

## THE RATIONAL LS-CATEGORY OF $k$ -TRIVIAL FIBRATIONS

MAXENCE CUVILLIEZ AND BARRY JESSUP

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ABSTRACT. We provide new upper and lower bounds for the rational LS-category of a rational fibration  $\xi : F \rightarrow E \rightarrow K(\mathbf{Q}, 2n)$  of simply connected spaces that depend on a measure of the triviality of  $\xi$  which is strictly finer than the vanishing of the higher holonomy actions. In particular, we prove that if  $\xi$  is  $k$ -trivial for some  $k \geq 0$  and  $H^*(F)$  enjoys Poincaré duality, then

$$\text{cat}_0 E \geq \text{cat}_0 F + k.$$

### §1. INTRODUCTION

The Lusternik-Schnirelmann category of a space  $S$ , denoted  $\text{cat } S$ , is the least number of open sets, less one, which cover  $S$  and are contractible in  $S$ . It is a subtle measure of the complexity of  $S$  which is difficult to compute except where it agrees with other well-known homotopy invariants, such as  $\dim S$  or the cup length in the cohomology ring. If  $S$  is simply connected and has the homotopy type of a CW complex of finite type, the *rational category of  $S$* ,  $\text{cat}_0 S := \text{cat} S_{\mathbf{Q}}$ , introduced by Berstein, is a lower bound for  $\text{cat } S$  that is more amenable to computation because Felix and Halperin [FH] provided a complete algebraic description of  $\text{cat}_0 S$  in terms of a Sullivan minimal model of  $S$ .

We now know that  $\text{cat}_0(S_1 \times S_2) = \text{cat}_0 S_1 + \text{cat}_0 S_2^1$  [FHL], so a natural question is to find conditions on a non-trivial fibration  $\xi : F \rightarrow E \rightarrow B$  which permit estimates of  $\text{cat}_0 E$  in terms of data associated to the “twisting”. Some recent upper bounds in this spirit are provided in [CFJP], [JS] and [C2]. Lower bounds include the Mapping Theorem [FH], which in particular guarantees that  $\text{cat}_0 E \geq \text{cat}_0 F$  when  $F \hookrightarrow E$  induces an injection in rational homotopy. Extending this, various estimates of the form  $\text{cat}_0 E \geq \text{cat}_0 F + k$  for  $k > 0$  have been obtained using additional hypotheses on either the holonomy action and  $B$  [J], on the homotopy Lie algebra of  $E$  [GJ], or on the higher holonomy actions of  $\xi$  itself [C1], [C2].

In this paper we attempt to unify and extend some of these ideas by introducing a new measure of the triviality of  $\xi$  which is strictly finer than the vanishing of the higher holonomy operations.

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<sup>1</sup>A fact *not* true for  $\text{cat}$ . See [I].

To describe this new measure of triviality, let  $G_k E \rightarrow E$  denote the  $k^{th}$  Ganea fibration over  $E$ , and let  $F \rtimes_E G_k E$  be the join of  $F$  and  $G_k E$  over  $E$ , with induced map  $F \xrightarrow{j_k} F \rtimes_E G_k E$ . (Recall that the join  $X \rtimes_Y Z$  of  $X \rightarrow Y \leftarrow Z$  may be obtained by a pull-back followed by a push-out.) Then, we make the following

**Definition.** The fibration  $\xi$  is *trivial of order  $k$*  (or  *$k$ -trivial*) if  $j_k$  has a homotopy retraction, i.e., there exists a map  $r$  in the diagram

$$\begin{array}{ccc}
 F & \xrightarrow{j_k} & F \rtimes_E G_k E \cdots \cdots \xrightarrow{r} F \\
 & \searrow & \nearrow \\
 & & \text{id}_F
 \end{array}$$

making it commute up to homotopy.

Since the map  $j_k$  factorizes through  $j_{k-1}$ ,  $k$ -trivial fibrations are also  $(k - 1)$ -trivial. Moreover, a trivial fibration in the usual sense is trivial of arbitrarily high order.

In rational homotopy, 0-triviality of the fibration is exactly the (single) hypothesis of the Mapping Theorem. When  $k = 1$ , it is precisely the first hypothesis of [GJ, Thm. 1] in that particular case.

The vanishing of the  $k^{th}$  (higher) holonomy action [C2] implies, but is not equivalent to, the  $k$ -triviality of the fibration. To see this, we recall that the  $k^{th}$  holonomy action of  $F \hookrightarrow E \rightarrow B$  vanishes iff

$$i : F \rightarrow G_{k+1}(E, F)$$

has a homotopy retraction (where  $G_k(E, F)$  is the homotopy pullback of  $E \rightarrow B$  and  $G_k B \rightarrow B$ ), and that there is a natural map  $F \rtimes_E G_k E \rightarrow G_{k+1}(E, F)$  which is compatible with  $i$  and  $j_k$ .

