PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 131, Number 7, Pages 2161–2170
S 0002-9939(02)06795-3
Article electronically published on November 6, 2002

PERIODIC SOLUTIONS FOR PLANAR 2N-BODY PROBLEMS

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(Communicated by Carmen C. Chicone)

ABSTRACT. In this paper we study some necessary conditions and sufficient conditions for the nested polygonal solutions of planar 2N-body problems.

1. MAIN RESULTS

This paper uses the same notations as those used in [6]. For $n \geq 2$ the equations of motion of the planar n-body problem ([1], [2], [3], [5], [6], [7]) can be written in the form

$$\ddot{z}_k = -\sum_{\substack{j=1 \\ j \neq k}}^{n} m_j \frac{z_k - z_j}{|z_k - z_j|^3}, \tag{1.1}$$

where z_k is the complex coordinate of the kth mass m_k in an inertial coordinate system. Let ρ_k denote the N complex kth roots of unity, i.e.,

$$\rho_k = \exp(2\pi i k/N). \tag{1.2}$$

This equation will also serve to define ρ_k for any number k. We assume that the mass m_k $(k = 1, \cdots, N)$ is located at the vertex ρ_k of a regular polygon inscribed on the unit circle, and $\tilde{m}_k$ $(k = 1, \cdots, N)$ is located at

$$\tilde{\rho}_k = a\rho_k \cdot e^{i\theta}, \tag{1.3}$$

where $a > 0, 0 \leq \theta \leq 2\pi$, and $a \neq 1$ when $\theta = 0$ or 2π.

Then the center of masses $m_1, \cdots, m_N; \tilde{m}_1, \cdots, \tilde{m}_N$ is

$$z_0 = \sum_j (m_j\rho_j + \tilde{m}_j\tilde{\rho}_j)/M \tag{1.4}$$

where $M = \sum_j (m_j + \tilde{m}_j)$. In (1.4) and throughout this paper, unless specially restricted, all indices and summations will range from 1 to N. The functions describing their rotation about z_0 with angular velocity ω are then given by

$$z_k(t) = (\rho_k - z_0)\exp(i\omega t), \qquad k = 1, \cdots, N, \tag{1.5}$$

Received by the editors November 7, 2001 and, in revised form, February 25, 2002.

2000 *Mathematics Subject Classification.* Primary 34C25, 34C15, 70F10.

Key words and phrases. 2N-body problems, nested regular polygon periodic solutions, circulant matrices.

This work was supported by the NSF of China.

and

$$\tilde{z}_k(t) = (a\rho_k e^{i\theta} - z_0)\exp(i\omega t), \qquad k = 1, \cdots, N. \tag{1.6}$$

Then the equations of motion of the planar 2N-body problem can be written in the following form:

$$\ddot{z}_k = \sum_{j\neq k} m_j \frac{z_j - z_k}{|z_j - z_k|^3} + \sum_j \tilde{m}_j \frac{\tilde{z}_j - z_k}{|\tilde{z}_j - z_k|^3} \tag{1.7}$$

and

$$\ddot{\tilde{z}}_k = \sum_j m_j \frac{z_j - \tilde{z}_k}{|z_j - \tilde{z}_k|^3} + \sum_{j\neq k} \tilde{m}_j \frac{\tilde{z}_j - \tilde{z}_k}{|\tilde{z}_j - \tilde{z}_k|^3}. \tag{1.8}$$

R. Moeckel and C. Simó ([5]) proved the following result:

Theorem (Moeckel-Simó)**.** *If $\theta = 0$ and $m_1 = \cdots = m_N$, $\tilde{m}_1 = \cdots = \tilde{m}_N$, then for every mass ratio $b = \frac{\tilde{m}_1}{m_1}$, there are exactly two planar central configurations consisting of two nested regular N-gons. For one of these, the ratio of the sizes of the two polygons is less than 1, and for the other it is greater than 1.*

In this paper, we study the inverse problem of the theorem (Moeckel-Simó) and the following results are established.

