot])$ the *rational homotopy Lie algebra of S* .

In particular, if $\{x_\alpha\}$ is a K-S basis for X , and $\{\hat{x}_\alpha\}$ is its dual basis, then

1. $[\hat{x}_\alpha, \hat{x}_\beta] = 0$ iff the coefficient of $x_\alpha x_\beta$ in $d_2 x_\gamma$ is zero for all γ , so that

2. \hat{x}_α is in the centre of L_S iff

$$d_2 : X \rightarrow \Lambda^2 \langle x_\beta \mid \beta \neq \alpha \rangle.$$

When no confusion will arise, we shall say that x_α belongs to the centre of L_S if \hat{x}_α does.

Thus, a rational fibration $F \hookrightarrow E \rightarrow K(\mathbf{Q}, 2n)$ is k -trivial iff the fundamental homotopy class of the base, when viewed in L_E , is in the center of L_E .

We can now proceed with the proof of Proposition 2.

Lemma 3. *Let*

$$\begin{array}{ccc} & (A, d) & \\ & \nearrow f & \downarrow p \\ (\Lambda V, \bar{d}) & \xrightarrow{g} & (B, d) \end{array}$$

be a commutative diagram with p surjective, and suppose that g' is an another map homotopic to g . Then there exists a map $f' : (\Lambda V, \bar{d}) \rightarrow (A, d)$, homotopic to f satisfying $p \circ f' = g'$.

The proof of this lemma is a standard lifting argument.

Lemma 4. *Let $\Lambda(a; 0) \rightarrow \Lambda(a, X; d) \xrightarrow{p} \Lambda(X; \bar{d})$ be any minimal model of ξ and suppose that W_k is a geometric realization of $(\Lambda(a, X)/\Lambda^{>k}(a, X), D)$. Then,*

$$(\Lambda(a, X)/a \cdot \Lambda^{>k}(a, X), D)$$

is a model of $F \rtimes_E W_k$.

Proof. Consider a relative model of $(\Lambda(a, X), d) \xrightarrow{q} (\Lambda(a, X)/\Lambda^{>k}(a, X), D)$

$$(1) \quad \begin{array}{ccc} (\Lambda(a, X), d) & \xrightarrow{q} & (\Lambda(a, X)/\Lambda^{>k}(a, X), D) \\ & \searrow j & \uparrow \simeq \varphi \\ & & (\Lambda(a, X) \otimes \Lambda Y, \delta) \end{array}$$

and let (A, d) be the join of $\Lambda(X; \bar{d})$ and $(\Lambda(a, X) \otimes \Lambda Y, \delta)$ as in the diagram

$$\begin{array}{ccc} (\Lambda(a, X), d) & \xrightarrow{j} & (\Lambda(a, X) \otimes \Lambda Y, \delta) \\ \downarrow p & \searrow & \downarrow \text{1. push } Y \\ & (A, d) & \\ & \swarrow \text{2. pull} & \downarrow \text{1. push } Y \\ \Lambda(X; \bar{d}) & \dashrightarrow & (B, d) \end{array}$$

Thus, $(\Lambda X \otimes \Lambda Y, \bar{\delta}) := (\Lambda X, \bar{d}) \otimes_{(\Lambda(a, X), d)} (\Lambda(a, X) \otimes \Lambda Y, \delta) = (B, d)$ is a model of the pull back of p and j , and

$$(A, d) = (\Lambda X \oplus_{\Lambda X \otimes \Lambda Y} \Lambda(a, X) \otimes \Lambda Y, \bar{d} + \delta),$$

which is isomorphic to the differential subspace $(\Lambda X \oplus (a \Lambda(a, X) \otimes \Lambda Y), \delta)$ of $(\Lambda(a, X) \otimes \Lambda Y, \delta)$. On the other hand,

$$(\Lambda(a, X)/a \Lambda^{>k}(a, X), D) = (\Lambda X \oplus a(\Lambda(a, X)/\Lambda^{>k}(a, X)), D)$$

is a differential subspace of $(\Lambda(a, X)/\Lambda^{>k}(a, X), D)$. Now, the surjective quasi-isomorphism φ in diagram (1) above induces a surjective map of differential algebras

$$\begin{aligned} \psi &:= (\text{id}_{\Lambda X} \oplus a\varphi) : (\Lambda X \oplus (a\Lambda(a, X) \otimes \Lambda Y), \delta) \\ &\rightarrow (\Lambda X \oplus a(\Lambda(a, X)/\Lambda^{>k}(a, X)), D) \end{aligned}$$

whose kernel is isomorphic to $a \cdot \ker \varphi$, so that $H^*(\ker \psi) = aH^*(\ker \varphi) = 0$. Thus, ψ is a quasi-isomorphism. Since the functor \mathcal{A}_{PL} and the geometric realization functor interchange homotopy pushouts and homotopy pullbacks, $(\Lambda(a, X)/\Lambda^{>k}(a, X), D)$ is indeed a model of $F \bowtie_E W_k$. \square

