

REGULAR COMPONENTS OF MODULI SPACES OF STABLE MAPS

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ABSTRACT. We construct regular components of the moduli space of stable maps from curves of genus g to a product of two projective spaces. These components are generically smooth and have the expected dimension predicted by deformation theory. This result can be seen as a general position theorem for loci in M_g consisting of curves carrying exceptional linear series.

1. INTRODUCTION

The purpose of this note is to prove the existence of ‘nice’ components of the Hilbert scheme of curves $C \subseteq \mathbb{P}^1 \times \mathbb{P}^r$ of genus $g \geq 2$ and bidegree (k, d) . We can also phrase our result using the Kontsevich moduli space of stable maps to $\mathbb{P}^1 \times \mathbb{P}^r$. We work over an algebraically closed field of characteristic zero.

For a smooth projective variety Y and a class $\beta \in H_2(Y, \mathbb{Z})$, one considers the moduli stack $\overline{\mathcal{M}}_g(Y, \beta)$ of stable maps $f : C \rightarrow Y$, with C a reduced connected nodal curve of genus g and $f_*([C]) = \beta$ (see [FP] for the construction of these stacks). The open substack $\mathcal{M}_g(Y, \beta)$ of $\overline{\mathcal{M}}_g(Y, \beta)$ parametrizes maps from smooth curves to Y . By $\overline{M}_g(Y, \beta)$ we denote the coarse moduli space corresponding to the stack $\overline{\mathcal{M}}_g(Y, \beta)$ and similarly \overline{M}_g is the moduli space corresponding to the stack $\overline{\mathcal{M}}_g$ of stable curves of genus g . We denote by $\pi : \overline{\mathcal{M}}_g(Y, \beta) \rightarrow \overline{M}_g$ the natural projection. The *expected dimension* of the stack $\overline{\mathcal{M}}_g(Y, \beta)$ is

$$\chi(g, Y, \beta) = \dim(Y) (1 - g) + 3g - 3 - \beta \cdot K_Y.$$

Since in general the geometry of $\overline{\mathcal{M}}_g(Y, \beta)$ is quite messy (e.g. existence of many components, some nonreduced and/or not of expected dimension), it is not obvious what the definition of a nice component of $\overline{\mathcal{M}}_g(Y, \beta)$ should be. Following Sernesi [Se] we introduce the following terminology:

Definition. A component V of $\overline{\mathcal{M}}_g(Y, \beta)$ is said to be *regular* if it is generically smooth and of dimension $\chi(g, Y, \beta)$. We say that V has the *expected number of moduli* if

$$\dim \pi(V) = \min(3g - 3, \chi(g, Y, \beta) - \dim \text{Aut}(Y)).$$

In this paper we only construct regular components of moduli spaces of stable maps. We study the stacks $\overline{\mathcal{M}}_g(Y, \beta)$ when $Y = \mathbb{P}^1 \times \mathbb{P}^r$, $r \geq 3$ and $\beta = (k, d) \in H_2(\mathbb{P}^1 \times \mathbb{P}^r, \mathbb{Z})$. We denote by $\rho(g, r, d) = g - (r + 1)(g - d + r)$ the *Brill-Noether*

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number governing the existence of \mathfrak{g}_d^r 's on curves of genus g . Our main result is the following theorem.

Theorem 1. *Let g, r, d and k be positive integers with $r \geq 3, \rho(g, r, d) < 0$ and*

$$(2 - \rho(g, r, d))r + 2 \leq k \leq (g + 2)/2.$$

Then there exists a regular component of the stack of maps $\mathcal{M}_g(\mathbb{P}^1 \times \mathbb{P}^r, (k, d))$.

We introduce the Brill-Noether locus $M_{g,d}^r = \{[C] \in M_g : C \text{ has a } \mathfrak{g}_d^r\}$, in the case $\rho(g, r, d) < 0$. The expected codimension of $M_{g,d}^r$ inside M_g is $-\rho(g, r, d)$. We view Theorem 1 as a tool in the study of the relative position of the loci $M_{g,k}^1$ and $M_{g,d}^r$ when $r \geq 3, \rho(g, 1, k) < 0 \Leftrightarrow k < (g + 2)/2$ and $\rho(g, r, d) < 0$. The stack $\mathcal{M}_g(\mathbb{P}^1 \times \mathbb{P}^r, (k, d))$ comes naturally into play when looking at the intersection in M_g of the loci $M_{g,k}^1$ and $M_{g,d}^r$. In such a setting, if V is a regular component of $M_g(\mathbb{P}^1 \times \mathbb{P}^r, (k, d))$, then $M_{g,k}^1$ and $M_{g,d}^r$ intersect properly along $\pi(V)$. It is very plausible that one has a similar statement to Theorem 1 when $\rho(g, r, d) \geq 0$ and/or $\rho(g, 1, k) \geq 0$. However, from our perspective that seems of less interest because it would essentially be a statement about linear series on the general curve of genus g with no implications on the problem of understanding the geography of the Brill-Noether loci inside M_g .