Theorem 1. *If, for $N \geq 2$, the functions $z_k(t)$ and $\tilde{z}_k(t)$ given by* (1.5) *and* (1.6) *are solutions of the 2N-body problem* (1.7) *and* (1.8)*, it follows that ω^2 satisfies*

$$\begin{aligned} &\left[\frac{1}{4}\sum_{j\neq N}\csc\left(\frac{\pi j}{N}\right) - \frac{\omega^2}{M}N\right]\cdot\left[\frac{1}{4}a^{-2}\sum_{j\neq N}\csc\left(\frac{\pi j}{N}\right) - \frac{\omega^2}{M}a\cdot N\right]e^{i\theta} \\ &\quad = \left[\sum_j \frac{1 - a\rho_{j-1}e^{i\theta}}{|1 - a\rho_{j-1}e^{i\theta}|^3} - \frac{N\omega^2}{M}\right]\cdot\left[\sum_j \frac{ae^{i\theta} - \rho_{j-1}}{|ae^{i\theta} - \rho_{j-1}|^3} - \frac{\omega^2}{M}Nae^{i\theta}\right]. \end{aligned} \tag{1.9}$$

Theorem 2. *For $N \geq 2$, $m_k, \tilde{m}_k > 0$, if the functions $z_k(t)$ and $\tilde{z}_k(t)$ given by* (1.5) *and* (1.6) *with ω^2 given by* (1.9) *are solutions of the 2N-body problem* (1.7) *and* (1.8)*, then $m_1 = m_2 = \cdots = m_N$ and $\tilde{m}_1 = \tilde{m}_2 = \cdots = \tilde{m}_N$.*

Conversely, if $m_1 = m_2 = \cdots = m_N$ and $\tilde{m}_1 = \tilde{m}_2 = \cdots = \tilde{m}_N$, let $b = \tilde{m}_1/m_1$. If b, a and ω^2 satisfy

$$\frac{N\omega^2}{M} = \frac{1}{1+b}\left[\sum_{j=1}^{N-1}\frac{1-\rho_j}{|1-\rho_j|^3} + \sum_{j=0}^{N-1}\frac{b(1 - a\rho_j e^{i\theta})}{|1 - a\rho_j e^{i\theta}|^3}\right], \tag{1.10}$$

$$\frac{N\omega^2}{M} = \frac{e^{-i\theta}}{a(1+b)}\left[\sum_j \frac{ae^{i\theta} - \rho_j}{|ae^{i\theta} - \rho_j|^3} + \sum_{j\neq N}\frac{b(a - a\rho_j)e^{i\theta}}{|a - a\rho_j|^3}\right], \tag{1.11}$$

then $z_k(t)$ and $\tilde{z}_k(t)$ are solutions of (1.7) *and* (1.8)*.*

Remark. It is not difficult to prove that "ω^2" and "a" are positive real numbers if $\theta = 0$ *or* $\frac{\pi}{N}$, and it seems that only $\theta = 0$ *or* $\frac{\pi}{N}$, "ω^2" and "a" are positive real numbers, but the proof seems difficult.

Corollary 1 (MacMillan-Bartky [3]). *Under the above assumptions,*

(i) *if* $N = 2, \theta = 0, a > 1$ *and* $z_k(t)$ *and* $\tilde{z}_k(t)$ *given by* (1.5) *and* (1.6) *with* ω^2 *satisfying*

$$\begin{aligned}\left[\frac{1}{4} - \frac{2\omega^2}{M}\right] \cdot \left[\frac{1}{4}a^{-2} - \frac{\omega^2}{M} \cdot 2a\right] \\ = \left[\frac{2(a^2+1)}{(a^2-1)^2} - \frac{\omega^2}{M} \cdot 2a\right] \cdot \left[\frac{-4a}{(a^2-1)^2} - \frac{2\omega^2}{M}\right]\end{aligned} \tag{1.12}$$

are solutions of the 2×2*-body problems* (1.7) *and* (1.8)*, then* $m_1 = m_2, \tilde{m}_1 = \tilde{m}_2$*. Conversely, if* $\omega^2 = (m_1 + m_2 + \tilde{m}_1 + \tilde{m}_2)\gamma/N$ *satisfying* (1.12) *and*

$$\frac{2\omega^2}{M} = \frac{1}{1+b}\left[\frac{1}{4} - \frac{4ab}{(a^2-1)^2}\right] \tag{1.13}$$

where

$$b = \frac{a^7 - 2a^5 - 8a^4 + a^3 - 8a^2}{17a^4 - 2a^2 + 1}, \tag{1.14}$$

then $z_k(t)$ *and* $\tilde{z}_k(t)$ *given by* (1.5) *and* (1.6) *are solutions of* (1.7) *and* (1.8).