To proceed, we retain the notation introduced in Lemma 4, and as usual, identify x with $q(x)$ for all $x \in X$. Then, we can see that the condition on the differential d is equivalent to $DX \subset \Lambda X$ in $(\Lambda(a, X)/\Lambda^{>k}(a, X), D)$. We next establish

Lemma 5. *Let $\Lambda(a; 0) \rightarrow \Lambda(a, V; d) \xrightarrow{p} \Lambda(V; \bar{d})$ be any model of ξ . There exists a minimal model $(\Lambda(a, X), d)$ of E with*

$$d : X \rightarrow \Lambda X \oplus \Lambda^+ a \cdot \Lambda^{>k}(a, X)$$

iff there exists a map σ in the diagram

$$\begin{array}{ccc} \Lambda(V; \bar{d}) & \xrightarrow{\sigma} & (\Lambda(a, V)/a\Lambda^{>k}(a, V), D) & \xrightarrow{\tilde{p}} & \Lambda(V; \bar{d}) \\ & \searrow & \text{id}_{\Lambda V} & \nearrow & \\ & & & & \end{array}$$

making it commute up to homotopy. (Here, \tilde{p} is the map induced by p .)

Proof. Suppose the condition on d is satisfied. By the remark preceding Lemma 5, if we define $\sigma(x) = q(x)$, σ will be a map of differential algebras making the diagram above (with $V = X$) commute exactly.

Conversely, suppose we have such a σ . Because \tilde{p} is surjective and $\Lambda(V; \bar{d})$ is minimal, we may suppose without loss of generality (by Lemma 3) that $\tilde{p}\sigma = \text{id}_{\Lambda V}$. We will use σ to make a change of basis as follows. Our assumptions imply that there is a subspace $X \subset \Lambda(a, V)$ such that

$$q : X \rightarrow \sigma(V)$$

and

$$\tilde{p}q : X \rightarrow V$$

are isomorphisms. This implies that $\Lambda(a, X) = \Lambda(a, V)$, and so

$$(\Lambda(a, V)/\Lambda^{>k}(a, V), D) = (\Lambda(a, X)/\Lambda^{>k}(a, X), D).$$

But then,

$$DX = D\sigma(V) = \rho\bar{d}(V) \subset \sigma(\Lambda V) = \Lambda\sigma(V) = \Lambda X,$$

so we are done. \square

We now relate $F \bowtie_E W_k$ and $F \bowtie_E G_k E$. As before, let $g_k : G_k E \rightarrow E$ denote the k^{th} Ganea fibration over E .

Lemma 6. *There exists a retraction r in the diagram*

$$\begin{array}{ccc}
 F & \longrightarrow & F \rtimes_E G_k E \xrightarrow{r} F \\
 & \searrow & \nearrow \\
 & & \text{id}_F
 \end{array}$$

making it commute up to homotopy iff there exists a retraction s in the diagram

$$\begin{array}{ccc}
 F & \longrightarrow & F \rtimes_E W_k \xrightarrow{s} F \\
 & \searrow & \nearrow \\
 & & \text{id}_F
 \end{array}$$

rendering it homotopy commutative.

Proof. Denote by $w_k : W_k \rightarrow E$ a geometric realization of the morphism $\Lambda(a, X) \rightarrow \Lambda(a, X)/\Lambda^{>k}(a, X)$ and by j (resp. by \bar{j}) the inclusion of F in $F \rtimes_E G_k E$ (resp. $F \rtimes_E W_k$). Recall [ST] that there exist maps $\beta : G_k E \rightarrow W_k$ and $\alpha : W_k \rightarrow G_k E$ such that $g_k \circ \alpha \simeq w_k$ and $w_k \circ \beta \simeq g_k$. These induce maps $\bar{\alpha} : F \rtimes_E G_k E \rightarrow F \rtimes_E W_k$ and $\bar{\beta} : F \rtimes_E W_k \rightarrow G_k E$ such that $\bar{j} \circ \bar{\alpha} \simeq j$ and $j \circ \bar{\alpha} \simeq \bar{j}$. Thus, j has a homotopy retraction if and only if \bar{j} does. \square