Regarding the problem of existence of regular components of $\mathcal{M}_g(Y, \beta)$, so far the spaces $\mathcal{M}_g(\mathbb{P}^r, d)$ have received the bulk of attention. When $r = 1, 2$ the problem boils down to the study of the Hurwitz scheme and of the Severi variety of plane curves which are known to be irreducible and regular. For $r \geq 3$ we have the following result of Sernesi (cf. [Se, p. 26]):

Proposition 1.1. *For all g, r, d such that $d \geq r + 1$ and*

$$-\frac{g}{r} + \frac{r+1}{r} \leq \rho(g, r, d) < 0,$$

there exists a regular component V of $\mathcal{M}_g(\mathbb{P}^r, d)$ which has the expected number of moduli. A general point of V corresponds to an embedding $C \hookrightarrow \mathbb{P}^r$ by a complete linear system (i.e. $h^0(C, \mathcal{O}_C(1)) = r + 1$), the normal bundle N_C satisfies $H^1(C, N_C) = 0$ and the Petri map

$$\mu_0(C) : H^0(C, \mathcal{O}_C(1)) \otimes H^0(C, K_C(-1)) \rightarrow H^0(C, K_C)$$

is surjective.

A. Lopez has obtained significant improvements on the range of g, r, d such that there exists a regular component of $\mathcal{M}_g(\mathbb{P}^r, d)$: if $h(r) = (4r^3 + 8r^2 - 9r + 3)/(r + 3)$, then for all g, r, d such that $-(2 - 6/(r + 3))g + h(r) \leq \rho(g, r, d) < 0$ there exists a regular component of $\mathcal{M}_g(\mathbb{P}^r, d)$ with the expected number of moduli (cf. [Lo]).

When Y is a smooth surface, methods from [AC] can be employed to show that if V is a component of $\mathcal{M}_g(Y, \beta)$ with $\dim(V) \geq g + 1$ and contains a point $[f : C \rightarrow Y]$ with $\deg(f) = 1$ (i.e. f is generically injective), then V is regular. Here it is crucial that the normal sheaf N_f is of rank 1, as then the Clifford Theorem provides an easy criterion for the vanishing of $H^1(C, N_f)$, which turns out to be a sufficient criterion for regularity (see Section 2).

2. DEFORMATIONS OF MAPS AND SMOOTHINGS OF SPACE CURVES

We review some facts about deformations of maps and smoothings of reducible nodal curves in \mathbb{P}^r . Our references are [Ran] and [Se].

We start by describing the deformation theory of maps between complex algebraic varieties when the source is (possibly) singular and the target is smooth. Let $f : X \rightarrow Y$ be a morphism between complex projective varieties, with Y being smooth. We denote by $\text{Def}(X, f, Y)$ the space of first-order deformations of the map f when X and Y are not considered fixed. The space of first-order deformations of X (resp. Y) is denoted by $\text{Def}(X)$ (resp. $\text{Def}(Y)$). We have the standard identification $\text{Def}(X) = \text{Ext}^1(\Omega_X, \mathcal{O}_X)$. The deformation space $\text{Def}(X, f, Y)$ fits in the following exact sequence:

$$(1) \quad \text{Hom}_{\mathcal{O}_X}(f^*\Omega_Y, \mathcal{O}_X) \longrightarrow \text{Def}(X, f, Y) \longrightarrow \text{Def}(X) \oplus \text{Def}(Y) \longrightarrow \text{Ext}_f^1(\Omega_Y, \mathcal{O}_X).$$

The second arrow is given by the natural forgetful maps, the space

$$\text{Hom}_{\mathcal{O}_X}(f^*\Omega_Y, \mathcal{O}_X) = H^0(X, f^*T_Y)$$

parametrizes first-order deformations of $f : X \rightarrow Y$ when both X and Y are fixed, while for A, B , respectively \mathcal{O}_X and \mathcal{O}_Y -modules, $\text{Ext}_f^i(B, A)$ denotes the derived functor of $\text{Hom}_f(B, A) = \text{Hom}_{\mathcal{O}_X}(f^*B, A) = \text{Hom}_{\mathcal{O}_Y}(B, f_*A)$. Under reasonable assumptions (trivially satisfied when f is a finite map between nodal curves), one has that $\text{Ext}_f^1(\Omega_Y, \mathcal{O}_X) = \text{Ext}^1(f^*\Omega_Y, \mathcal{O}_X)$. Using (1) it follows that when X is smooth and irreducible and Y is rigid (e.g. a product of projective spaces), $\text{Def}(X, f, Y) = H^0(X, N_f)$, where $N_f = \text{Coker}\{T_X \rightarrow f^*T_Y\}$ is the normal sheaf of the map f .

For a smooth variety Y , a class $\beta \in H_2(Y, \mathbb{Z})$ and a point $[f : C \rightarrow Y] \in \mathcal{M}_g(Y, \beta)$, we have that $T_{[f]}(\overline{\mathcal{M}}_g(Y, \beta)) = H^0(C, N_f)$. If moreover $\text{deg}(f) = 1$ and $H^1(C, N_f) = 0$, then every class in $H^0(C, N_f)$ is unobstructed, f is an immersion (cf. [AC, Lemma 1.4]) and $\overline{\mathcal{M}}_g(Y, \beta)$ is smooth and of the expected dimension at the point $[f]$, that is, $[f]$ belongs to a regular component of $\overline{\mathcal{M}}_g(Y, \beta)$.