Recall the definition of relative LS-category in the sense of Fadell and Husseini [FaHu]. If  $(M, A)$  is an NDR pair, the relative category of  $(M, A)$ , denoted  $cat(M, A)$ , is the least integer  $n$  such that there exists an open covering  $(M_j)_{0 \leq j \leq n}$  of  $M$  such that  $M_0 \hookrightarrow M$  factorizes up to homotopy (relative to  $A$ ) through  $A$  and that the open sets  $M_j, j \geq 1$ , are contractible in  $M$ . Moreover,  $cat(M, A) \leq n + 1$  iff the map  $i \rtimes g_n : A \rtimes_M G_n M \rightarrow M$  has a homotopy section  $s$  satisfying  $s \circ i \simeq j_k$ , where  $j_k$  is the inclusion  $A \hookrightarrow A \rtimes_M G_k M$  [M]. Note that we have

$$cat M/A \leq cat(M, A) \leq cat M + 1.$$

*Remark 1.* Note that when  $cat(E, F) \leq k + 1$  and the fibration  $\xi : F \hookrightarrow E \rightarrow B$  is  $k$ -trivial, then  $\xi$  is in fact homotopically trivial, since the composition of the two homotopy retractions guaranteed by these conditions shows that  $F \hookrightarrow E$  has a homotopy retraction. This has two interesting consequences:

- If  $\xi$  is a  $k$ -trivial fibration which is not homotopically trivial, then  $cat(E, F) > k + 1$  and, in general, for any  $k$ -trivial fibration, we have

$$cat(E, F) > min(cat B - 1, k + 1).$$

Indeed, assume  $cat(E, F) \leq k + 1$ . Then the fibration  $\xi$  is homotopically trivial and

$$cat(E, F) = cat(B \times F, F) \geq cat B.$$

- Since  $cat(E, F) \leq cat E + 1$ , if  $\xi$  is a  $k$ -trivial fibration with  $cat E \leq k$ , then the fibration is homotopically trivial in the usual sense.

With  $K(\mathbf{Q}, 2n)$  denoting as usual an Eilenberg-Mac Lane space and  $\text{cat}_0(E, F)$  denoting  $\text{cat}(E_{\mathbf{Q}}, F_{\mathbf{Q}})$ , we can now state

**Theorem 1.** *Suppose  $F \rightarrow E \rightarrow K(\mathbf{Q}, 2n)$  is a rational  $k$ -trivial fibration for some  $k \geq 0$ . Then:*

- $\text{cat}_0 E \leq \text{cat}_0(E, F) + \max(\text{cat}_0 F, k + 1) - k - 1$ .
- *If in addition  $H^*(F; \mathbf{Q})$  satisfies Poincaré duality, we have  $\text{cat}_0 E \geq \text{cat}_0 F + k$ .*

In the general case, we have  $\text{cat } M \leq \text{cat}(M, A) + \text{cat } A$  and so the first inequality is an improvement on this bound. Similarly, the second point of Remark 1 implies that  $\text{cat } E > k$  and the second inequality is an improvement on this bound.

Stated this way, the second inequality relies on the equality  $\text{cat}_0 = e_0$  for Poincaré duality spaces [FHL]. Here,  $e_0$  denotes Toomer’s invariant [T], which is the largest  $p$  such that in the spectral sequence of Milnor and Moore,  $E_{\infty}^{p,*} \neq 0$ . Under our assumptions,  $H^*(E; \mathbf{Q})$  is also a P.d.a [FHT2, Thms. 4.3 & 3.1], so what we actually prove is that  $e_0 E \geq e_0 F + k$ . As already mentioned, second point of Theorem 1 is a generalization of [GJ, Thm. 1], and, once we have algebraically characterized the condition of  $k$ -triviality, it is proved in the same way.

Theorem 1 is proven using the standard methods of rational homotopy, and we remark that it is a straightforward matter to check for  $k$ -triviality when one has a minimal model of  $E$ , especially when  $E$  is an elliptic space, i.e., when  $\pi_* E \otimes \mathbf{Q}$  and  $H^*(E; \mathbf{Q})$  are finite dimensional.

This paper is organized as follows. In the next section, we give a characterization of our hypotheses at the level of minimal models, and we state a proposition that will imply the second point of Theorem 1. In section 3, we will prove the two points of Theorem 1. In the final section, we present 3 examples to illustrate our results.

§2. RATIONAL HOMOTOPY AND  $\text{cat}_0$

All our spaces will be simply connected with the homotopy type of CW complexes with rational cohomology of finite type. We will work with  $\mathbf{Q}$  as ground field and our principal tools are Sullivan models. A detailed description of these and the standard tools of rational homotopy can be found in [FHT1]. For our purposes, we recall the following.

Sullivan [S] defined a contravariant functor  $\mathcal{A}_{PL}$  which associates to each space  $S$  a commutative graded differential algebra (hereafter cgda)  $\mathcal{A}_{PL}(S)$  which represents the rational homotopy type of  $S$ . He also constructed, for each simply connected cgda  $(A, d)$  (i.e. satisfying  $H^0(A, d) = H^1(A, d) = 0$ ), another cgda  $(\Lambda X, d)$  and a map

$$(\Lambda X, d) \xrightarrow{\cong} (A, d)$$

which induces an isomorphism in cohomology (hereafter called a *quasi-isomorphism*), where  $\Lambda X$  denotes the free commutative-graded algebra on the graded vector space  $X = \sum_{n \geq 2} X^n$ , which has a well ordered, homogeneous basis  $\{x_\alpha\}$  such that, if  $X_{<\alpha}$  denotes  $\text{span}\{x_\beta \mid \beta < \alpha\}$ , we have  $dx_\alpha \in \Lambda^{\geq 2}(X_{<\alpha})$ . The cgda  $(\Lambda X, d)$  is called a (*minimal*) *Sullivan model* of  $(A, d)$  or a *Sullivan model* of  $S$  if  $(A, d) = \mathcal{A}_{PL}(S)$ .