Proof of Proposition 2. This is immediate from Lemmas 4, 5 and 6. \square

§3. PROOF OF THE THEOREM

With the results and notation of the previous section, the second point of Theorem 1 now follows directly from

Proposition 7. *Suppose the degree of a is even, $\Lambda(a; 0) \rightarrow \Lambda(a, X; d) \rightarrow \Lambda(X; \bar{d})$ is a minimal K - S extension and that $H^* \Lambda(X; \bar{d})$ is a Poincaré duality algebra. Then, if $k \geq 0$ and $d : X \rightarrow \Lambda X \oplus \Lambda^+ a. \Lambda^{>k}(a, X)$, we have*

$$\text{cat}_0 \Lambda(a, X; d) \geq \text{cat}_0 \Lambda(X; \bar{d}) + k.$$

Proof. Though this closely follows the proof of [GJ, Thm. 1], we present it here for the sake of completeness.

Since the fibre is a Poincaré duality space and the base is Gorenstein, by [FHT2, Thm. 4.3], the total space is also Gorenstein. Moreover, since we may assume that $\text{cat}_0 \Lambda(a, X; d) < \infty$, by Theorem 3.1 of the same article, we conclude that $H^* \Lambda(a, X; \bar{d})$ is a Poincaré duality algebra. Then, by Propositions 5.1 and 5.3, again of [FHT2], the formal dimension of the fibre is strictly greater than that of the total space.

Now note that we can write $d = \bar{d} + \sum_i a^i \eta_i$, where each η_i is a derivation of ΛX of odd degree. The assumption $d : X \rightarrow \Lambda X \oplus \Lambda^+ a. \Lambda^{>k}(a, X)$ then implies that $\eta_i : \Lambda^p X \rightarrow \Lambda^{\geq \max\{p-1, p+k+1-i\}} X$. In particular, if $\beta \in \Lambda^{\geq p} X$ satisfies $\bar{d}\beta = 0$, then $d\beta = a\gamma$, where $\gamma \in \Lambda^{\geq p+k}(a, X)$.

Denote $e_0 \Lambda(X; \bar{d}) = e$, and let $\beta \in \Lambda^{\geq e} X$ be a cycle of degree equal to the formal dimension, representing the longest non-zero class. Since $\bar{d}\beta = 0$, $d\beta = a\alpha$, where $d\alpha = 0$ and $\alpha \in \Lambda^{\geq e+k}(a, X)$. Suppose that α is exact in $\Lambda(a, X; d)$, say $\alpha = d\gamma$. Then, $\beta - a\gamma$ is a d -cycle of degree greater than the formal dimension of $\Lambda(a, X; d)$, so $\beta - a\gamma = dz$ for some $z = z_0 + az_1$, where $z_0 \in \Lambda X$. Comparison of coefficients of powers of a on both sides of $\beta - a\gamma = dz$ yields $\bar{d}z_0 = \beta$, which is a contradiction. Thus, the class of α is nonzero in $H^* \Lambda(a, X; d)$, and so $e_0 \Lambda(a, X; d) \geq e + k$. \square

It is hoped that one can eventually remove the assumption of Poincaré duality in the above. For the moment however, we content ourselves by remarking that Theorem 1 is *not* valid for a fibration over an *odd* Eilenberg-Mac Lane space (i.e. an odd rational sphere)

$$F \rightarrow E \rightarrow \mathbf{S}_{\mathbf{Q}}^{2n+1}.$$

Consider the fibration $\Lambda(u; 0) \rightarrow \Lambda(u, v, w, x, y; d) \rightarrow \Lambda(v, w, x, y; 0)$ where all generators are of odd degree and the only non-zero differential is $dy = uvwx$. This satisfies all the hypotheses of Theorem 2 with $k = 2$ except for the parity, but

$$\text{cat}_0\Lambda(u, v, w, x, y; d) = 5 < 6 = \text{cat}_0\Lambda(v, w, x, y; 0) + 2.$$

We now proceed with the proof of the first part of Theorem 1. Let (M, A) be an NDR pair, denote by i the inclusion $A \hookrightarrow M$, and as before let $g_k : G_k M \rightarrow M$ denote the k^{th} Ganea fibration over M . Recall that for a map $f : S \rightarrow M$, $\text{cat} f \leq n$ iff f factors through g_n iff there is an open cover $(S_l)_{1 \leq l \leq n+1}$ of S such that for each l , the composition $S_l \hookrightarrow S \xrightarrow{f} M$ is homotopic to the constant map.