Let $C \subseteq \mathbb{P}^r$ be a stable curve of genus g and degree d . If \mathcal{I}_C is the ideal sheaf of C we denote by $N_C := \text{Hom}(\mathcal{I}_C/\mathcal{I}_C^2, \mathcal{O}_C)$ the normal sheaf of C in \mathbb{P}^r . Assume that $H^1(C, N_C) = 0$ and that $h^0(C, \mathcal{O}_C(1)) = r + 1$, that is, C is embedded by a complete linear system. The differential of the map $\pi : \overline{\mathcal{M}}_g(\mathbb{P}^r, d) \rightarrow \overline{\mathcal{M}}_g$ at the point $[C \hookrightarrow \mathbb{P}^r]$ is given by the natural map $H^0(C, N_C) \rightarrow \text{Ext}^1(\Omega_C, \mathcal{O}_C)$. If ω_C denotes the dualizing sheaf of C , then $\text{rk}(d\pi)_{[C \hookrightarrow \mathbb{P}^r]} = 3g - 3 - \dim \text{Ker} \mu_0(C)$, where

$$\mu_0(C) : H^0(C, \mathcal{O}_C(1)) \otimes H^0(C, \omega_C(-1)) \rightarrow H^0(C, \omega_C)$$

is the Petri map. In particular $(d\pi)_{[C \hookrightarrow \mathbb{P}^r]}$ has rank $3g - 3 + \rho(g, r, d)$ if and only if $\mu_0(C)$ is surjective.

In the same setting, via the standard identification

$$T_{[C]}(\overline{\mathcal{M}}_g)^\vee = H^0(C, \omega_C \otimes \Omega_C),$$

the annihilator $(\text{Im}(d\pi)_{[C \hookrightarrow \mathbb{P}^r]})^\perp \subseteq H^0(C, \omega_C \otimes \Omega_C)$ can be naturally identified with $\text{Im}(\mu_1(C))$, where

$$\mu_1(C) : \text{Ker} \mu_0(C) \rightarrow H^0(C, \Omega_C \otimes \omega_C)$$

is the Gaussian map obtained from taking the ‘derivative’ of $\mu_0(C)$ (cf. [CGGH, p. 163]).

In Section 3 we will smooth curves $X \subseteq \mathbb{P}^r$ which are unions of two smooth curves C and E meeting quasi-transversally (i.e. having distinct tangent lines) at a finite set Δ . For such a curve one has the exact sequences (cf. [Se, p. 35])

$$(2) \quad 0 \longrightarrow \mathcal{O}_E(-\Delta) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0$$

and

$$(3) \quad 0 \longrightarrow \Omega_E \longrightarrow \omega_X \longrightarrow \Omega_C(\Delta) \longrightarrow 0.$$

Also in Section 3 we will use an inductive procedure to construct curves $C \subseteq \mathbb{P}^1 \times \mathbb{P}^r$ with $H^1(C, N_{C/\mathbb{P}^1 \times \mathbb{P}^r}) = 0$. The induction step uses the following result (cf. [BE, Lemma 2.3]):

Proposition 2.1. *Let $C \subseteq \mathbb{P}^r$ be a smooth curve with $H^1(C, N_C) = 0$. We take $r + 2$ points $p_1, \dots, p_{r+2} \in C$ in general linear position and a smooth rational curve $E \subseteq \mathbb{P}^r$ of degree r which meets C quasi-transversally at p_1, \dots, p_{r+2} . Then $X = C \cup E$ is smoothable in \mathbb{P}^r and $H^1(X, N_X) = 0$.*

3. EXISTENCE OF REGULAR COMPONENTS OF $\mathcal{M}_g(\mathbb{P}^1 \times \mathbb{P}^r, (k, d))$

In this section we prove the existence of regular components of $\overline{\mathcal{M}}_g(\mathbb{P}^1 \times \mathbb{P}^r, (k, d))$ in the case $k \geq r + 2, d \geq r \geq 3$, and $\rho(g, r, d) < 0$. We achieve this by constructing smooth curves $C \subseteq \mathbb{P}^1 \times \mathbb{P}^r$ of bidegree (k, d) satisfying $H^1(C, N_{C/\mathbb{P}^1 \times \mathbb{P}^r}) = 0$.

Let us fix integers $g \geq 2, d \geq r \geq 3$ and $k \geq 2$, as well as a smooth curve C of genus g with maps $f_1 : C \rightarrow \mathbb{P}^1, f_2 : C \rightarrow \mathbb{P}^r$, such that $\deg(f_1) = k, \deg(f_2(C)) = d$ and f_2 is generically injective. Let us denote by $f : C \rightarrow \mathbb{P}^1 \times \mathbb{P}^r$ the product map. As usual we denote by $G_d^r(C)$ the scheme parametrizing \mathfrak{g}_d^r 's on C .

There is a commutative diagram of exact sequences:

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & & & T_C & & \\
 & & & & \downarrow & & \\
 0 & \longrightarrow & T_C & \longrightarrow & f^*(T_{\mathbb{P}^1 \times \mathbb{P}^r}) & \longrightarrow & N_f \longrightarrow 0 \\
 & & \downarrow & & \downarrow = & & \downarrow \\
 0 & \longrightarrow & T_C \oplus T_C & \longrightarrow & f_1^*(T_{\mathbb{P}^1}) \oplus f_2^*(T_{\mathbb{P}^r}) & \longrightarrow & N_{f_1} \oplus N_{f_2} \longrightarrow 0 \\
 & & & & & & \downarrow \\
 & & & & & & 0
 \end{array}$$