(ii) *if* $N = 2, \theta = 0, 0 < a < 1$ *and* $z_k(t)$ *and* $\tilde{z}_k(t)$ *given by* (1.5) *and* (1.6) *with* ω^2 *satisfying*

$$\begin{aligned}\left[\frac{1}{4} - \frac{2\omega^2}{M}\right] \cdot \left[\frac{1}{4}a^{-2} - \frac{\omega^2}{M} \cdot 2a\right] \\ = \left[\frac{-4a}{(1-a^2)^2} - \frac{\omega^2}{M} \cdot 2a\right] \cdot \left[\frac{2(1+a^2)}{(1-a^2)^2} - \frac{2\omega^2}{M}\right]\end{aligned} \tag{1.15}$$

are solutions of (1.7) *and* (1.8)*, then* $m_1 = m_2, \tilde{m}_1 = \tilde{m}_2$*. Conversely, if* $\omega^2 = M\gamma/N$ *satisfying* (1.15) *and*

$$\gamma = \frac{1}{1+b}\left[\frac{1}{4} + \frac{2b(a^2+1)}{(a^2-1)^2}\right] \tag{1.16}$$

where

$$b = \frac{a^7 - 2a^5 + 17a^3}{-8a^5 + a^4 - 8a^3 - 2a^2 + 1}, \tag{1.17}$$

then $z_k(t)$ *and* $\tilde{z}_k(t)$ *given by* (1.5) *and* (1.6) *are solutions of* (1.7) *and* (1.8).

Corollary 2 (MacMillan-Bartky [3]). (i) *For* $N = 2$ *and* $\theta = \frac{\pi}{2}$*, then* (1.10) *and* (1.11) *imply*

$$b = [(1+a^2)^{-3/2} - 2^{-3}] \cdot [(1+a^2)^{-3/2} - (2a)^{-3}]^{-1}. \tag{1.18}$$

(ii) *For* $N = 2$ *and* $\theta = \frac{\pi}{2}$*,* $b = 1$ *if and only if* $a = 1$*. That is,* $z_k, \tilde{z}_k$ *are solutions of* (1.7) *and* (1.8)*. If* $m_1 = m_2, \tilde{m}_1 = \tilde{m}_2$*, then* $m_1 = m_2 = \tilde{m}_1 = \tilde{m}_2$ *if and only if* $m_1, m_2, \tilde{m}_1, \tilde{m}_2$ *are at the vertices of a square.*

(iii) *For* $N = 2$ *and* $\theta = \frac{\pi}{2}$*,* $m_1 = m_2, \tilde{m}_1 = \tilde{m}_2$ *are at the vertices of a diamond. Then* $m_1, m_2, \tilde{m}_1, \tilde{m}_2$ *form a central configuration if and only if* $\frac{\sqrt{3}}{3} < a < \sqrt{3}$*, where* a *stands for the ratio of its diagonals and the ratio* b *of masses is determined uniquely by* (1.18)*; especially considering the degenerate central configuration, when* $b = \infty$*,* $a = \frac{\sqrt{3}}{3}$*, and when* $a = \sqrt{3}, b = 0$.

Corollary 3 (Perko-Walter [6])**.** *For $N \geq 2, m_k, \tilde{m}_k > 0$, if $a = 1$ and $\theta = \frac{\pi}{N}$, then $z_k(t)$ and $\tilde{z}_k(t)$ given by* (1.5) *and* (1.6) *with $\omega^2 = M\gamma/N$ and ω^2 given by* (1.9) *are solutions of the 2N-body problem* (1.7) *and* (1.8). *Then $m_1 = m_2 = \cdots = m_N = \tilde{m}_1 = \cdots = \tilde{m}_N$.*

Corollary 4. *If $N = 3$, $\theta = 0$. If $z_k(t)$ and $\tilde{z}_k(t)$ given by* (1.5) *and* (1.6) *with ω^2 satisfying*

$$\begin{aligned}\omega^2 = {} & [\frac{\sqrt{3}}{M}(a + a^{-2}) - \frac{3}{M}(\sum_{j=1}^{3}\frac{a-\rho_{j-1}}{|a-\rho_{j-1}|^3} + a\sum_{j=1}^{3}\frac{1-a\rho_{j-1}}{|1-a\rho_{j-1}|^3})]^{-1} \\ & \cdot [\frac{1}{3}a^{-2} - (\sum_{j=1}^{3}\frac{a-\rho_{j-1}}{|a-\rho_{j-1}|^3})(\sum_{j=1}^{3}\frac{1-a\rho_{j-1}}{|1-a\rho_{j-1}|^3})]\end{aligned} \tag{1.19}$$

are the solution of the nested 2×3-body problems (1.7) *and* (1.8), *then $m_1 = m_2 = m_3, \tilde{m}_1 = \tilde{m}_2 = \tilde{m}_3$.*