He also defined a *geometric realization* functor  $|\cdot|$  which converts a (minimal) Sullivan model  $(\Lambda X, d)$  (of finite type) into a rational space  $|\Lambda X, d|$  so that  $(\Lambda X, d)$

is a Sullivan model for  $|(\Lambda X, d)|$ . These functors define bijections

$$\left\{ \begin{array}{l} \text{rational homotopy} \\ \text{types of spaces} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{isomorphism classes of} \\ \text{minimal Sullivan models} \end{array} \right\},$$

$$\left\{ \begin{array}{l} \text{homotopy classes of} \\ \text{maps between rational spaces} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{homotopy classes of maps} \\ \text{between minimal Sullivan models} \end{array} \right\}.$$

When two cgda’s have isomorphic minimal models, we say that they are *quasi-isomorphic cgda’s*, even though there may not be a quasi-isomorphism between them.

Let  $\Lambda(t, dt)$  be the cgda generated as an algebra by  $t$  in degree 0, and  $dt$  in degree 1, with  $d(t) = dt$  and  $d(dt) = 0$ , and let  $\varepsilon_0, \varepsilon_1 : \Lambda(t, dt) \rightarrow (\mathbf{Q}, 0)$  denote the unique maps satisfying  $\varepsilon_i(t) = it$ . If  $(\Lambda X, d)$  is a minimal model and  $(A, d_A)$  is any 0-connected cgda, two maps  $\phi_0, \phi_1 : (\Lambda X, d) \rightarrow (A, d_A)$  are *homotopic* if there is a map

$$\Phi : (\Lambda X, d) \rightarrow (A, d_A) \otimes \Lambda(t, dt)$$

with  $\phi_i = \varepsilon_i \Phi$ ,  $i = 0, 1$ .

If  $\phi : (A, d) \rightarrow (B, d)$  is a morphism of 1-connected cgda’s, a *Sullivan* or *relative model* of  $\phi$  is a factoring  $\phi = \psi i$  in

$$(A, d_A) \xrightarrow{i} (A \otimes \Lambda X, d) \xrightarrow{\psi} (B, d_B)$$

where  $i(a) = a \otimes 1$  for  $a \in A$ ,  $\psi$  is a quasi-isomorphism and

$$(\Lambda X, \bar{d}) := (A \otimes \Lambda X, d) / (A^+ \otimes \Lambda X, d)$$

is a minimal Sullivan model.

For every Serre fibration  $\xi : F \xrightarrow{i} E \xrightarrow{p} B$  of simply connected spaces, there is a commutative diagram of augmented cgda’s

$$\begin{array}{ccccc} \mathcal{A}_{PL}(B) & \xrightarrow{\mathcal{A}_{PL}(p)} & \mathcal{A}_{PL}(E) & \xrightarrow{\mathcal{A}_{PL}(i)} & \mathcal{A}_{PL}(F) \\ \simeq \uparrow \phi_B & & \simeq \uparrow & & \simeq \uparrow \\ (\Lambda X, d) & \longrightarrow & (\Lambda X \otimes \Lambda Y, d) & \longrightarrow & (\Lambda Y, \bar{d}) \end{array}$$

in which  $(\Lambda X, d)$  and  $(\Lambda Y, \bar{d})$  are Sullivan models for  $B$  and  $F$  respectively, and the bottom row is the Sullivan model of  $\mathcal{A}_{PL}(p) \circ \phi_B$ . The bottom row of this diagram is called a minimal K-S extension and a minimal model of the fibration. In general, the middle cgda need not be a minimal model of  $E$ , but will be precisely when the kernel of the homomorphism  $\pi_k(F) \rightarrow \pi_k(E)$  is strictly torsion.

Now let  $S$  be a space and  $(\Lambda X, d)$  a minimal model of  $S$ . The projection  $\Lambda X \rightarrow \Lambda X / \Lambda^{>m} X$  induces a differential  $D$  in  $\Lambda X / \Lambda^{>m} X$  which makes it a map of differential algebras. Let

$$(\Lambda X, d) \rightarrow (\Lambda X \otimes \Lambda V, d) \xrightarrow{\cong} (\Lambda X / \Lambda^{>m} X, D)$$

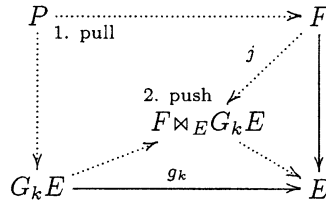
be a relative model for this projection. The *rational category* of  $(\Lambda X, d)$ , denoted  $\text{cat}_0 S$ , is the least  $m$  such that there is a map  $r : (\Lambda X \otimes \Lambda V, d) \rightarrow (\Lambda X, d)$  of cgda’s which satisfies  $r(x) = x$  for all  $x \in X$ . Félix and Halperin [FH] proved that  $\text{cat}_0 S = \text{cat} S_{\mathbf{Q}}$ .

