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LOCAL COMPLETE INTERSECTIONS IN $\mathbb{P}^2$ AND KOSZUL SYZYGIES

DAVID COX AND HAL SCHENCK

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Abstract. We study the syzygies of a codimension two ideal $I = \langle f_1, f_2, f_3 \rangle \subseteq k[x, y, z]$. Our main result is that the module of syzygies vanishing (scheme-theoretically) at the zero locus $Z = \mathbf{V}(I)$ is generated by the Koszul syzygies iff Z is a local complete intersection. The proof uses a characterization of complete intersections due to Herzog. When I is saturated, we relate our theorem to results of Weyman and Simis and Vasconcelos. We conclude with an example of how our theorem fails for four generated local complete intersections in $k[x, y, z]$ and we discuss generalizations to higher dimensions.

1. Introduction

Let $R = k[x, y, z]$ be the coordinate ring of $\mathbb{P}^2$, and consider the ideal $I = \langle f_1, f_2, f_3 \rangle \subseteq R$, where f_i is homogeneous of degree d_i. It is well known that the f_i form a regular sequence in R if and only if the Koszul complex

$$0 \longrightarrow R(-\sum_{j=1}^{3} d_j) \xrightarrow{\begin{bmatrix} f_3 \\ -f_2 \\ f_1 \end{bmatrix}} \bigoplus_{j<k} R(-d_j - d_k) \xrightarrow{\begin{bmatrix} f_2 & f_3 & 0 \\ -f_1 & 0 & f_3 \\ 0 & -f_1 & -f_2 \end{bmatrix}} \bigoplus_{j=1}^{3} R(-d_j) \xrightarrow{[f_1\ f_2\ f_3]} I \longrightarrow 0 \tag{1.1}$$

is exact. (As usual $R(i)$ denotes a shift in grading, i.e., $R(i)_j = R_{i+j}$.)

This paper will study the situation when $I = \langle f_1, f_2, f_3 \rangle$ has codimension two in R. Thus $Z = \mathbf{V}(I) \subseteq \mathbb{P}^2$ is a zero-dimensional subscheme. We will call Z the *base point locus* of f_1, f_2, f_3.

Since $Z \neq \emptyset$, f_1, f_2, f_3 no longer form a regular sequence, so that (1.1) fails to be exact. We can pinpoint the failure of exactness as follows.

Lemma 1.1. *If $I = \langle f_1, f_2, f_3 \rangle$ has codimension two in R, then* (1.1) *is exact except at $\bigoplus_{j=1}^{3} R(-d_j)$. In particular, the Koszul complex of f_1, f_2, f_3 gives the*

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exact sequences

$$0 \to R(-\sum_{j=1}^{3} d_j) \to \bigoplus_{j<k} R(-d_j - d_k) \to \bigoplus_{j=1}^{3} R(-d_j) \quad \textit{and} \quad \bigoplus_{j=1}^{3} R(-d_j) \to I \to 0.$$

Proof. The exactness of the first sequence follows from $\operatorname{depth}(I) = \operatorname{codim}(I) = 2$ and the Buchsbaum-Eisenbud exactness criterion [1], and the second sequence is obviously exact. □

To describe how (1.1) behaves at $\bigoplus_{j=1}^{3} R(-d_i)$, we make the following definition.

Definition 1.2. A *Koszul syzygy* on f_1, f_2, f_3 is an element of the submodule

$$K \subseteq \bigoplus_{j=1}^{3} R(-d_i)$$

generated by the columns of the matrix

$$\begin{bmatrix} f_2 & f_3 & 0 \\ -f_1 & 0 & f_3 \\ 0 & -f_1 & -f_2 \end{bmatrix}.$$

This leads to the following corollary of Lemma 1.1.

Corollary 1.3. If $I = \langle f_1, f_2, f_3 \rangle$ is a codimension two ideal, then

$$0 \longrightarrow R(-\sum_{j=1}^{3} d_j) \longrightarrow \bigoplus_{j<k} R(-d_j - d_k) \longrightarrow K \longrightarrow 0$$

is exact.

By Lemma 1.1, K is a proper submodule of the syzygy module S defined by the exact sequence

$$0 \longrightarrow S \longrightarrow \bigoplus_{j=1}^{3} R(-d_j) \longrightarrow I \longrightarrow 0. \tag{1.2}$$

Here is a simple example of how K and S can differ.

