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RESIDUES FOR AKIZUKI'S ONE-DIMENSIONAL LOCAL DOMAIN

I-CHIAU HUANG AND JAN-LI LIN

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Abstract. For a one-dimensional local domain C_M constructed by Akizuki, we find residue maps which give rise to a local duality. The completion of C_M is described using these residue maps.

Injective hulls of a given module are all isomorphic. For this reason, people often speak of *the* injective hull to indicate its "*uniqueness*". However, isomorphisms between these injective hulls are not canonical. In fact, they are a part of the structure of the given module. For instance, a local duality for a power series ring [2, (5.9)] is interpreted as an isomorphism between two injective hulls—one given by local cohomology and another by continuous homomorphisms. This isomorphism is induced by a residue map, which was not observed from the viewpoint of "*uniqueness*" of injective hulls. In this article, our philosophy is taken up again by a Noetherian local ring C_M constructed by Akizuki [1]. Although C_M behaves beyond geometric expectation, we can still define certain maps, which give rise to a local duality as an identification of local cohomology classes and continuous homomorphisms. These maps, also called residue maps, determine all endomorphisms of an injective hull of the residue field of C_M. So we are able to describe the completion of C_M.

We recall Akizuki's construction. Let A be a discrete valuation ring with the maximal ideal $\mathfrak{m} = tA$, let $\hat{A}$ be its completion, and let K (resp. $\hat{K}$) be the quotient field of A (resp. $\hat{A}$). Assume that there is an element

$$z = a_0 + a_1 t^{n_1} + a_2 t^{n_2} + \cdots \in \hat{A} \qquad (a_i \in A \setminus \mathfrak{m})$$

transcendental over A with the condition

$$n_r \geq 2n_{r-1} + 2 \qquad (r \geq 1)$$

on exponents, where $n_0 = 0$. Let

$$z_r = a_r + a_{r+1} t^{n_{r+1} - n_r} + \cdots \qquad (r \geq 0)$$

and

$$C = A[t(z_0 - a_0), \{(z_i - a_i)^2\}_{i=0}^{\infty}].$$

C_M is defined to be the localization of C at the maximal ideal M generated by t and $t(z_0 - a_0)$. Akizuki [1] showed that C_M is a one-dimensional Noetherian local domain, whose normalization is not a finite C_M-module.

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The quotient field of C_M is $K(z)$, which equals $(C_M)_t$ as t is a system of parameter of C_M. We can use the exact sequence

$$0 \to C_M \xrightarrow{\text{localization}} (C_M)_t \to \mathrm{H}^1_{MC_M}(C_M) \to 0 \tag{1}$$

to describe elements of the first local cohomology module $\mathrm{H}^1_{MC_M}(C_M)$ of C_M supported at the maximal ideal of C_M: For $f \in C_M$ and $n > 0$, the generalized fraction

$$\begin{bmatrix} f \\ t^n \end{bmatrix}_{C_M}$$

is defined as the image of f/t^n in $\mathrm{H}^1_{MC_M}(C_M)$ under the map in (1). The description of top local cohomology modules by generalized fractions is essentially used in the concrete realizations of Grothendieck duality [2, 4, 5]. It can be applied to combinatorial analysis [3, 6]. See also [9] for an alternate treatment to generalized fractions.

For $f \in \hat{A}$ and $n > 0$, we denote by

$$\begin{bmatrix} f \\ t^n \end{bmatrix}_A \tag{2}$$

the image of f/t^n in $\hat{K}/\hat{A}$ under the map in the exact sequence

$$0 \to \hat{A} \xrightarrow{\text{inclusion}} \hat{K} \to \hat{K}/\hat{A} \to 0.$$

We remark that there is a canonical isomorphism between $\hat{K}/\hat{A}$ and $\mathrm{H}^1_{\mathfrak{m}\hat{A}}(\hat{A})$, with which the representation (2) of elements in $\hat{K}/\hat{A}$ and the representation of elements in $\mathrm{H}^1_{\mathfrak{m}\hat{A}}(\hat{A})$ by generalized fractions agree.

Vanishing of elements in $\hat{K}/\hat{A}$ and $\mathrm{H}^1_{MC_M}(C_M)$ can be described in terms of ideal membership:

$$\begin{bmatrix} f \\ t^n \end{bmatrix}_A = 0 \qquad \left(\text{resp. } \begin{bmatrix} f \\ t^n \end{bmatrix}_{C_M} = 0\right)$$

if and only if f is contained in the ideal of $\hat{A}$ (resp. C_M) generated by t^n. For instance,

$$\begin{bmatrix} t(z_0 - a_0) \\ t \end{bmatrix}_A = 0$$

but

$$\begin{bmatrix} t(z_0 - a_0) \\ t \end{bmatrix}_{C_M} \neq 0.$$

The canonical map $K/A \to \hat{K}/\hat{A}$ is an isomorphism. We use the notation in (2) to represent elements in K/A under this isomorphism.