Toomer’s invariant, denoted  $e_0S$ , may also be defined as the least  $m$  for which there is a map  $r : (\Lambda X \otimes \Lambda V, d) \rightarrow (\Lambda X, d)$  of graded differential *vector spaces* which satisfies  $r(x) = x$  for all  $x \in X$ . Clearly,  $e_0S \leq \text{cat}_0S$  and it is straightforward that

$$e_0S = \sup\{k \mid \exists \alpha \in \Lambda^{\geq k} X \text{ with } 0 \neq [\alpha] \in H^*(S; \mathbf{Q})\}.$$

Moreover, if  $S$  is a Poincaré duality space, the top class is the ‘longest’ class, that is, we may assume there exists  $\alpha \in \Lambda^{\geq e_0S} X$  such that  $[\alpha] \neq 0$  and  $H^{>|\alpha|}(S; \mathbf{Q}) = 0$ .

We are now in a position to interpret the  $k$ -triviality of the fibration on the level of minimal models. Let  $E$  and  $F$  be simply connected rational spaces and  $\xi : F \rightarrow E \rightarrow K(Q, 2n)$  a rational fibration. Suppose  $k \geq 0$ , that  $g_k : G_k E \rightarrow E$  is the  $k^{\text{th}}$  Ganea fibration of  $E$  and let  $F \rtimes_E G_k E$  be the join of  $g_k$  and the inclusion  $i : F \rightarrow E$ . That is, if  $P$  denotes the homotopy pullback of  $g_k$  and  $i$ , then  $F \rtimes_E G_k E$  is the homotopy pushout of the projection  $P \rightarrow F$  and  $P \rightarrow G_k E$ . Let  $i \rtimes g_k : F \rtimes_E G_k E \rightarrow E$  be the induced map as in the diagram



where we have denoted by  $j$  the inclusion  $F \rightarrow F \rtimes_E G_k E$ .

**Proposition 2.** *The fibration  $\xi$  is  $k$ -trivial iff there exists a minimal model*

$$\Lambda(a; 0) \rightarrow \Lambda(a, X; d) \rightarrow \Lambda(X; \bar{d})$$

of  $\xi$  with

$$d : X \rightarrow \Lambda X \oplus \Lambda^+ a. \Lambda^{>k}(a, X).$$

We remark [C2] that the vanishing of the  $k^{\text{th}}$ -higher holonomy is equivalent, on this level, to the existence of a model of  $\xi$  as above with

$$d : X \rightarrow \Lambda X \oplus \Lambda^{>k+1} a. \Lambda(a, X),$$

and an example exhibiting that this is stronger than the above is presented in the last section.

The proof of Proposition 2 will follow directly from Lemmas 3–6. However, before proceeding with them, we indicate why 1-triviality of the fibration is equivalent to the hypotheses of [GJ, Theorem 1].

If  $(\Lambda X, d)$  is the minimal model of  $S$ , then as graded vector spaces,  $L_S^n := \text{Hom}(X^n, \mathbf{Q}) \cong \pi_n(S) \otimes \mathbf{Q}$ . Then,  $L_S^n \cong \pi_n(\Omega S) \otimes \mathbf{Q}$ , and the homotopy Lie algebra of  $S_{\mathbf{Q}}$  is encoded in the minimal model  $(\Lambda X, d)$  as follows.

The differential  $d$  can be written as a sum of derivations  $d = d_2 + d_3 + \dots$  where  $d_i : X \rightarrow \Lambda^i X$ . The fact that  $d^2 = 0$  implies the same for  $d_2$ . The dual of  $d_2 : X \rightarrow \Lambda^2 X$  induces a bilinear and antisymmetric  $[\cdot, \cdot] : L_S \otimes L_S \rightarrow L_S$ , which represents the Samelson product in  $\pi(\Omega S) \otimes \mathbf{Q}$ . The Jacobi identity for  $[\cdot, \cdot]$  is equivalent to  $d_2^2 = 0$ . One calls  $(L_S, [\cdot, \cdot])$  the *rational homotopy Lie algebra of  $S$* .

In particular, if  $\{x_\alpha\}$  is a K-S basis for  $X$ , and  $\{\hat{x}_\alpha\}$  is its dual basis, then

1.  $[\hat{x}_\alpha, \hat{x}_\beta] = 0$  iff the coefficient of  $x_\alpha x_\beta$  in  $d_2 x_\gamma$  is zero for all  $\gamma$ , so that

2.  $\hat{x}_\alpha$  is in the centre of  $L_S$  iff

$$d_2 : X \rightarrow \Lambda^2 \langle x_\beta \mid \beta \neq \alpha \rangle.$$

When no confusion will arise, we shall say that  $x_\alpha$  belongs to the centre of  $L_S$  if  $\hat{x}_\alpha$  does.