Example 1.4. Consider the ideal $I = \langle xy, xz, yz \rangle$. Then the syzygy module S is generated by the columns of the matrix

$$\phi = \begin{bmatrix} z & 0 \\ -y & y \\ 0 & -x \end{bmatrix}.$$

The generators of S are clearly not Koszul since they have degree 1.

In this example, note that the generators of S do not vanish on the base point locus $Z = \{(1,0,0), (0,1,0), (0,0,1)\}$. Since the Koszul syzygies clearly vanish at the base points, we get another way to see that the generators are not Koszul.

The observation that Koszul syzygies vanish at the basepoints leads one to ask the question: *Is every syzygy which vanishes at the base points a Koszul syzygy?* Before answering this question, we need to recall what "vanishing" means.

Definition 1.5. A syzygy (a_1, a_2, a_3) on the generators of $I = \langle f_1, f_2, f_3\rangle$ *vanishes at the basepoint locus* $Z = \mathbf{V}(I)$ if $a_i \in I^{sat}$ for all i. We will let V denote the module of syzygies which vanish at the basepoint locus Z. (See [2, Def. 5.4] for a sheaf-theoretic version of this definition.)

In this terminology, we have $K \subseteq V$ since Koszul syzygies vanish at the base points, and the above question can be rephrased as: *Is* $K = V$*?* In Example 1.4, one can compute that $K = V$, but $K \neq V$ can also occur; see [2, (5.4)] for an example. This leads to the central question of this paper: *When is* $K = V$*?*

To answer this question, we need the following definition.

Definition 1.6. An ideal is a *local complete intersection* (*lci*) if it is locally generated by a regular sequence. A codimension two lci in $\mathbb{P}^2$ is *curvilinear* if it is locally of the form $\langle x, y^k\rangle$, where x, y are local coordinates.

In [2], toric methods were used to prove $K = V$ whenever $I = \langle f_1, f_2, f_3\rangle \subseteq R$ is a codimension two curvilinear local complete intersection. The paper [2] also conjectured that $K = V$ should hold in the lci case. Our main result is the following strong form of this conjecture.

Theorem 1.7. *If* $I = \langle f_1, f_2, f_3\rangle \subseteq R$ *has codimension two, then* $K = V$ *if and only if* I *is a local complete intersection.*

The proof of this theorem will be given in the next section.

2. Proof of the main theorem

We begin with some preliminary definitions and results which will be used in the proof of Theorem 1.7. As in Section 1, $R = k[x, y, z]$ with the usual grading.

Definition 2.1. A submodule M of a finitely generated graded free R-module F is *saturated* if

$$M = \{u \in F \mid \langle x, y, z\rangle u \subseteq M\}.$$

Recall that the *Hilbert polynomial* $H(M)$ of a finitely generated graded R-module M is the unique polynomial such that

$$H(M)(n) = \dim_k M_n$$

for $n \gg 0$, where M_n is the graded piece of M in degree n.

We omit the straightforward proof of the following result.

Lemma 2.2. *Let* $M \subseteq N$ *be saturated submodules of a finitely generated graded free* R*-module. Then* $M = N$ *if and only if* $H(M) = H(N)$*.*

Given a codimension two ideal $I = \langle f_1, f_2, f_3\rangle \subseteq R$, we define the modules K and V as in the previous section. These modules satisfy

$$K \subseteq V \subseteq \bigoplus_{j=1}^{3} R(-d_j).$$

We next show that they are saturated.

Lemma 2.3. *K and V are saturated submodules of* $\bigoplus_{j=1}^3 R(-d_j)$*.*

Proof. We first consider V. Definition 1.5 implies that

$$V = S \cap \Big(\bigoplus_{j=1}^{3} I^{sat}(-d_j) \Big). \tag{2.1}$$

It is easy to see that S is saturated. Since the intersection of saturated submodules is saturated, we conclude that V is saturated.

Turning to K, suppose that $u \in \bigoplus_{j=1}^{3} R(-d_j)$ satisfies $\langle x, y, z\rangle u \subseteq K$. We may assume that u is homogeneous. We need to prove that $u \in K$. For this purpose, let $L = K + Ru$. Then consider the short exact sequence

$$0 \longrightarrow K \longrightarrow L \longrightarrow L/K \longrightarrow 0,$$

and note that L/K has finite length since $L_n = K_n$ for $n \geq \deg(u) + 1$. Let $\mathfrak{m} = \langle x, y, z\rangle$ denote the irrelevant ideal of R. We obtain a long exact sequence in local cohomology

$$0 \longrightarrow H^0_{\mathfrak{m}}(K) \longrightarrow H^0_{\mathfrak{m}}(L) \longrightarrow H^0_{\mathfrak{m}}(L/K) \longrightarrow H^1_{\mathfrak{m}}(K) \longrightarrow \cdots .$$