Conversely, if $m_1 = m_2 = m_3, \tilde{m}_1 = \tilde{m}_2 = \tilde{m}_3$, let $b = \frac{\tilde{m}_1}{m_1}$. If ω^2 satisfies (1.19) *and a and b have the relation*

$$b = (\frac{\sqrt{3}}{3}a^{-3} - \sum_{j=1}^{3}\frac{1-a\rho_{j-1}}{|1-a\rho_{j-1}|^3})^{-1}(\frac{\sqrt{3}}{3} - a^{-1}\sum_{j=1}^{3}\frac{a-\rho_{j-1}}{|a-\rho_{j-1}|^3}), \tag{1.20}$$

then $z_k(t)$ and $\tilde{z}_k(t)$ defined by (1.5) *and* (1.6) *are solutions of* (1.7) *and* (1.8).

Corollary 5. *Assume $N = 3, \theta = \frac{\pi}{3}$. If $z_k(t)$ and $\tilde{z}_k(t)$ given by* (1.5) *and* (1.6) *with ω^2 satisfying*

$$\begin{aligned}\omega^2 = {} & \frac{M}{3}[\frac{\sqrt{3}}{3}(a + a^{-2}) - (\sum_{j=1}^{3}\frac{a-e^{-\frac{\pi}{3}\sqrt{-1}}\rho_{j-1}}{|a-e^{-\frac{\pi}{3}\sqrt{-1}}\rho_{j-1}|^3} + a\sum_{j=1}^{3}\frac{1-ae^{\frac{\pi}{3}\sqrt{-1}}\rho_{j-1}}{|1-ae^{\frac{\pi}{3}\sqrt{-1}}\rho_{j-1}|^3})]^{-1} \\ & \cdot [\frac{1}{3}a^{-2}e^{\frac{\pi}{3}\sqrt{-1}} - (\sum_{j=1}^{3}\frac{a-e^{-\frac{\pi}{3}\sqrt{-1}}\rho_{j-1}}{|a-e^{-\frac{\pi}{3}\sqrt{-1}}\rho_{j-1}|^3})(\sum_{j=1}^{3}\frac{1-ae^{\frac{\pi}{3}\sqrt{-1}}\rho_{j-1}}{|1-ae^{\frac{\pi}{3}\sqrt{-1}}\rho_{j-1}|^3})]\end{aligned} \tag{1.21}$$

are the solution of the twisted 2×3-body problems (1.7) *and* (1.8), *then $m_1 = m_2 = m_3, \tilde{m}_1 = \tilde{m}_2 = \tilde{m}_3$.*

Conversely, if $m_1 = m_2 = m_3, \tilde{m}_1 = \tilde{m}_2 = \tilde{m}_3$, let $b = \frac{\tilde{m}_1}{m_1}$. If ω^2 satisfies (1.21) *and a and b have the following relation:*

$$b = (\frac{\sqrt{3}}{3}a^{-2} - \sum_{j=1}^{3}\frac{a(1-ae^{\frac{\pi}{3}\sqrt{-1}}\rho_{j-1})}{|1-ae^{\frac{\pi}{3}\sqrt{-1}}\rho_{j-1}|^3})^{-1}(\frac{\sqrt{3}}{3}a - \sum_{j=1}^{3}\frac{a-e^{-\frac{\pi}{3}\sqrt{-1}}\rho_{j-1}}{|a-e^{-\frac{\pi}{3}\sqrt{-1}}\rho_{j-1}|^3}), \tag{1.22}$$

then $z_k(t)$ and $\tilde{z}_k(t)$ defined by (1.5) *and* (1.6) *are solutions of* (1.7) *and* (1.8).

2. Eigenvalues and eigenvectors for circulant matrices

Definition 2.1 ([4])**.** If $N \times N$ matrix $A = (a_{i,j})$ satisfies

$$a_{i,j} = a_{i-1,j-1},\ 1 \leq i,\ j \leq N, \tag{2.1}$$

where we assume $a_{i,0} = a_{i,N}$ and $a_{0,j} = a_{N,j}$. Then we call A a circulant matrix.