Thus, a rational fibration  $F \hookrightarrow E \rightarrow K(\mathbf{Q}, 2n)$  is  $k$ -trivial iff the fundamental homotopy class of the base, when viewed in  $L_E$ , is in the center of  $L_E$ .

We can now proceed with the proof of Proposition 2.

**Lemma 3.** *Let*

$$\begin{array}{ccc} & (A, d) & \\ & \nearrow f & \downarrow p \\ (\Lambda V, \bar{d}) & \xrightarrow{g} & (B, d) \end{array}$$

be a commutative diagram with  $p$  surjective, and suppose that  $g'$  is an another map homotopic to  $g$ . Then there exists a map  $f' : (\Lambda V, \bar{d}) \rightarrow (A, d)$ , homotopic to  $f$  satisfying  $p \circ f' = g'$ .

The proof of this lemma is a standard lifting argument.

**Lemma 4.** *Let  $\Lambda(a; 0) \rightarrow \Lambda(a, X; d) \xrightarrow{p} \Lambda(X; \bar{d})$  be any minimal model of  $\xi$  and suppose that  $W_k$  is a geometric realization of  $(\Lambda(a, X)/\Lambda^{>k}(a, X), D)$ . Then,*

$$(\Lambda(a, X)/a \cdot \Lambda^{>k}(a, X), D)$$

is a model of  $F \rtimes_E W_k$ .

*Proof.* Consider a relative model of  $(\Lambda(a, X), d) \xrightarrow{q} (\Lambda(a, X)/\Lambda^{>k}(a, X), D)$

$$(1) \quad \begin{array}{ccc} (\Lambda(a, X), d) & \xrightarrow{q} & (\Lambda(a, X)/\Lambda^{>k}(a, X), D) \\ & \searrow j & \uparrow \simeq \varphi \\ & & (\Lambda(a, X) \otimes \Lambda Y, \delta) \end{array}$$

and let  $(A, d)$  be the join of  $\Lambda(X; \bar{d})$  and  $(\Lambda(a, X) \otimes \Lambda Y, \delta)$  as in the diagram

$$\begin{array}{ccc} (\Lambda(a, X), d) & \xrightarrow{j} & (\Lambda(a, X) \otimes \Lambda Y, \delta) \\ \downarrow p & \searrow & \downarrow \text{1. push } Y \\ & (A, d) & \\ & \swarrow \text{2. pull} & \downarrow \text{1. push } Y \\ \Lambda(X; \bar{d}) & \dashrightarrow & (B, d) \end{array}$$

Thus,  $(\Lambda X \otimes \Lambda Y, \bar{\delta}) := (\Lambda X, \bar{d}) \otimes_{(\Lambda(a, X), d)} (\Lambda(a, X) \otimes \Lambda Y, \delta) = (B, d)$  is a model of the pull back of  $p$  and  $j$ , and

$$(A, d) = (\Lambda X \oplus_{\Lambda X \otimes \Lambda Y} \Lambda(a, X) \otimes \Lambda Y, \bar{d} + \delta),$$

which is isomorphic to the differential subspace  $(\Lambda X \oplus (a \Lambda(a, X) \otimes \Lambda Y), \delta)$  of  $(\Lambda(a, X) \otimes \Lambda Y, \delta)$ . On the other hand,

$$(\Lambda(a, X)/a \Lambda^{>k}(a, X), D) = (\Lambda X \oplus a(\Lambda(a, X)/\Lambda^{>k}(a, X)), D)$$

is a differential subspace of  $(\Lambda(a, X)/\Lambda^{>k}(a, X), D)$ . Now, the surjective quasi-isomorphism  $\varphi$  in diagram (1) above induces a surjective map of differential algebras

$$\begin{aligned} \psi &:= (\text{id}_{\Lambda X} \oplus a\varphi) : (\Lambda X \oplus (a\Lambda(a, X) \otimes \Lambda Y), \delta) \\ &\rightarrow (\Lambda X \oplus a(\Lambda(a, X)/\Lambda^{>k}(a, X)), D) \end{aligned}$$

whose kernel isomorphic to  $a \cdot \ker \varphi$ , so that  $H^*(\ker \psi) = aH^*(\ker \varphi) = 0$ . Thus,  $\psi$  is a quasi-isomorphism. Since the functor  $\mathcal{A}_{PL}$  and the geometric realization functor interchange homotopy pushouts and homotopy pullbacks,  $(\Lambda(a, X)/a\Lambda^{>k}(a, X), D)$  is indeed a model of  $F \bowtie_E W_k$ .  $\square$

To proceed, we retain the notation introduced in Lemma 4, and as usual, identify  $x$  with  $q(x)$  for all  $x \in X$ . Then, we can see that the condition on the differential  $d$  is equivalent to  $DX \subset \Lambda X$  in  $(\Lambda(a, X)/\Lambda^{>k}(a, X), D)$ . We next establish