Lemma 1. *Every element of* $\mathrm{H}^1_{MC_M}(C_M)$ *can be written as*

$$\begin{bmatrix} X + Yt(z_0 - a_0) \\ t^n \end{bmatrix}_{C_M}$$

for some $X, Y \in A$ *and* $n > 0$.

Proof. It suffices to show that, for any specified $n > 0$, any element $f \in C_M$ can be written as

$$f = X + Yt(z_0 - a_0) + t^n Z \tag{3}$$

with $X, Y \in A$ and $Z \in C_M$. Write f as

$$f = \frac{f_1}{1 - f_2}$$

for some $f_1 \in C$ and $f_2 \in M$. Then

$$f - f_1(1 + f_2 + f_2^2 + \cdots + f_2^n) = \frac{f_1 f_2^{n+1}}{1 - f_2} \in M^{n+1} C_M.$$

Since $M^{n+1} C_M \subset t^n C_M$,

$$f - f_1(1 + f_2 + f_2^2 + \cdots + f_2^n) = t^n Z_2$$

for some $Z_2 \in C_M$. By [8, Section 9.5, equation (5)], there exist $X, Y \in A$ and $Z_1 \in C$ such that

$$f_1(1 + f_2 + f_2^2 + \cdots + f_2^n) = X + Yt(z_0 - a_0) + t^n Z_1.$$

The representation

$$f = X + Yt(z_0 - a_0) + t^n (Z_1 + Z_2)$$

is of the required form. □

Lemma 2. *If* $X, Y \in A$ *and*

$$\left[\begin{array}{c} X + Yt(z_0 - a_0) \\ t^n \end{array} \right]_{C_M} = 0,$$

then $X, Y \in t^n A$.

Proof.

$$X + Yt(z_0 - a_0) = t^n Z$$

for some $Z \in C_M$. Write $X = t^\ell X_1$ and $Y = t^m Y_1$ with invertible $X_1, Y_1 \in A$. If $\ell \leq m$, then $n \leq \ell$, otherwise $X_1 = -t^{m-\ell} Y_1 t(z_0 - a_0) + t^{n-\ell} Z \in t\hat{A}$. If $\ell > m$, then $m \geq n$, otherwise $t(z_0 - a_0) = -t^{\ell - m} X_1 Y_1^{-1} + t^{n-m} Z Y_1^{-1} \in tC_M$. In either case, $X, Y \in t^n A$. □

With these lemmas, we are able to define the following map for any $\sigma, \rho \in \hat{A}$.

Definition 3.

$$\mathrm{res}_{\sigma,\rho} \colon \mathrm{H}^1_{MC_M}(C_M) \to K/A$$

is defined to be the A-linear map given by

$$\left[\begin{array}{c} X + Yt(z_0 - a_0) \\ t^n \end{array} \right]_{C_M} \mapsto \left[\begin{array}{c} X\sigma + Y\rho \\ t^n \end{array} \right]_A.$$

Adopting the terminology of [2], we call $\mathrm{res}_{\sigma,\rho}$ a residue map. Let

$$\mathrm{Hom}^c_A(C_M, K/A) = \{\varphi \in \mathrm{Hom}_A(C_M, K/A) \mid \varphi(M^n C_M) = 0 \text{ for some } n\}$$

be the C_M-module of continuous homomorphism. As a special case of J. Lipman's result [2, Proposition 3.4], $\mathrm{Hom}^c_A(C_M, K/A)$ is an injective hull of the residue field of C_M. Note that $t^{n+1} C_M \subset M^{n+1} C_M \subset t^n C_M$. Hence an A-linear map $\varphi : C_M \to K/A$ is continuous if and only if $\varphi(t^n C_M) = 0$ for some n. Using the representation

(3) of elements of C_M, we see that a continuous homomorphism is determined by an integer n with which $t^n C_M$ is in the kernel and by its values at 1 and $t(z_0 - a_0)$.

Definition 4.
$$\Phi_{\sigma,\rho}\colon \mathrm{H}^1_{MC_M}(C_M) \to \mathrm{Hom}^c_A(C_M, K/A)$$
is defined to be the C_M-linear map given by
$$\Phi_{\sigma,\rho}(\omega)(f) = \mathrm{res}_{\sigma,\rho}(f\omega),$$
where $\omega \in \mathrm{H}^1_{MC_M}(C_M)$ and $f \in C_M$.

Local Duality. If ρ is invertible, then $\Phi_{\sigma,\rho}$ is an isomorphism.