From [3, App. A.4], we know that for a graded R-module M the local cohomology $H^i_{\mathfrak{m}}(M)$ vanishes for $i <$ depth M, so the exact sequence of Corollary 1.3 implies that $H^0_{\mathfrak{m}}(K) = H^1_{\mathfrak{m}}(K) = \{0\}$. Since $L \hookrightarrow \bigoplus_{j=1}^{3} R(-d_j)$, $H^0_{\mathfrak{m}}(L) = \{0\}$. This forces

$$H^0_{\mathfrak{m}}(L/K, R) = \{0\}.$$

However, since L/K is a module of finite length, we also have $H^0_{\mathfrak{m}}(L/K) = L/K$. We conclude that $L/K = \{0\}$, so that $K = L$. This implies $u \in K$, as desired. □

The final ingredient we need for the proof is the following result of Herzog [4] which characterizes complete intersections in the local case.

Theorem 2.4. *Let $\mathcal{O}_p$ be the local ring of a point $p \in \mathbb{P}^2$, and let $\mathcal{I}_p \subseteq \mathcal{O}_p$ be a codimension two ideal. Then*

$$\dim_k \mathcal{I}_p/\mathcal{I}_p^2 \geq 2 \dim_k \mathcal{O}_p/\mathcal{I}_p. \tag{2.2}$$

Furthermore, equality holds if and only if $\mathcal{I}_p$ is a complete intersection in $\mathcal{O}_p$.

We can now prove our main result.

Proof of Theorem 1.7. By Lemmas 2.2 and 2.3, we know that $K = V$ if and only if they have the same Hilbert polynomials. In other words,

$$K = V \iff H(K) = H(V). \tag{2.3}$$

We next compute $H(K)$ and $H(V)$.

By Lemma 1.3, $H(K)$ is given by

$$H(K) = \sum_{j<k} H(R(-d_j - d_k)) - H(R(-\textstyle\sum_{j=1}^{3} d_j)).$$

Using the exactness of the Koszul complex of the regular sequence $x^{d_1}, y^{d_2}, z^{d_3}$, this formula simplifies to

$$H(K) = \sum_{j=1}^{3} H(R(-d_j)) - H(R). \tag{2.4}$$

We next consider $H(V)$. Using (2.1) and the exact sequence (1.2), we obtain the exact sequence

$$0 \longrightarrow V \longrightarrow \bigoplus_{j=1}^{3} I^{sat}(-d_j) \longrightarrow I \cdot I^{sat} \longrightarrow 0.$$

This gives the Hilbert polynomial

$$H(V) = \sum_{j=1}^{3} H(I^{sat}(-d_j)) - H(I \cdot I^{sat}).$$

However, since $0 \to I^{sat} \to R \to R/I^{sat} \to 0$ is exact and $\mathbf{V}(I^{sat}) = \mathbf{V}(I) = Z$ is zero dimensional, we also have

$$H(I^{sat}) = H(R) - \deg Z.$$

Hence the above formula for $H(V)$ can be written

$$H(V) = \sum_{j=1}^{3} H(R(-d_j)) - 3\deg Z - H(I \cdot I^{sat}).$$

We next observe that I^2 and $I \cdot I^{sat}$ have the same saturation. To prove this, note that $I^2 \subseteq I \cdot I^{sat}$, so that it suffices to show that $I \cdot I^{sat} \subseteq (I^2)^{sat}$. Let $f \in I \cdot I^{sat}$, so $f = \sum_{i=1}^{k} f_i g_i$ with $f_i \in I, g_i \in I^{sat}$. But then there exists an n such that $\langle x, y, z\rangle^n g_i \subseteq I$ for all i, hence $f \in (I^2)^{sat}$.