**Lemma 5.** *Let  $\Lambda(a; 0) \rightarrow \Lambda(a, V; d) \xrightarrow{p} \Lambda(V; \bar{d})$  be any model of  $\xi$ . There exists a minimal model  $(\Lambda(a, X), d)$  of  $E$  with*

$$d : X \rightarrow \Lambda X \oplus \Lambda^+ a \cdot \Lambda^{>k}(a, X)$$

*iff there exists a map  $\sigma$  in the diagram*

$$\begin{array}{ccc} \Lambda(V; \bar{d}) & \xrightarrow{\sigma} & (\Lambda(a, V)/a\Lambda^{>k}(a, V), D) & \xrightarrow{\tilde{p}} & \Lambda(V; \bar{d}) \\ & \searrow & \text{id}_{\Lambda V} & \nearrow & \\ & & & & \end{array}$$

*making it commute up to homotopy. (Here,  $\tilde{p}$  is the map induced by  $p$ .)*

*Proof.* Suppose the condition on  $d$  is satisfied. By the remark preceding Lemma 5, if we define  $\sigma(x) = q(x)$ ,  $\sigma$  will be a map of differential algebras making the diagram above (with  $V = X$ ) commute exactly.

Conversely, suppose we have such a  $\sigma$ . Because  $\tilde{p}$  is surjective and  $\Lambda(V; \bar{d})$  is minimal, we may suppose without loss of generality (by Lemma 3) that  $\tilde{p}\sigma = \text{id}_{\Lambda V}$ . We will use  $\sigma$  to make a change of basis as follows. Our assumptions imply that there is a subspace  $X \subset \Lambda(a, V)$  such that

$$q : X \rightarrow \sigma(V)$$

and

$$\tilde{p}q : X \rightarrow V$$

are isomorphisms. This implies that  $\Lambda(a, X) = \Lambda(a, V)$ , and so

$$(\Lambda(a, V)/\Lambda^{>k}(a, V), D) = (\Lambda(a, X)/\Lambda^{>k}(a, X), D).$$

But then,

$$DX = D\sigma(V) = \rho\bar{d}(V) \subset \sigma(\Lambda V) = \Lambda\sigma(V) = \Lambda X,$$

so we are done.  $\square$

We now relate  $F \bowtie_E W_k$  and  $F \bowtie_E G_k E$ . As before, let  $g_k : G_k E \rightarrow E$  denote the  $k^{\text{th}}$  Ganea fibration over  $E$ .

**Lemma 6.** *There exists a retraction  $r$  in the diagram*

$$\begin{array}{ccc}
 F & \longrightarrow & F \rtimes_E G_k E \xrightarrow{r} F \\
 & \searrow & \uparrow \\
 & & \text{id}_F
 \end{array}$$

*making it commute up to homotopy iff there exists a retraction  $s$  in the diagram*

$$\begin{array}{ccc}
 F & \longrightarrow & F \rtimes_E W_k \xrightarrow{s} F \\
 & \searrow & \uparrow \\
 & & \text{id}_F
 \end{array}$$

*rendering it homotopy commutative.*

*Proof.* Denote by  $w_k : W_k \rightarrow E$  a geometric realization of the morphism  $\Lambda(a, X) \rightarrow \Lambda(a, X)/\Lambda^{>k}(a, X)$  and by  $j$  (resp. by  $\bar{j}$ ) the inclusion of  $F$  in  $F \rtimes_E G_k E$  (resp.  $F \rtimes_E W_k$ ). Recall [ST] that there exist maps  $\beta : G_k E \rightarrow W_k$  and  $\alpha : W_k \rightarrow G_k E$  such that  $g_k \circ \alpha \simeq w_k$  and  $w_k \circ \beta \simeq g_k$ . These induce maps  $\bar{\alpha} : F \rtimes_E G_k E \rightarrow F \rtimes_E W_k$  and  $\bar{\beta} : F \rtimes_E W_k \rightarrow G_k E$  such that  $\bar{j} \circ \bar{\alpha} \simeq j$  and  $j \circ \bar{\alpha} \simeq \bar{j}$ . Thus,  $j$  has a homotopy retraction if and only if  $\bar{j}$  does.  $\square$

*Proof of Proposition 2.* This is immediate from Lemmas 4, 5 and 6.  $\square$

§3. PROOF OF THE THEOREM

With the results and notation of the previous section, the second point of Theorem 1 now follows directly from

**Proposition 7.** *Suppose the degree of  $a$  is even,  $\Lambda(a; 0) \rightarrow \Lambda(a, X; d) \rightarrow \Lambda(X; \bar{d})$  is a minimal  $K$ - $S$  extension and that  $H^* \Lambda(X; \bar{d})$  is a Poincaré duality algebra. Then, if  $k \geq 0$  and  $d : X \rightarrow \Lambda X \oplus \Lambda^+ a. \Lambda^{>k}(a, X)$ , we have*

$$\text{cat}_0 \Lambda(a, X; d) \geq \text{cat}_0 \Lambda(X; \bar{d}) + k.$$

*Proof.* Though this closely follows the proof of [GJ, Thm. 1], we present it here for the sake of completeness.