By taking cohomology in the last column, we see that the condition $H^1(C, N_f) = 0$ is equivalent to $H^1(C, N_{f_1}) = 0$ (automatic), $H^1(C, N_{f_2}) = 0$, and

$$(4) \quad \text{Im}\{\delta_1 : H^0(C, N_{f_1}) \rightarrow H^1(C, T_C)\} + \text{Im}\{\delta_2 : H^0(C, N_{f_2}) \rightarrow H^1(C, T_C)\} = H^1(C, T_C),$$

where δ_1 and δ_2 are coboundary maps. Condition (4) is equivalent (cf. Section 2) to

$$(5) \quad (d\pi_1)_{[f_1]} (T_{[f_1]}(\mathcal{M}_g(\mathbb{P}^1, k))) + (d\pi_2)_{[f_2]} (T_{[f_2]}(\mathcal{M}_g(\mathbb{P}^r, d))) = T_{[C]}(\mathcal{M}_g),$$

where the projections $\pi_1 : \mathcal{M}_g(\mathbb{P}^1, k) \rightarrow \mathcal{M}_g$ and $\pi_2 : \mathcal{M}_g(\mathbb{P}^r, d) \rightarrow \mathcal{M}_g$ are the natural forgetful maps. Slightly abusing terminology, if C is a smooth curve and $(l_1, l_2) \in G_k^1(C) \times G_d^r(C)$ is a pair of base point free linear series on C , we say that

(C, l_1, l_2) satisfies (5) if (C, f_1, f_2) satisfies (5), where f_1 and f_2 are maps associated to l_1 and l_2 .

Recall that a base point free pencil \mathfrak{g}_k^1 is said to be *simple* if the induced covering $f : C \rightarrow \mathbb{P}^1$ has a single ramification point x over each branch point and moreover $e_x(f) = 2$.

We prove the existence of regular components of $\mathcal{M}_g(\mathbb{P}^1 \times \mathbb{P}^r, (k, d))$ using the following inductive procedure:

Proposition 3.1. *Fix positive integers g, r, d and k with $d \geq r \geq 3, k \geq r + 2$ and $\rho(g, r, d) < 0$. Let us assume that $C \subseteq \mathbb{P}^r$ is a smooth nondegenerate curve of degree d and genus g , such that $h^1(C, \mathcal{N}_C) = 0, h^0(C, \mathcal{O}_C(1)) = r + 1$ and the Petri map*

$$\mu_0(C) = \mu_0(C, \mathcal{O}_C(1)) : H^0(C, \mathcal{O}_C(1)) \otimes H^0(C, \mathcal{K}_C(-1)) \rightarrow H^0(C, \mathcal{K}_C)$$

is surjective. Assume furthermore that C possesses a simple base point free pencil \mathfrak{g}_k^1 say l , such that $|\mathcal{O}_C(1)|(-l) = \emptyset$ and $(C, l, |\mathcal{O}_C(1)|)$ satisfies (5).

Then there exists a smooth nondegenerate curve $Y \subseteq \mathbb{P}^r$ with $g(Y) = g + r + 1, \deg(Y) = d + r$ and a simple base point free pencil $l' \in G_k^1(Y)$, so that Y enjoys exactly the same properties: $h^1(Y, \mathcal{N}_Y) = 0, h^0(Y, \mathcal{O}_Y(1)) = r + 1$, the Petri map $\mu_0(Y)$ is surjective, $|\mathcal{O}_Y(1)|(-l') = \emptyset$ and $(Y, l', |\mathcal{O}_Y(1)|)$ satisfies (5).

Proof. We first construct a reducible k -gonal nodal curve $X \subseteq \mathbb{P}^r$, with $p_a(X) = g + r + 1, \deg(X) = d + r$, having all the required properties. Then we prove that X can be smoothed in \mathbb{P}^r preserving all of the properties that we want.

Let $f_1 : C \rightarrow \mathbb{P}^1$ be the degree k map corresponding to the pencil l . The covering f_1 is simple, hence the monodromy of f_1 is the full symmetric group. Then since $|\mathcal{O}_C(1)|(-l) = \emptyset$, we have that for a general $\lambda \in \mathbb{P}^1$ the fibre $f_1^{-1}(\lambda) = p_1 + \dots + p_k$ consists of k distinct points in general linear position. Let $\Delta = \{p_1, \dots, p_{r+2}\}$ be a subset of $f_1^{-1}(\lambda)$ and let $E \subseteq \mathbb{P}^r$ be a rational normal curve ($\deg(E) = r$) passing through p_1, \dots, p_{r+2} . (Through any $r + 3$ points in general linear position in \mathbb{P}^r , there passes a unique rational normal curve.) We set $X := C \cup E$, with C and E meeting quasi-transversally at Δ . Of course $p_a(X) = g + r + 1$ and $\deg(X) = d + r$. Note that $\rho(g, r, d) = \rho(g + r + 1, r, d + r)$.

We first prove that $[X] \in \overline{\mathcal{M}}_{g+r+1, k}^1$ (that is, X is a limit of smooth k -gonal curves), by constructing an admissible covering of degree k having as its domain a curve X' , stably equivalent to X . Let $X' := X \cup D_{r+3} \cup \dots \cup D_k$, where $D_i \simeq \mathbb{P}^1$ and $D_i \cap X = \{p_i\}$, for $i = r + 3, \dots, k$. Take $Y := (\mathbb{P}^1)_1 \cup_\lambda (\mathbb{P}^1)_2$ a union of two lines identified at λ . We construct a degree k admissible covering $f' : X' \rightarrow Y$ as follows: take $f'|_C = f_1 : C \rightarrow (\mathbb{P}^1)_1, f'|_E = f_2 : E \rightarrow (\mathbb{P}^1)_2$ a map of degree $r + 2$ sending the points p_1, \dots, p_{r+2} to λ , and finally $f'|_{D_i} : D_i \simeq (\mathbb{P}^1)_2$ isomorphisms sending p_i to λ . Clearly f' is an admissible covering, so X , which is stably equivalent to X' , is a k -gonal curve.