Lemma 2.1 ([4]). *If A and B are $N\times N$ circulant matrices, then $A+B, A-B, A\cdot B$ are also circulant matrices and $AB = BA$.*

Lemma 2.2 ([4]). *Let $A = (a_{i,j})$ be an $N \times N$ circulant matrix. The eigenvalues λ_k and the eigenvectors $\vec{v}_k$ of A are*

$$\lambda_k(A) = \sum_j a_{1,j}\rho_{k-1}^{j-1} \tag{2.2}$$

and

$$\vec{v}_k = (\rho_{k-1}, \rho_{k-1}^2, \cdots, \rho_{k-1}^N)^T. \tag{2.3}$$

Lemma 2.3 ([4]). *Let A, B be circulant matrices, where $\lambda_k(A)$, $\lambda_k(B)$ are eigenvalues of A, B. Then the eigenvalues of $A+B, A-B, A\cdot B$ are $\lambda_k(A)+\lambda_k(B), \lambda_k(A)-\lambda_k(B), \lambda_k(A)\cdot\lambda_k(B)$.*

The following useful lemma can be simply proved using the properties of circulant matrices.

Lemma 2.4. *If $A = (a_{i,j})$ is an $N \times N$ circulant matrix, and $A \cdot X = 0$, where $X = (x_1, \cdots, x_n)^T$, $\sum_i x_i \neq 0$, then*

$$a_{1,j} + \cdots + a_{N,j} = 0,\ 1 \leq j \leq N,$$

$$a_{i,1} + \cdots + a_{i,N} = 0,\ 1 \leq i \leq N. \tag{2.4}$$

3. Proof of the main results

For two nested regular polygons, we define

$$\rho_k = \exp(2\pi ik/N), \tag{3.1}$$

$$\tilde{\rho}_k = a\exp(2\pi ik/N)e^{i\theta}, \tag{3.2}$$

$$z_0 = \sum_j (m_j\rho_j + \tilde{m}_j\tilde{\rho}_j)/M, \tag{3.3}$$

where

$$M = \sum_j (m_j + \tilde{m}_j), \tag{3.4}$$

$$z_k(t) = (\rho_k - z_0)\exp(i\omega t), \qquad k = 1, \cdots, N, \tag{3.5}$$

and

$$\tilde{z}_k(t) = (a\rho_k e^{i\theta} - z_0)\exp(i\omega t), \qquad k = 1, \cdots, N. \tag{3.6}$$

Proof of Theorem 1. (3.1)–(3.6) imply that the $z_k(t)$ and $\tilde{z}_k(t)$ are the solutions of (1.7) and (1.8) if and only if

(3.7)

$$(\rho_k - z_0)\omega^2\exp(i\omega t) = \left(\sum_{j\neq k} m_j \frac{\rho_k - \rho_j}{|\rho_k - \rho_j|^3} + \sum_j \tilde{m}_j \frac{\rho_k - \tilde{\rho}_j}{|\rho_k - \tilde{\rho}_j|^3}\right)\exp(i\omega t)$$

and

(3.8)
$$(\tilde{\rho}_k - z_0)\omega^2 \exp(i\omega t) = \left(\sum_j m_j \frac{\tilde{\rho}_k - \rho_j}{|\tilde{\rho}_k - \rho_j|^3} + \sum_{j\neq k} \tilde{m}_j \frac{\tilde{\rho}_k - \tilde{\rho}_j}{|\tilde{\rho}_k - \tilde{\rho}_j|^3}\right) \exp(i\omega t)$$

or if and only if

(3.9)
$$\sum_{j\neq k} m_j \left(\frac{1}{|\rho_k - \rho_j|^3} - \frac{\omega^2}{M}\right)(\rho_k - \rho_j) + \sum_j \tilde{m}_j \left(\frac{1}{|\rho_k - \tilde{\rho}_j|^3} - \frac{\omega^2}{M}\right)(\rho_k - \tilde{\rho}_j) = 0$$

and

(3.10)
$$\sum_j m_j \left(\frac{1}{|\tilde{\rho}_k - \rho_j|^3} - \frac{\omega^2}{M}\right)(\tilde{\rho}_k - \rho_j) + \sum_{j\neq k} \tilde{m}_j \left(\frac{1}{|\tilde{\rho}_k - \tilde{\rho}_j|^3} - \frac{\omega^2}{M}\right)(\tilde{\rho}_k - \tilde{\rho}_j) = 0.$$