The first part of Theorem 1 now follows immediately from

Proposition 8. *Suppose $F \rightarrow E \rightarrow K(\mathbf{Q}, 2n)$ is a fibration with model $\Lambda(a; 0) \rightarrow \Lambda(a, X; d) \rightarrow \Lambda(X; \bar{d})$. If $k \geq 0$ and $d : X \rightarrow \Lambda X \oplus \Lambda^+ a \cdot \Lambda^{>k}(a, X)$, then*

$$\text{cat}_0 E \leq \text{cat}_0(E, F) + \max(\text{cat}_0 F, k + 1) - k - 1.$$

Proof. Suppose that $\text{cat}(E, F) = n + 1$. By [M], E is dominated by $F \rtimes_E G_n E$ and so E is dominated by $F \rtimes_E G_n E \simeq F \rtimes_E G_k E \rtimes_E G_{n-k-1} E$ [C2]. Using the map β used in the proof of Lemma 6, we also see that E is dominated by $F \rtimes_E W_k \rtimes_E G_{n-k-1} E$. Hence,

$$\begin{aligned} \text{cat} E &\leq \text{cat}(F \rtimes_E W_k \rtimes_E G_{n-k-1} E) \\ &\leq \text{cat}(F \rtimes_E W_k) + \text{cat} G_{n-k-1} E + 1 \\ &\leq \text{cat}(F \rtimes_E W_k) + n - k. \end{aligned}$$

The above inequalities also apply to the \mathbf{Q} -localizations of all the spaces, so we may replace cat by cat_0 . Now suppose that $\text{cat}_0 F = m$, so that $\Lambda(X; \bar{d})$ is a homotopy retract of $\Lambda X / \Lambda^{>m} X$. The condition on the differential implies that $A := (\Lambda X \oplus a \cdot (\Lambda(a, X) / \Lambda^{>k}(a, X)), D)$ is a homotopy retract of

$$B := (\Lambda X / \Lambda^{>m} X \oplus a \cdot (\Lambda(a, X) / \Lambda^{>k}(a, X)), D).$$

Since the nilpotency of B is at most $\max(m, k + 1)$, its rational category also has the same upper bound [FH]. Moreover, by Lemma 4, A is a model of $F \rtimes_E W_k$, so we obtain $\text{cat}_0(F \rtimes_E W_k) \leq \max(\text{cat}_0 F, k + 1)$, which completes the proof. \square

Note that this proof does not use the relativity of the homotopy in the definition of $\text{cat}(E, F)$.

§4. EXAMPLES

Example 1. Consider the rational fibration $(\mathbf{CP}^2)^3 \times (\mathbf{S}^7)^2 \rightarrow E \rightarrow \mathbf{CP}^\infty$ with model

$$(3) \quad (\Lambda a, 0) \rightarrow (\Lambda(a, b, c, e, u, v, w, x, y), d) \rightarrow (\Lambda(b, c, e, u, v, w, x, y), \bar{d}),$$

where a, b, c and e are cycles of degree 2, $du = a^4$, $dv = b^3$, $dw = c^3$, $dx = e^3$, and $dy = abce$. The fibration is 2-trivial (while even the 0-holonomy does not

vanish), so the second point of Theorem 1 implies that $\text{cat}_0 E \geq 10$. The best lower bound that the Mapping Theorem or [C1] can provide is 8, while [J] and [GJ] can both be massaged to yield 9. In fact, if we apply [FH, lemma 6.6] to the fibration $\mathbf{S}^7 \rightarrow E \rightarrow \mathbf{CP}^3 \times (\mathbf{CP}^2)^3$, we obtain $\text{cat}_0 E \leq 10$, so indeed $\text{cat}_0 E = 10$ and the lower bound of Theorem 1 is sharp here.

Theorem 1 applied to (3) tells us that $\text{cat}_0(E, (\mathbf{CP}^2)^3 \times (\mathbf{S}^7)^2) \geq 5$.

Example 2. Consider the rational fibration $\mathbf{S}^7 \times \mathbf{CP}^2 \rightarrow E \rightarrow \mathbf{CP}^\infty$ with model $(\Lambda a, 0) \rightarrow (\Lambda(a, b, x, y), d) \rightarrow (\Lambda(b, x, y), \bar{d})$ where a and b are cycles of degree 2 and $dx = b^3$ and $dy = a^4 + b^2 a^2$. Again, the fibration is 2-trivial and so Theorem 1 gives $\text{cat}_0 E \geq 5$. In fact, $\text{cat}_0 E = 5$, because a simple length-degree argument shows that $e_0 E = 5$. Theorem 1 implies also that $\text{cat}_0(E, \mathbf{S}^7 \times \mathbf{CP}^2) \geq 5$.