Let us now consider the space $\overline{\mathcal{H}}_{g+r+1, k}$ of Harris-Mumford admissible coverings of degree k (cf. [HM]) and denote by $\pi_1 : \overline{\mathcal{H}}_{g+r+1, k} \rightarrow \overline{\mathcal{M}}_{g+r+1}$ the natural projection which sends a covering to the stable model of its source. We assume for simplicity that $\text{Aut}(C) = \{Id_C\}$ which implies that $\text{Aut}(f') = \{Id_{X'}\}$, so $[f']$ is a smooth point of $\overline{\mathcal{H}}_{g+r+1, k}$. In the case when C has nontrivial automorphisms the argument carries through without change if we replace the space of admissible coverings with the space of twisted covers of Abramovich, Corti and Vistoli (cf. [ACV]).

We compute the differential of the map π_1 at $[f']$. We have $T_{[f']}(\overline{\mathcal{H}}_{g+r+1,k}) = \text{Def}(X', f', Y) = \text{Def}(X, f, Y)$, where $f = f'_X : X \rightarrow Y$. The differential $(d\pi_1)_{[f']}$ is the forgetful map $\text{Def}(X, f, Y) \rightarrow \text{Def}(X)$ and from the sequence (1) we get that $\text{Im}(d\pi_1)_{[f']} = u_1^{-1}(\text{Im } u_2)$, where $u_1 : \text{Def}(X) \rightarrow \text{Ext}^1(f^*\Omega_Y, \mathcal{O}_X)$ and $u_2 : \text{Def}(Y) \rightarrow \text{Ext}^1(f^*\Omega_Y, \mathcal{O}_X)$ are the dual maps of $u_1^\vee : H^0(X, \omega_X \otimes f^*\Omega_Y) \rightarrow H^0(X, \omega_X \otimes \Omega_X)$ and $u_2^\vee : H^0(X, \omega_X \otimes f^*\Omega_Y) \rightarrow H^0(Y, \omega_Y \otimes \Omega_Y)$. Here u_2^\vee is induced by the trace map $\text{tr} : f_*\omega_X \rightarrow \omega_Y$. Starting with the exact sequence on X ,

$$0 \longrightarrow \text{Tors}(\omega_X \otimes \Omega_X) \longrightarrow \omega_X \otimes \Omega_X \longrightarrow \Omega_C^{\otimes 2}(\Delta) \oplus \Omega_E^{\otimes 2}(\Delta) \longrightarrow 0,$$

we can write the following commutative diagram of sequences:

$$\begin{array}{ccccc} 0 & & 0 & & 0 \\ \downarrow & & \downarrow & & \downarrow \\ H^0(\text{Tors}(\omega_X \otimes f^*\Omega_Y)) & \hookrightarrow & H^0(\omega_X \otimes f^*\Omega_Y) & \rightarrow & H^0(2K_C - R_1 + \Delta) \oplus H^0(2K_E - R_2 + \Delta) \\ \downarrow (u_1^\vee)_{\text{tors}} & & \downarrow u_1^\vee & & \downarrow \\ H^0(\text{Tors}(\omega_X \otimes \Omega_X)) & \hookrightarrow & H^0(\omega_X \otimes \Omega_X) & \rightarrow & H^0(2K_C + \Delta) \oplus H^0(2K_E + \Delta) \end{array}$$

where R_1 (resp. R_2) is the ramification divisor of the map f_1 (resp. f_2). Taking into account that $H^0(E, 2K_E - R_2 + \Delta) = 0$ and that

$$H^0(Y, \omega_Y \otimes \Omega_Y) = H^0(\text{Tors}(\omega_Y \otimes \Omega_Y)),$$

we obtain

$$(6) \quad \text{Im}(d\pi_1)_{[f']} = (H^0(C, 2K_C - R_1 + \Delta) \oplus \text{Ker}(u_2^\vee)_{\text{tors}})^\perp,$$

where $(u_2^\vee)_{\text{tors}} : H^0(\text{Tors}(\omega_X \otimes f^*\Omega_Y)) \rightarrow H^0(\text{Tors}(\omega_Y \otimes \Omega_Y))$ is the restriction of u_2^\vee . The space $\text{Ker}(u_2^\vee)_{\text{tors}}$ is just a hyperplane in $H^0(\text{Tors}(\omega_X \otimes f^*\Omega_Y)) \simeq \mathbb{C}^{r+2}$.

Intermezzo. If we also assume that $\rho(g, 1, k) < 0$ and that $[C]$ is a smooth point of $M_{g,k}^1$ (which happens precisely when $\text{Aut}(C) = \{Id_C\}$, C has exactly one \mathfrak{g}_k^1 and $\dim|2\mathfrak{g}_k^1| = 2$), then we can prove that the locus $\overline{M}_{g+r+1,k}^1$ is smooth at $[X]$ as well. Indeed, since $\Delta \in C_{r+2}$ was chosen generically in a fibre of the \mathfrak{g}_k^1 on C , from Riemann-Roch we have that $h^0(C, 2K_C - R_1 + \Delta) = g - 2k + 3 + r = \text{codim}(\overline{M}_{g+r+1,k}^1, \overline{M}_{g+r+1})$. The fibre over $[X]$ of the map $\pi_1 : \overline{\mathcal{H}}_{g+r+1,k} \rightarrow \overline{M}_{g+r+1}$ is identified with the space of degree $r + 1$ maps $f_2 : E \rightarrow \mathbb{P}^1$ such that $f_2(p_1) = \dots = f_2(p_{r+2}) = \lambda$, hence it is $r + 1$ dimensional. We compute the tangent cone

$$TC_{[X]}(\overline{M}_{g+r+1,k}^1) = \bigcup \{ \text{Im}(d\pi_1)_z : z \in \pi_1^{-1}([X]) \} = H^0(C, 2K_C - R_1 + \Delta)^\perp,$$

which shows that $[X]$ is a smooth point of the locus $\overline{M}_{g+r+1,k}^1$.