Multiplying both sides by ρ_{N-k} and noting that $|\rho_k - \rho_j| = |\rho_k||1-\rho_{j-k}| = |1-\rho_{j-k}|$ and using $\tilde{\rho}_k = a\rho_k e^{i\theta}$,

$$\begin{aligned} &\sum_{j\neq k} m_j \left(\frac{1}{|1-\rho_{j-k}|^3} - \frac{\omega^2}{M}\right)(1-\rho_{j-k}) \\ &\quad + \sum_j \tilde{m}_j \left(\frac{1}{|1 - a\rho_{j-k}e^{i\theta}|^3} - \frac{\omega^2}{M}\right)(1 - a\rho_{j-k}e^{i\theta}) = 0 \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} &\sum_j m_j \left(\frac{1}{|ae^{i\theta} - \rho_{j-k}|^3} - \frac{\omega^2}{M}\right)(ae^{i\theta} - \rho_{j-k}) \\ &\quad + \sum_{j\neq k} \tilde{m}_j \left(\frac{1}{|a - a\rho_{j-k}|^3} - \frac{\omega^2}{M}\right)(a - a\rho_{j-k})e^{i\theta} = 0. \end{aligned} \tag{3.12}$$

Notice that every step from (3.7) to (3.12) can be conversed, respectively. Now we define the $N\times N$ circulant matrices $C = [c_{k,j}], A = [a_{k,j}], B = [b_{k,j}], D = [d_{k,j}]$ as follows:

$$c_{k,j} = 0, \qquad \text{for} \quad k = j,$$

$$c_{k,j} = \left(\frac{1}{|1-\rho_{j-k}|^3} - \frac{\omega^2}{M}\right)(1-\rho_{j-k}), \qquad \text{for} \quad k \neq j, \tag{3.13}$$

$$a_{k,j} = \left(\frac{1}{|1 - a\rho_{j-k}e^{i\theta}|^3} - \frac{\omega^2}{M}\right)(1 - a\rho_{j-k}e^{i\theta}), \tag{3.14}$$

$$b_{k,j} = \left(\frac{1}{|ae^{i\theta} - \rho_{j-k}|^3} - \frac{\omega^2}{M}\right)(ae^{i\theta} - \rho_{j-k}), \tag{3.15}$$

$$d_{k,j} = 0, \qquad \text{for} \quad k = j,$$

$$d_{k,j} = \left(\frac{1}{|a - a\rho_{j-k}|^3} - \frac{\omega^2}{M}\right)(a - a\rho_{j-k})e^{i\theta} \quad \text{for} \quad k \neq j. \tag{3.16}$$

Then (3.11) and (3.12) hold if and only if the matrix equation

$$\begin{pmatrix} C & A \\ B & D \end{pmatrix} \begin{pmatrix} m_1 \\ \vdots \\ m_N \\ \tilde{m}_1 \\ \vdots \\ \tilde{m}_N \end{pmatrix} = 0 \tag{3.17}$$

has a positive solution.

Let

$$m = (m_1, \cdots, m_N)^T, \tilde{m} = (\tilde{m}_1, \cdots, \tilde{m}_N)^T. \tag{3.18}$$

Then (3.17) is equivalent to

$$C \cdot m + A \cdot \tilde{m} = 0, \tag{3.19}$$

$$B \cdot m + D \cdot \tilde{m} = 0. \tag{3.20}$$

By (3.19) and (3.20) we have

$$(CD - BA) \cdot m = 0, \tag{3.21}$$

$$(AB - CD) \cdot \tilde{m} = 0. \tag{3.22}$$

We notice that (3.21) and (3.22) have a solution is equivalent to that $AB-CD$ has a positive real eigenvector corresponding to eigenvalue 0. By Lemma 2.3 we have

$$\begin{aligned} &\lambda_k(AB - CD) \\ &= \lambda_k(AB) - \lambda_k(CD) \\ &= \lambda_k(A)\lambda_k(B) - \lambda_k(C)\lambda_k(D). \end{aligned} \tag{3.23}$$