It follows that $H(I^2) = H(I \cdot I^{sat})$ since the two ideals have the same saturation. This allows us to write the above formula for $H(V)$ in the form

$$H(V) = \sum_{j=1}^{3} H(R(-d_j)) - 3\deg Z - H(I^2).$$

The exact sequences $0 \to I^2 \to R \to R/I^2 \to 0$ and $0 \to I/I^2 \to R/I^2 \to R/I \to 0$ show that

$$H(I^2) = H(R) - H(R/I^2) = H(R) - H(I/I^2) - \deg Z.$$

Then the previous formula for $H(V)$ can be written as

$$H(V) = \sum_{j=1}^{3} H(R(-d_j)) - H(R) + H(I/I^2) - 2\deg Z. \tag{2.5}$$

Comparing this to (2.4), we obtain

$$H(K) = H(V) \iff H(I/I^2) = 2\deg Z. \tag{2.6}$$

It remains to compute $\deg Z$ and $H(I/I^2)$. If $\mathcal{I}$ is the ideal sheaf of Z, then

$$\deg Z = \dim_k H^0(Z, \mathcal{O}_Z) = \dim_k H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}/\mathcal{I}) = \sum_{p \in Z} \dim_k \mathcal{O}_p/\mathcal{I}_p,$$

where $\mathcal{O}_p = \mathcal{O}_{\mathbb{P}^2,p}$ and $\mathcal{I}_p$ is the localization of I at $p \in Z$. Since I/I^2 has zero dimensional support, we also have

$$H(I/I^2) = \dim_k H^0(\mathbb{P}^2, \mathcal{I}/\mathcal{I}^2) = \sum_{p \in Z} \dim_k \mathcal{I}_p/\mathcal{I}_p^2.$$

By (2.2), we know that

$$\dim_k \mathcal{I}_p/\mathcal{I}_p^2 \geq 2\dim_k \mathcal{O}_p/\mathcal{I}_p$$

for every $p \in Z$. It follows easily that

$$H(I/I^2) = 2\deg Z \iff \dim_k \mathcal{I}_p/\mathcal{I}_p^2 = 2\dim_k \mathcal{O}_p/\mathcal{I}_p \text{ for all } p \in Z,$$

and by the final assertion of Theorem 2.4, we conclude that

$$H(I/I^2) = 2\deg Z \iff I \text{ is lci.} \tag{2.7}$$

Theorem 1.7 follows immediately from (2.3), (2.6) and (2.7). □

3. The saturated case

This section will consider the case where $I \subset R = k[x,y,z]$ is a saturated ideal of codimension two. Our goal is to explain how I being a local complete intersection relates to the results of Weyman [7] and Simis and Vasconcelos [6].

With the above hypothesis, I is Cohen-Macaulay (see [2, Proposition 5.2]) of codimension 2, so that by the Hilbert-Burch Theorem (see [3]), the free resolution of I has the form

$$0 \longrightarrow \bigoplus_{i=1}^{m} R(-b_i) \xrightarrow{\phi} \bigoplus_{j=1}^{m+1} R(-a_j) \xrightarrow{\psi} I \longrightarrow 0.$$

We will write this resolution as

$$0 \longrightarrow F \xrightarrow{\phi} G \xrightarrow{\psi} I \longrightarrow 0, \tag{3.1}$$

where F and G are free graded R-modules of ranks m and $m+1$, respectively.

We first consider the results of [7]. From (3.1), Weyman constructs the complex

$$0 \longrightarrow \wedge^2 F \longrightarrow F \otimes G \longrightarrow \operatorname{Sym}_2 G \longrightarrow \operatorname{Sym}_2 I \longrightarrow 0. \tag{3.2}$$

By Theorem 1 of [7], it follows easily that this complex is exact in our situation.

Using the natural map $\operatorname{Sym}_2 I \to I^2$, (3.2) gives the complex

$$0 \longrightarrow \wedge^2 F \longrightarrow F \otimes G \longrightarrow \operatorname{Sym}_2 G \longrightarrow I^2 \longrightarrow 0. \tag{3.3}$$

The results of Simis and Vasconcelos [6] lead to the following theorem.

Theorem 3.1. *Let $I \subseteq R$ be Cohen-Macaulay of codimension two. Then the following conditions are equivalent:*

1. *I is a local complete intersection.*
2. *The natural map $\operatorname{Sym}_2 I \to I^2$ is an isomorphism.*
3. *$\operatorname{Sym}_2 I$ is torsion free.*
4. *The complex (3.3) is a free resolution of I^2.*

Proof. The paper [6] defines an invariant $\delta(I)$ and, in its Corollary 1.2, shows that $\delta(I) = \ker(\operatorname{Sym}_2 I \to I^2)$. The first remark following Theorem 2.2 in [6] implies that $\delta(I) = \{0\}$ if and only if I is generically a complete intersection (i.e., the localization of R/I at each minimal prime is a complete intersection). Since $\mathbf{V}(I)$ has dimension zero, the latter is equivalent to being a local complete intersection. This proves (1) $\iff$ (2).