Since the fibre is a Poincaré duality space and the base is Gorenstein, by [FHT2, Thm. 4.3], the total space is also Gorenstein. Moreover, since we may assume that  $\text{cat}_0 \Lambda(a, X; d) < \infty$ , by Theorem 3.1 of the same article, we conclude that  $H^* \Lambda(a, X; \bar{d})$  is a Poincaré duality algebra. Then, by Propositions 5.1 and 5.3, again of [FHT2], the formal dimension of the fibre is strictly greater than that of the total space.

Now note that we can write  $d = \bar{d} + \sum_i a^i \eta_i$ , where each  $\eta_i$  is a derivation of  $\Lambda X$  of odd degree. The assumption  $d : X \rightarrow \Lambda X \oplus \Lambda^+ a. \Lambda^{>k}(a, X)$  then implies that  $\eta_i : \Lambda^p X \rightarrow \Lambda^{\geq \max\{p-1, p+k+1-i\}} X$ . In particular, if  $\beta \in \Lambda^{\geq p} X$  satisfies  $\bar{d}\beta = 0$ , then  $d\beta = a\gamma$ , where  $\gamma \in \Lambda^{\geq p+k}(a, X)$ .

Denote  $e_0 \Lambda(X; \bar{d}) = e$ , and let  $\beta \in \Lambda^{\geq e} X$  be a cycle of degree equal to the formal dimension, representing the longest non-zero class. Since  $\bar{d}\beta = 0$ ,  $d\beta = a\alpha$ , where  $d\alpha = 0$  and  $\alpha \in \Lambda^{\geq e+k}(a, X)$ . Suppose that  $\alpha$  is exact in  $\Lambda(a, X; d)$ , say  $\alpha = d\gamma$ . Then,  $\beta - a\gamma$  is a  $d$ -cycle of degree greater than the formal dimension of  $\Lambda(a, X; d)$ , so  $\beta - a\gamma = dz$  for some  $z = z_0 + az_1$ , where  $z_0 \in \Lambda X$ . Comparison of coefficients of powers of  $a$  on both sides of  $\beta - a\gamma = dz$  yields  $\bar{d}z_0 = \beta$ , which is a contradiction. Thus, the class of  $\alpha$  is nonzero in  $H^* \Lambda(a, X; d)$ , and so  $e_0 \Lambda(a, X; d) \geq e + k$ .  $\square$



It is hoped that one can eventually remove the assumption of Poincaré duality in the above. For the moment however, we content ourselves by remarking that Theorem 1 is *not* valid for a fibration over an *odd* Eilenberg-Mac Lane space (i.e. an odd rational sphere)

$$F \rightarrow E \rightarrow \mathbf{S}_{\mathbf{Q}}^{2n+1}.$$

Consider the fibration  $\Lambda(u; 0) \rightarrow \Lambda(u, v, w, x, y; d) \rightarrow \Lambda(v, w, x, y; 0)$  where all generators are of odd degree and the only non-zero differential is  $dy = uvwx$ . This satisfies all the hypotheses of Theorem 2 with  $k = 2$  except for the parity, but

$$\text{cat}_0\Lambda(u, v, w, x, y; d) = 5 < 6 = \text{cat}_0\Lambda(v, w, x, y; 0) + 2.$$

We now proceed with the proof of the first part of Theorem 1. Let  $(M, A)$  be an NDR pair, denote by  $i$  the inclusion  $A \hookrightarrow M$ , and as before let  $g_k : G_k M \rightarrow M$  denote the  $k^{\text{th}}$  Ganea fibration over  $M$ . Recall that for a map  $f : S \rightarrow M$ ,  $\text{cat} f \leq n$  iff  $f$  factors through  $g_n$  iff there is an open cover  $(S_l)_{1 \leq l \leq n+1}$  of  $S$  such that for each  $l$ , the composition  $S_l \hookrightarrow S \xrightarrow{f} M$  is homotopic to the constant map.

The first part of Theorem 1 now follows immediately from

**Proposition 8.** *Suppose  $F \rightarrow E \rightarrow K(\mathbf{Q}, 2n)$  is a fibration with model  $\Lambda(a; 0) \rightarrow \Lambda(a, X; d) \rightarrow \Lambda(X; \bar{d})$ . If  $k \geq 0$  and  $d : X \rightarrow \Lambda X \oplus \Lambda^+ a \cdot \Lambda^{>k}(a, X)$ , then*

$$\text{cat}_0 E \leq \text{cat}_0(E, F) + \max(\text{cat}_0 F, k + 1) - k - 1.$$

*Proof.* Suppose that  $\text{cat}(E, F) = n + 1$ . By [M],  $E$  is dominated by  $F \rtimes_E G_n E$  and so  $E$  is dominated by  $F \rtimes_E G_n E \simeq F \rtimes_E G_k E \rtimes_E G_{n-k-1} E$  [C2]. Using the map  $\beta$  used in the proof of Lemma 6, we also see that  $E$  is dominated by  $F \rtimes_E W_k \rtimes_E G_{n-k-1} E$ . Hence,

$$\begin{aligned} \text{cat} E &\leq \text{cat}(F \rtimes_E W_k \rtimes_E G_{n-k-1} E) \\ &\leq \text{cat}(F \rtimes_E W_k) + \text{cat} G_{n-k-1} E + 1 \\ &\leq \text{cat}(F \rtimes_E W_k) + n - k. \end{aligned}$$