Hence

$$\lambda_k(AB - CD) = 0 \tag{3.24}$$

for some $1 \le k \le N$ if and only if

$$\lambda_k(A)\lambda_k(B) = \lambda_k(C)\lambda_k(D). \tag{3.25}$$

Also, we notice by Lemma 2.2 and Lemma 2.4 $\lambda_1(AB-CD) = 0$. Hence by Lemma 2.2 we have

(3.26)

$$\begin{aligned} &\left[\sum_{j\neq 1}\left(\frac{1}{|1-\rho_{j-1}|^3} - \frac{\omega^2}{M}\right)(1-\rho_{j-1})\right] \cdot \left[\sum_{j\neq 1}\left(\frac{1}{|a-a\rho_{j-1}|^3} - \frac{\omega^2}{M}\right)(a-a\rho_{j-1})e^{i\theta}\right] \\ &\quad = \left[\sum_{j}\left(\frac{1}{|1-a\rho_{j-1}e^{i\theta}|^3} - \frac{\omega^2}{M}\right)(1-a\rho_{j-1}e^{i\theta})\right] \\ &\qquad \cdot \left[\sum_{j}\left(\frac{1}{|ae^{i\theta}-\rho_{j-1}|^3} - \frac{\omega^2}{M}\right)(ae^{i\theta}-\rho_{j-1})\right]. \end{aligned}$$

We notice that

(3.27)
$$\begin{aligned}&\left[\sum_{j\neq 1}\frac{1-\rho_{j-1}}{|1-\rho_{j-1}|^3}-\frac{\omega^2}{M}\sum_j(1-\rho_{j-1})\right]\\&\qquad\cdot\left[a^{-2}\sum_{j\neq 1}\frac{1-\rho_{j-1}}{|1-\rho_{j-1}|^3}-\frac{\omega^2}{M}a\cdot\sum_j(1-\rho_{j-1})\right]\cdot e^{i\theta}\\&\quad=\left[\frac{1}{4}\sum_{j\neq N}\csc\left(\frac{\pi j}{N}\right)-\frac{\omega^2}{M}\cdot N\right]\cdot\left[\frac{1}{4}a^{-2}\sum_{j\neq N}\csc\left(\frac{\pi j}{N}\right)-\frac{\omega^2}{M}\cdot a\cdot N\right]\cdot e^{i\theta},\end{aligned}$$

(3.28)
$$\begin{aligned}&\sum_j\left[\frac{1-ae^{i\theta}\rho_{j-1}}{|1-ae^{i\theta}\rho_{j-1}|^3}-\frac{\omega^2}{M}\cdot(1-ae^{i\theta}\rho_{j-1})\right]\\&\qquad\cdot\sum_j\left[\frac{ae^{i\theta}-\rho_{j-1}}{|ae^{i\theta}-\rho_{j-1}|^3}-\frac{\omega^2}{M}(ae^{i\theta}-\rho_{j-1})\right]\\&\quad=\left[\sum_j\frac{1-ae^{i\theta}\rho_{j-1}}{|1-ae^{i\theta}\rho_{j-1}|^3}-\frac{\omega^2\cdot N}{M}\right]\cdot\left[\sum_j\frac{ae^{i\theta}-\rho_{j-1}}{|ae^{i\theta}-\rho_{j-1}|^3}-\frac{\omega^2}{M}\cdot Nae^{i\theta}\right].\end{aligned}$$

Proof of Theorem 2. $\lambda_1(AB-CD)=0$ is simple so $\vec{v}_1=(1,1,\cdots,1)^T$ is the only positive real eigenvector for $\lambda_1=0$.

Assume $m_1=m_2=\cdots=m_N>0$ and $\tilde{m}_1=m_2=\cdots=\tilde{m}_N>0$, ω^2 is determined by (1.9) and a,b is determined by (1.10) and (1.11). Then $(m_1,\cdots,m_1)^T$ is a solution of (3.11) and (3.12) or (1.7) and (1.8), since

(3.29)
$$\begin{aligned}&\sum_{j\neq k}m_j\left(\frac{1}{|1-\rho_{j-k}|^3}-\frac{\omega^2}{M}\right)(1-\rho_{j-k})\\&\qquad+\sum_j bm_j\left(\frac{1}{|1-a\rho_{j-k}e^{i\theta}|^3}-\frac{\omega^2}{M}\right)(1-a\rho_{j-k}e^{i\theta})\end{aligned}$$