We now compute the differential

$$(d\pi_2)_{[X]} : T_{[X]}(\text{Hilb}_{d+r,g+r+1,r}) \rightarrow T_{[X]}(\overline{\mathcal{M}}_{g+r+1}),$$

which is the same thing as the differential at the point $[X \hookrightarrow \mathbb{P}^r]$ of the projection $\pi_2 : \overline{\mathcal{M}}_{g+r+1}(\mathbb{P}^r, d+r) \rightarrow \overline{\mathcal{M}}_{g+r+1}$. We start by noticing that X is smoothable in \mathbb{P}^r and that $H^1(X, N_X) = 0$ (apply Proposition 2.1). We also have that X is embedded in \mathbb{P}^r by a complete linear system, that is, $h^0(X, \mathcal{O}_X(1)) = r + 1$. Indeed, on one hand, since X is nondegenerate, $h^0(X, \mathcal{O}_X(1)) \geq h^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r(1)}) = r + 1$; on the other hand from the sequence (2) we have that $h^0(X, \mathcal{O}_X(1)) \leq h^0(C, \mathcal{O}_C(1)) = r + 1$.

If X is embedded in \mathbb{P}^r by a complete linear system, we know (cf. Section 2) that

$$\text{Im}(d\pi_2)_{[X]} = (\text{Im}\mu_1(X))^\perp,$$

where $\mu_1(X) : \text{Ker}\mu_0(X) \rightarrow H^0(X, \omega_X \otimes \Omega_X)$ is the ‘derivative’ of the Petri map $\mu_0(X) : H^0(X, \mathcal{O}_X(1)) \otimes H^0(X, \omega_X(-1)) \rightarrow H^0(X, \omega_X)$. We compute the kernel of $\mu_0(X)$ and show that $\mu_0(X)$ is surjective in a way that resembles the proof of Proposition 2.3 in [Se].

From the sequence (3) we obtain $H^0(X, \omega_X) = H^0(C, K_C + \Delta)$, while from (2) we have that $H^0(X, \mathcal{O}_X(1)) = H^0(E, \mathcal{O}_E(1))$ (keeping in mind that $H^0(C, \mathcal{O}_C(1)(-\Delta)) = 0$, as p_1, \dots, p_{r+2} are in general linear position). Finally, using (3) again, we have that $H^0(X, \omega_X(-1)) = H^0(C, K_C(-1) + \Delta)$. Therefore we can write the following commutative diagram:

$$\begin{array}{ccc} H^0(C, \mathcal{O}_C(1)) \otimes H^0(C, K_C(-1)) & \xrightarrow{\mu_0(C)} & H^0(C, K_C) \\ \downarrow & & \downarrow \\ H^0(C, \mathcal{O}_C(1)) \otimes H^0(C, K_C(-1) + \Delta) & \longrightarrow & H^0(C, K_C + \Delta) \\ \downarrow = & & \downarrow = \\ H^0(X, \mathcal{O}_X(1)) \otimes H^0(X, \omega_X(-1)) & \xrightarrow{\mu_0(X)} & H^0(X, \omega_X) \end{array}$$

It follows that $\text{Ker}\mu_0(C) \subseteq \text{Ker}\mu_0(X)$. By Corollary 1.6 from [CR], our assumptions ($\mu_0(C)$ surjective and $\text{card}(\Delta) \geq 4$) enable us to conclude that $\mu_0(X)$ is surjective, too. Then $\text{Ker}\mu_0(C) = \text{Ker}\mu_0(X)$ for dimension reasons, hence also $\text{Im}\mu_1(X) = \text{Im}\mu_1(C) \subseteq H^0(C, 2K_C) \subseteq H^0(X, \omega_X \otimes \Omega_X)$. We thus get that $\text{Im}(d\pi_2)_{[X]} = (\text{Im}\mu_1(X))^\perp = (\text{Im}\mu_1(C))^\perp$.

The assumption that (C, f_1, f_2) satisfies (5) can be rewritten by passing to duals as

$$\begin{aligned} H^0(C, 2K_C - R_1)^\perp + (\text{Im}\mu_1(C))^\perp &= H^1(C, T_C) \\ \iff H^0(C, 2K_C - R_1) \cap \text{Im}\mu_1(C) &= 0. \end{aligned}$$

Then it follows that $\text{Im}\mu_1(X) \cap (H^0(C, 2K_C - R_1 + \Delta) \oplus \text{Ker}((u_2^\vee)_{\text{tors}})) = 0$, which is the same thing as

$$(7) \quad \begin{aligned} &(d\pi_1)_{[f']} (T_{[f']}(\overline{\mathcal{H}}_{g+r+1,k})) + (d\pi_2)_{[X \hookrightarrow \mathbb{P}^r]} (T_{[X \hookrightarrow \mathbb{P}^r]}(\overline{\mathcal{M}}_{g+r+1}(\mathbb{P}^r, d+r))) \\ &= \text{Ext}^1(\Omega_X, \mathcal{O}_X). \end{aligned}$$

This means that the images of $\overline{\mathcal{H}}_{g+r+1,k}$ under the map π_1 and of $\overline{\mathcal{M}}_{g+r+1}(\mathbb{P}^r, d+r)$ under the map π_2 , meet transversally at the point $[X] \in \overline{\mathcal{M}}_{g+r+1}$.