Example 3 . Consider the rational fibration $\mathbf{S}^2 \times \mathbf{S}^7 \times \mathbf{S}^7 \rightarrow E \rightarrow \mathbf{CP}^\infty$ with Sullivan model given by $(\Lambda b, 0) \rightarrow (\Lambda(a, b, x, y), d) \rightarrow (\Lambda(a, x, y), \bar{d})$ where $|a| = |b| = 2$, $|x| = 3$, $|y| = |z| = 7$, $da = db = 0$, $dx = a^2$, $dy = b^4$ and $dz = a^3 b$. Theorem 1 implies that $\text{cat}_0 E \geq 5$ and that $\text{cat}_0(E, S^2 \times S^2 \times S^5) \geq 2$. If we apply [C2] to the fibration $S^5 \rightarrow E \rightarrow CP(2) \times S^2$ with model $(\Lambda b, a, x, y, d) \rightarrow (\Lambda(a, b, x, y), d) \rightarrow (\Lambda z, 0)$, we get $\text{cat}_0 E \leq 5$.

The authors would like to thank Yves Félix for many helpful discussions. The paper was also much improved as a result of the referee's comments, and we thank the referee for his/her help.

REFERENCES

- [C1] Maxence Cuvilliez, *Properties of semi-trivial fibrations*, preprint of the Centre de Recherche mathématique (Barcelona) **468** (2001).
- [C2] ———, *L-S category of the total space in a fibration and k-monomorphisms*, *Topology and its Applications* **109** (2001), 127–146. MR **2001m**:55008
- [CFJP] Maxence Cuvilliez, Yves Félix, Barry Jessup & Paul-Eugène Parent, *Rational Lusternik-Schnirelmann category of fibrations*, to appear in J.P.A.A.
- [FH] Y. Félix and S. Halperin, *Rational L.-S. category and its applications*, *Trans. Amer. Math. Soc.* **237** (1982), 1–37. MR **84h**:55011
- [FHL] Y. Félix, S. Halperin and Jean-Michel Lemaire, *The rational LS category of products and Poincaré duality complexes*, *Topology* **37**, No. 4 (1998), 749–756. MR **99a**:55003
- [FHT1] Yves Félix, Stephen Halperin and Jean-Claude Thomas, *Rational Homotopy Theory*, Springer-Verlag, GTM. MR **2002d**:55014
- [FHT2] ———, *Gorenstein Spaces*, *Advances in Mathematics* **71** (1) (1988), 92–112. MR **89k**:55019
- [GJ] Sonia Ghorbal and Barry Jessup, *Estimating the rational LS-category of elliptic spaces*, *Proc. Amer. Math. Soc.* **129** (2001), 1833–1842. MR **2001k**:55032
- [FaHu] E. Fadell, S. Y. Husseini, *Relative category, product and coproducts*, *Rend. Sem. Mat. Fis. Milano* **64** (1994), 99–115. MR **97f**:55003
- [I] N. Iwase, *Ganea's conjecture on Lusternik-Schnirelmann category*, *Bull. London Math. Soc.* **30** (1998), 623–634. MR **99j**:55003
- [J] Barry Jessup, *Holonomy-Nilpotent Fibrations and Rational Lusternik-Schnirelmann Category*, *Topology* **34** (4) (1995), 759–770. MR **96k**:55014
- [JS] I.M. James and W. Singhoff, *On the category of fibre bundles, Lie groups and Frobenius maps*, *Contemp. Math.* **227** (1996), 177–189. MR **2000a**:55007
- [M] Pierre-Marie Moyaux, *Lower bounds for relative Lusternik-Schnirelmann Category*, *Manuscripta Math.* **101** (2000), 533–542. MR **2001f**:55006
- [S] D. Sullivan, *Infinitesimal computations in topology*, *Inst. Hautes Études Sci. Publ. Math.* **47** (1978), 269–331. MR **58**:31119

- [ST] Hans Scheerer and Daniel Tanré, *Fibrations à la Ganea (French)*, Bull. Belg. Math. Soc. **4** (3) (1997), 333–353. MR **98g**:55003
- [T] G.H. Toomer, *Lusternik-Schnirelmann category and the Moore spectral sequence*, Math. Z. **138** (1974), 123–143. MR **50**:8509

CENTRE DE RECERCA MATEMÀTICA, BARCELONA, SPAIN
E-mail address: mcuvilli@crm.es

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF OTTAWA, OTTAWA, ONTARIO,
CANADA K1N 6N5
E-mail address: bjessup@uottawa.ca