(3.30)
$$\begin{aligned}&=m_1\left[\sum_{j\neq k}\left(\frac{1}{|1-\rho_{j-k}|^3}-\frac{\omega^2}{M}\right)(1-\rho_{j-k})\right.\\&\qquad\left.+\sum_j b\left(\frac{1}{|1-a\rho_{j-k}e^{i\theta}|^3}-\frac{\omega^2}{M}\right)(1-a\rho_{j-k}e^{i\theta})\right]\end{aligned}$$

(3.31)

$$= m_1 \left[\sum_{j \neq N} \left(\frac{1}{|1-\rho_j|^3} - \frac{\omega^2}{M} \right) (1-\rho_j) \right.$$
$$\left. + \sum_j b \left(\frac{1}{|1-a\rho_j e^{i\theta}|^3} - \frac{\omega^2}{M} \right) (1 - a\rho_j e^{i\theta}) \right] = 0,$$

and the same results for (3.12) are easily obtained. Thus the proof is completed.

Corollary 1 is obvious from Theorem 2.

Proof of Corollary 2. (i) For $N = 2$ and $\theta = \frac{\pi}{2}$, b and a have the following relationship:

$$b = \frac{\left(\sum_{j=0}^{N-1} \frac{ae^{i\theta}-\rho_j}{|ae^{i\theta}-\rho_j|^3} - \sum_{j=1}^{N-1} \frac{ae^{i\theta}(1-\rho_j)}{|1-\rho_j|^3}\right)}{\left(\sum_{j=0}^{N-1} \frac{ae^{i\theta}(1-a\rho_j e^{i\theta})}{|1-a\rho_j e^{i\theta}|^3} - \sum_{j=1}^{N-1} \frac{ae^{i\theta}(1-\rho_j)}{a^3|1-\rho_j|^3}\right)}. \tag{3.32}$$

Then

(3.33)

$$\sum_{j=0}^{N-1} \frac{ae^{i\theta}-\rho_j}{|ae^{i\theta}-\rho_j|^3} - \sum_{j=1}^{N-1} \frac{ae^{i\theta}(1-\rho_j)}{|1-\rho_j|^3} = \frac{ai-1}{|ai-1|^3} + \frac{ai+1}{|ai+1|^3} - \frac{ai(1+1)}{|2|^3}$$
$$= \frac{ai-1}{(a^2+1)^{3/2}} + \frac{ai+1}{(a^2+1)^{3/2}} - \frac{2ai}{8}$$
$$= \frac{2ai}{(a^2+1)^{3/2}} - \frac{2ai}{8} = 2ai(\frac{1}{(a^2+1)^{3/2}} - \frac{1}{8})$$

and

$$\begin{aligned} &\sum_{j=0}^{N-1} \frac{ae^{i\theta}(1-a\rho_j e^{i\theta})}{|1-a\rho_j e^{i\theta}|^3} - \sum_{j=1}^{N-1} \frac{ae^{i\theta}(1-\rho_j)}{a^3|1-\rho_j|^3} \\ &= \frac{ai(1-ai)}{|1-ai|^3} + \frac{ai(1-a(-1)i)}{|1-a(-1)i|^3} - \frac{ai(1+1)}{|2a|^3} \\ &= 2ai(\frac{1}{(a^2+1)^{3/2}} - \frac{1}{8a^3}), \end{aligned} \tag{3.34}$$

so

$$b = [(1+a^2)^{-3/2} - 2^{-3}] \cdot [(1+a^2)^{-3/2} - (2a)^{-3}]^{-1}. \tag{3.35}$$

(ii) That $b = 1$ if and only if $a = 1$ can be obtained directly from (1.18).

(iii) Since (1.11) holds and b must be greater than zero, we have

$$\begin{cases} (1+a^2)^{-3/2} - 2^{-3} & > 0, \\ (1+a^2)^{-3/2} - (2a)^{-3} & > 0, \end{cases} \tag{3.36}$$

or

$$\begin{cases} (1+a^2)^{-3/2} - 2^{-3} & < 0, \\ (1+a^2)^{-3/2} - (2a)^{-3} & < 0. \end{cases} \tag{3.37}$$

Then the result of (iii) is easily obtained.

Proof of Corollary 3. Under the assumptions of Corollary 3, we have $b = 1$.

Acknowledgments

The authors sincerely thank the referee for some helpful suggestions, which improved the paper.

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