Claim. The curve X can be smoothed in such a way that the \mathfrak{g}_k^1 and the very ample \mathfrak{g}_{d+r}^r are preserved (while (7) is an open condition on $\overline{\mathcal{H}}_{g+r+1,k} \times \overline{\mathcal{M}}_{g+r+1}(\mathbb{P}^r, d+r)$).

Indeed, the tangent directions that fail to smooth at least one node of X are those in $\bigcup_{i=1}^{r+2} H^0(\text{Tors}_{p_i}(\omega_X \otimes \Omega_X))^\perp$, whereas the tangent directions that preserve both the \mathfrak{g}_k^1 and the \mathfrak{g}_{d+r}^r are those in

$$((\text{Im}\mu_1(C) + H^0(C, 2K_C - R_1 + \Delta)) \oplus \text{Ker}(u_2^\vee)_{\text{tors}})^\perp.$$

Since $H^0(\text{Tors}_{p_i}(\omega_X \otimes \Omega_X)) \not\subseteq \text{Ker}(u_2^\vee)_{\text{tors}}$ for $i = 1, \dots, r + 2$, by moving in a suitable direction in the tangent space at $[f']$ of $\pi_1^{-1}\pi_2(\overline{\mathcal{M}}_{g+r+1}(\mathbb{P}^r, d+r))$, we

finally obtain a smooth curve $Y \subseteq \mathbb{P}^r$ with $g(Y) = g + r + 1, \deg(Y) = d + r$ and satisfying all the required properties. \square

Using the previous result together with Proposition 1.1 we now construct regular components of $\mathcal{M}_g(\mathbb{P}^1 \times \mathbb{P}^r, (k, d))$.

Theorem 1. *Let g, r, d and k be positive integers such that $r \geq 3, \rho(g, r, d) < 0$ and*

$$(2 - \rho(g, r, d))r + 2 \leq k \leq (g + 2)/2.$$

Then there exists a regular component of the stack of maps $\mathcal{M}_g(\mathbb{P}^1 \times \mathbb{P}^r, (k, d))$.

Proof. All integer solutions (g_0, d_0) of the equation $\rho(g_0, r, d_0) = \rho(g, r, d)$ with $g_0 \leq g$ and $d_0 \leq d$ are of the form $g_0 = g - a(r + 1)$ and $d_0 = d - ar$ with $a \geq 0$. Using our numerical assumptions, by a routine check we find that there exists $a > 0$ such that $g_0 = g - a(r + 1) > 0, d_0 = d - ar \geq r + 1, k \geq g_0 + 1$ and

$$-\frac{g_0}{r} + \frac{r + 1}{r} \leq \rho(g_0, r, d_0) < 0.$$

By Proposition 1.1 there exists a smooth curve $C_0 \subseteq \mathbb{P}^r$ of genus g_0 and degree d_0 , with $H^1(C_0, N_{C_0/\mathbb{P}^r}) = 0,$

$h^0(C_0, \mathcal{O}_{C_0}(1)) = r + 1$ and $\mu_0(C_0)$ surjective. Moreover, since $k \geq g_0 + 1,$ there exists an open dense subset $U \subseteq \text{Pic}^k(C_0)$ such that for each $L_1 \in U$ there exists a pencil $l_1 = (L_1, V_1) \in G_k^1(C_0)$ with $V_1 \in \text{Gr}(2, H^0(C_0, L))$, such that l_1 is simple and base point free (cf. [Fu, Proposition 8.1]).

We denote by $\pi_1 : \mathcal{M}_{g_0}(\mathbb{P}^1, k_0) \rightarrow \mathcal{M}_{g_0}$ the natural projection and by $f_1 : C \rightarrow \mathbb{P}^1$ the map corresponding to l_1 . By Riemann-Roch we have $H^1(C_0, L_1^{\otimes 2}) = 0,$ hence using Section 2 $(d\pi_1)_{[f_1]} : T_{[f_1]}(\mathcal{M}_{g_0}(\mathbb{P}^1, k)) \rightarrow T_{[C_0]}(\mathcal{M}_{g_0})$ is surjective since $\text{Coker}(d\pi_1)_{[f_1]} = H^1(C_0, f_1^* T_{\mathbb{P}^1}) = 0.$ It follows that $(C_0, |\mathcal{O}_{C_0}(1)|, l_1)$ satisfies (5).

We claim that if $L \in U$ is general, then $|\mathcal{O}_{C_0}(1) \otimes L^\vee| = \emptyset.$ Suppose not, that is, $\mathcal{O}_{C_0}(1) \otimes L^\vee \in W_{d_0-k}(C_0)$ for a general $L \in \text{Pic}^k(C_0).$ This is possible only for $d_0 - k \geq g_0,$ hence

$$r + 2 \leq k \leq d_0 - g_0 < r \text{ (because } \rho(g_0, r, d_0) = \rho(g, r, d) < 0),$$

a contradiction. Thus $(C_0, |\mathcal{O}_{C_0}(1)|, l_1)$ satisfies all conditions required by Proposition 3.1 which we can now apply a times to get a smooth curve $C \subseteq \mathbb{P}^1 \times \mathbb{P}^r$ of genus g and bidegree (k, d) such that $H^1(C, N_{C/\mathbb{P}^1 \times \mathbb{P}^r}) = 0.$ The conclusion of Theorem 1 now follows. \square

In the special case $\rho(g, r, d) = -1$ we can extend the range of possible g, r, d and k for which there is a regular component:

Theorem 2. *Let g, r, d, k be positive integers such that $r \geq 3, \rho(g, r, d) = -1$ and*

$$\frac{2r^2 + r + 1}{r - 1} \leq k \leq \frac{g + 2}{2}.$$

Then there exists a regular component of the stack of maps $\mathcal{M}_g(\mathbb{P}^1 \times \mathbb{P}^r, (k, d))$.