The equivalence (2) $\iff$ (3) is obvious since $\operatorname{Sym}_2 I \to I^2$ is onto and its kernel is torsion by [3, Ex. A2.4, p. 574]. Finally, (2) $\iff$ (4) follows from the exactness of (3.2). □

We conclude this section by showing that there are geometrically interesting ideals $I \subset R$ which are codimension two Cohen-Macaulay local complete intersections. The ideals we will consider arise in the study of line arrangements in $\mathbb{P}^2$. Let

$$Q = \prod_{i=1}^{d} \ell_i,$$

where the ℓ_i are distinct linear forms in $R = k[x, y, z]$. Localizing and applying the Euler relation shows that the Jacobian ideal

$$J_Q = \langle Q_x, Q_y, Q_z \rangle \subseteq R$$

is a local complete intersection, generated by three forms of degree $d-1$. An open problem in the study of line arrangements consists in determining when R/J_Q is Cohen-Macaulay. In this situation the arrangement is called a *free* arrangement; for more on this, see [5].

There are certain cases where freeness is known. For example, let

$$L_1 = \prod_{i=1}^{m} y - a_i x,$$
$$L_2 = \prod_{j=1}^{n} z - b_j x,$$

where a_i, b_j are nonzero and distinct. Then put

$$Q = xL_1L_2.$$

The Addition Theorem ([5, Thm. 4.50]) implies that these arrangements are free. It follows that the corresponding Jacobian ideal J_Q is a Cohen-Macaulay local complete intersection, and it has codimension two since $\mathbf{V}(J_Q)$ is the singular locus of $\mathbf{V}(Q)$, which is union of distinct lines.

For these arrangements, the free resolution is given by

$$0 \longrightarrow R(-m-2n) \oplus R(-2m-n) \longrightarrow R^3(-m-n) \longrightarrow J_Q \longrightarrow 0.$$

One can show that $H(R/J_Q) = m^2 + n^2 + mn$ and $H(J_Q/J_Q^2) = 2(m^2 + n^2 + mn)$. Notice how these numbers are consistent with the proof of Theorem 1.7. Also observe that Example 1.4 from Section 1 is the special case of this construction corresponding to $L_1 = y$ and $L_2 = z$.

4. Final remarks

We first give an example to show that Theorem 1.7 can fail if an ideal in $R = k[x, y, z]$ has more than three generators.

Example 4.1. Let $J \subseteq R$ be the ideal of all forms vanishing on five general points in $\mathbb{P}^2$. Then J is generated by a conic and two cubics, so the degree three piece of J has five generators. If we take four generic elements $f_1, f_2, f_3, f_4 \in J_3$, then one can show the following:

- $I = \langle f_1, f_2, f_3, f_4 \rangle$ and J define the same subscheme of $\mathbb{P}^2$. It follows that I is an lci of codimension two.
- Not every syzygy on f_1, f_2, f_3, f_4 which vanishes at the basepoints is Koszul.

So three generators is the number needed to make things work for codimension two ideals of $R = k[x, y, z]$.

In proving Theorem 1.7, we showed that the Hilbert polynomial of the Koszul syzygies K is determined by the degrees of the generators (see (2.4)), while we need more subtle information in order to compute the Hilbert polynomial of the syzygies V vanishing at the basepoints (see (2.5)). The miracle is that if I is a three generated ideal of codimension two, then exactly the right numerology occurs so that $V = K$ is equivalent to being a local complete intersection.

It makes sense to ask if a similar "numerology" occurs in higher dimensions. A natural place to start would be to consider ideals

$$I = \langle f_1, \ldots, f_{n+1} \rangle \subset k[x_0, \ldots, x_n]$$

whose base point locus has codimension n. In the local case, such ideals are called *almost complete intersections*. It would be interesting to see if the results of Sections 2 and 3 extend to this situation.

Finally, we should mention that evidence for Theorem 1.7 was provided by many Macaulay2 computations. Macaulay2 is available at the URL

`http://www.math.uiuc.edu/Macaulay2/`.

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Department of Mathematics and Computer Science, Amherst College, Amherst, Massachusetts 01002-5000

E-mail address: dac@cs.amherst.edu

Department of Mathematics, Harvard University, Cambridge, Massachusetts 02138

Current address: Department of Mathematics, Texas A&M University, College Station, Texas 77843

E-mail address: schenck@math.tamu.edu