The above inequalities also apply to the  $\mathbf{Q}$ -localizations of all the spaces, so we may replace  $\text{cat}$  by  $\text{cat}_0$ . Now suppose that  $\text{cat}_0 F = m$ , so that  $\Lambda(X; \bar{d})$  is a homotopy retract of  $\Lambda X / \Lambda^{>m} X$ . The condition on the differential implies that  $A := (\Lambda X \oplus a \cdot (\Lambda(a, X) / \Lambda^{>k}(a, X)), D)$  is a homotopy retract of

$$B := (\Lambda X / \Lambda^{>m} X \oplus a \cdot (\Lambda(a, X) / \Lambda^{>k}(a, X)), D).$$

Since the nilpotency of  $B$  is at most  $\max(m, k + 1)$ , its rational category also has the same upper bound [FH]. Moreover, by Lemma 4,  $A$  is a model of  $F \rtimes_E W_k$ , so we obtain  $\text{cat}_0(F \rtimes_E W_k) \leq \max(\text{cat}_0 F, k + 1)$ , which completes the proof.  $\square$

Note that this proof does not use the relativity of the homotopy in the definition of  $\text{cat}(E, F)$ .

#### §4. EXAMPLES

**Example 1.** Consider the rational fibration  $(\mathbf{CP}^2)^3 \times (\mathbf{S}^7)^2 \rightarrow E \rightarrow \mathbf{CP}^\infty$  with model

$$(3) \quad (\Lambda a, 0) \rightarrow (\Lambda(a, b, c, e, u, v, w, x, y), d) \rightarrow (\Lambda(b, c, e, u, v, w, x, y), \bar{d}),$$

where  $a, b, c$  and  $e$  are cycles of degree 2,  $du = a^4$ ,  $dv = b^3$ ,  $dw = c^3$ ,  $dx = e^3$ , and  $dy = abce$ . The fibration is 2-trivial (while even the 0-holonomy does not

vanish), so the second point of Theorem 1 implies that  $\text{cat}_0 E \geq 10$ . The best lower bound that the Mapping Theorem or [C1] can provide is 8, while [J] and [GJ] can both be massaged to yield 9. In fact, if we apply [FH, lemma 6.6] to the fibration  $\mathbf{S}^7 \rightarrow E \rightarrow \mathbf{CP}^3 \times (\mathbf{CP}^2)^3$ , we obtain  $\text{cat}_0 E \leq 10$ , so indeed  $\text{cat}_0 E = 10$  and the lower bound of Theorem 1 is sharp here.

Theorem 1 applied to (3) tells us that  $\text{cat}_0(E, (\mathbf{CP}^2)^3 \times (\mathbf{S}^7)^2) \geq 5$ .

**Example 2.** Consider the rational fibration  $\mathbf{S}^7 \times \mathbf{CP}^2 \rightarrow E \rightarrow \mathbf{CP}^\infty$  with model  $(\Lambda a, 0) \rightarrow (\Lambda(a, b, x, y), d) \rightarrow (\Lambda(b, x, y), \bar{d})$  where  $a$  and  $b$  are cycles of degree 2 and  $dx = b^3$  and  $dy = a^4 + b^2 a^2$ . Again, the fibration is 2-trivial and so Theorem 1 gives  $\text{cat}_0 E \geq 5$ . In fact,  $\text{cat}_0 E = 5$ , because a simple length-degree argument shows that  $e_0 E = 5$ . Theorem 1 implies also that  $\text{cat}_0(E, \mathbf{S}^7 \times \mathbf{CP}^2) \geq 5$ .

**Example 3 .** Consider the rational fibration  $\mathbf{S}^2 \times \mathbf{S}^7 \times \mathbf{S}^7 \rightarrow E \rightarrow \mathbf{CP}^\infty$  with Sullivan model given by  $(\Lambda b, 0) \rightarrow (\Lambda(a, b, x, y), d) \rightarrow (\Lambda(a, x, y), \bar{d})$  where  $|a| = |b| = 2$ ,  $|x| = 3$ ,  $|y| = |z| = 7$ ,  $da = db = 0$ ,  $dx = a^2$ ,  $dy = b^4$  and  $dz = a^3 b$ . Theorem 1 implies that  $\text{cat}_0 E \geq 5$  and that  $\text{cat}_0(E, S^2 \times S^2 \times S^5) \geq 2$ . If we apply [C2] to the fibration  $S^5 \rightarrow E \rightarrow CP(2) \times S^2$  with model  $(\Lambda b, a, x, y, d) \rightarrow (\Lambda(a, b, x, y), d) \rightarrow (\Lambda z, 0)$ , we get  $\text{cat}_0 E \leq 5$ .

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CENTRE DE RECERCA MATEMÀTICA, BARCELONA, SPAIN  
*E-mail address*: [mcuvilli@crm.es](mailto:mcuvilli@crm.es)

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF OTTAWA, OTTAWA, ONTARIO,  
CANADA K1N 6N5  
*E-mail address*: [bjessup@uottawa.ca](mailto:bjessup@uottawa.ca)