Proof. We find a solution $(g_0 = g - a(r + 1), d_0 = d - ar)$ of the equation $\rho(g_0, r, d_0) = \rho(g, r, d)$ with $a \in \mathbb{Z}_{\geq 0}$ such that $d_0 \geq k + r$ and $\rho(g_0, 1, k) \geq r - 1.$ Our numerical assumptions ensure that such an $a \geq 0$ exists. Note that in this case $k \leq g_0 + 1,$ so we are not in the situation covered by Theorem 1.

It also follows that $-\frac{g_0}{r} + \frac{r+1}{r} \leq -1 = \rho(g_0, r, d_0)$ and $d_0 \geq r + 1$, hence by Proposition 1.1 there exists an irreducible smooth open subset U of $\mathcal{M}_{g_0}(\mathbb{P}^r, d_0)$ of the expected dimension, such that all points of U correspond to embeddings of smooth curves $C \hookrightarrow \mathbb{P}^r$, with $h^1(C, \mathcal{N}_C) = 0$, $h^0(C, \mathcal{O}_C(1)) = r + 1$ and $\mu_0(C)$ surjective.

Since we are in the case $\rho(g_0, r, d_0) = -1$, a combination of results by Eisenbud, Harris and Steffen gives that the Brill-Noether locus M_{g_0, d_0}^r is an irreducible divisor in M_{g_0} (see [St, Theorem 0.2]). It follows that the natural projection $\pi_2 : U \rightarrow M_{g_0, d_0}^r$ is dominant.

To apply Proposition 3.1 we now find a curve $[C_0] \in M_{g_0, d_0}^{r_0}$ having a complete base point free \mathfrak{g}_k^1 such that $2\mathfrak{g}_k^1$ is nonspecial. Then by semicontinuity we get that the general $[C] \in U$ also possesses a pencil \mathfrak{g}_k^1 with these properties. To find one particular such curve we proceed as follows: take C_0 a general $(r + 1)$ -gonal curve of genus g_0 . These curves will have rather few moduli ($r + 1 < [(g + 3)/2]$) but we still have that $[C_0] \in M_{g_0, d_0}^r$. Indeed, according to [CM] we can construct a $\mathfrak{g}_{d_0}^r = |\mathfrak{g}_{r+1}^1 + F|$ on C_0 , where F is an effective divisor on C_0 with $h^0(C_0, F) = 1$. Since $k \leq g_0$, using Corollary 2.2.3 from [CKM] we find that C_0 also carries a complete base point free \mathfrak{g}_k^1 , not composed with the \mathfrak{g}_{r+1}^1 computing $\text{gon}(C_0)$, and such that $2\mathfrak{g}_k^1$ is nonspecial. Since these are open conditions, they will hold generically along a component of $G_k^1(C_0)$. Applying semicontinuity, for a general element $[C] \in M_{g_0, d_0}^r$ (hence also for a general element $[C] \in U$), the variety $G_k^1(C)$ will contain a component A with general point $l \in A$ being complete, base point free and with $2l$ nonspecial.

We claim that for a general $l \in A$ we have that $|\mathcal{O}_C(1)|(-l) = \emptyset$. Suppose not. Then if we denote by $V_{d_0-k}^{r-1}(|\mathcal{O}_C(1)|)$ the variety of effective divisors of degree $d_0 - k$ on C imposing $\leq r - 1$ conditions on $|\mathcal{O}_C(1)|$, we obtain

$$\dim V_{d_0-k}^{r-1}(|\mathcal{O}_C(1)|) \geq \dim A \geq \rho(g_0, 1, k) \geq r - 1.$$

Therefore $C \subseteq \mathbb{P}^r$ has at least ∞^{r-1} $(d_0 - k)$ -secant $(r - 2)$ -planes, hence also at least ∞^{r-1} r -secant $(r - 2)$ -planes (because $d_0 - k \geq r$). This last statement contradicts the Uniform Position Theorem (see [ACGH, p. 112]), hence the general point $[C] \in U$ enjoys all properties required to make Proposition 3.1 work. \square

Remark. From the proof of Theorem 2 the following question appears naturally: let us fix g, k such that $g/2 + 1 \leq k \leq g$. One knows (cf. [ACGH]) that if $l \in G_k^1(C)$ is a complete, base point free pencil, then $\dim T_l(G_k^1(C)) = \rho(g, 1, k) + h^1(C, 2l)$. Therefore if A is a component of $G_k^1(C)$ such that $\dim A = \rho(g, 1, k)$ and the general $l \in A$ is base point free such that $2l$ is special, then A is nonreduced. Then what is the dimension of the locus

$$V_{g,k} := \{[C] \in M_g : \text{every component of } G_k^1(C) \text{ is nonreduced}\}?$$

A result of Coppens (cf. [Co]) says that for a curve C , if the scheme $W_k^1(C)$ is reduced and of dimension $\rho(g, 1, k)$, then the scheme $W_{k+1}^1(C)$ is reduced, too, and of dimension $\rho(g, 1, k + 1)$. It would then make sense to determine $\dim(V_{g,k})$ when $\rho(g, 1, k) \in \{0, 1\}$ (depending on the parity of g). We suspect that $V_{g,k}$ depends on very few moduli and if g is suitably large we expect that $V_{g,k} = \emptyset$.

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