Proof. The inverse map of $\Phi_{\sigma,\rho}$ can be written explicitly. Let
$$s_r := a_1 t^{n_1} + a_2 t^{n_2} + \cdots + a_r t^{n_r} \in A.$$
Then we have
$$t(z_0 - a_0) = t^{n_r+1}(z_r - a_r) + ts_r$$
and

(4) $$t^2(z_0 - a_0)^2 + t^2 s_r^2 - 2ts_r t(z_0 - a_0) = t^{2n_r+2}(z_r - a_r)^2 \in t^r C_M.$$

Given $\varphi \in \mathrm{Hom}^c_A(C_M, K/A)$ with $\varphi(t^r C_M) = 0$ and
$$\begin{cases} \varphi(1) = \begin{bmatrix} \alpha \\ t^r \end{bmatrix}_A, \\ \varphi(t(z_0 - a_0)) = \begin{bmatrix} \beta \\ t^r \end{bmatrix}_A, \end{cases}$$
the system of equations
$$\begin{cases} \begin{bmatrix} X\sigma + Y\rho \\ t^r \end{bmatrix}_A = \begin{bmatrix} \alpha \\ t^r \end{bmatrix}_A, \\ \begin{bmatrix} X\rho - Yt^2 s_r^2 \sigma + 2Yts_r\rho \\ t^r \end{bmatrix}_A = \begin{bmatrix} \beta \\ t^r \end{bmatrix}_A \end{cases}$$
can be solved by choosing $X, Y \in A$ such that
$$\begin{cases} X - \dfrac{\alpha t s_r(\sigma t s_r - 2\rho) + \beta\rho}{(\rho - ts_r\sigma)^2} \in t^r \hat{A}, \\ Y - \dfrac{\alpha\rho - \beta\sigma}{(\rho - ts_r\sigma)^2} \in t^r \hat{A}. \end{cases}$$
We define
$$\Phi^{-1}(\varphi) := \begin{bmatrix} X + Yt(z_0 - a_0) \\ t^r \end{bmatrix}_{C_M},$$
which is independent of the choices of r, α, β, X or Y. Then
$$\Phi_{\sigma,\rho}(\Phi^{-1}(\varphi))(1) = \begin{bmatrix} \alpha \\ t^r \end{bmatrix}_A$$

and

$$\begin{aligned} & \Phi_{\sigma,\rho}(\Phi^{-1}(\varphi))(t(z_0-a_0)) \\ = \; & \operatorname{res}_{\sigma,\rho}\left[\begin{array}{c} Xt(z_0-a_0)-Yt^2s_r^2+2Yts_rt(z_0-a_0) \\ t^r \end{array}\right]_{C_M} \quad \text{(by (4))} \\ = \; & \left[\begin{array}{c} \beta \\ t^r \end{array}\right]_A . \end{aligned}$$

Hence $\Phi_{\sigma,\rho}(\Phi^{-1}(\varphi)) = \varphi$. For any $\omega \in \mathrm{H}^1_{MC_M}(C_M)$, it is also straightforward to check that $\Phi^{-1}(\Phi_{\sigma,\rho}(\omega)) = \omega$. So Φ^{-1} is indeed the inverse of $\Phi_{\sigma,\rho}$. □

Corollary 5. *C_M is Gorenstein.*

Proof. $\mathrm{H}^1_{MC_M}(C_M)$ is injective. Hence (1) is a finite injective resolution of C_M. □

We remark that a Noetherian local ring R whose maximal ideal is generated by $1 + \operatorname{depth} R$ elements is always Gorenstein [7, p. 163, Exercise 1].

Proposition 6. *Any C_M-linear map*

$$\Phi\colon \mathrm{H}^1_{MC_M}(C_M) \to \operatorname{Hom}^c_A(C_M, K/A)$$

equals $\Phi_{\sigma,\rho}$ for some $\sigma, \rho \in \hat{A}$.

Proof. For each n, there exist $\sigma_n, \rho_n \in A$ such that

$$\begin{cases} \left[\begin{array}{c} \sigma_n \\ t^n \end{array}\right]_A = \Phi\left(\left[\begin{array}{c} 1 \\ t^n \end{array}\right]_{C_M}\right)(1), \\ \\ \left[\begin{array}{c} \rho_n \\ t^n \end{array}\right]_A = \Phi\left(\left[\begin{array}{c} 1 \\ t^n \end{array}\right]_{C_M}\right)(t(z_0-a_0)). \end{cases}$$

Since $\sigma_n - \sigma_{n+1}$ and $\rho_n - \rho_{n+1}$ are contained in t^nA, the limits

$$\begin{cases} \sigma = \lim_{n\to\infty} \sigma_n, \\ \rho = \lim_{n\to\infty} \rho_n \end{cases}$$

exist in $\hat{A}$. For any $X, Y \in A$,

$$\Phi\left(\left[\begin{array}{c} X+Yt(z_0-a_0) \\ t^n \end{array}\right]_{C_M}\right)(1) = \left[\begin{array}{c} X\sigma_n + Y\rho_n \\ t^n \end{array}\right]_A = \left[\begin{array}{c} X\sigma + Y\rho \\ t^n \end{array}\right]_A .$$

Hence

$$(\Phi(\omega))(f) = \Phi(f\omega)(1) = \Phi_{\sigma,\rho}(f\omega)(1) = (\Phi_{\sigma,\rho}(\omega))(f)$$

for any $\omega \in \mathrm{H}^1_{MC_M}(C_M)$ and $f \in C_M$. That is, $\Phi = \Phi_{\sigma,\rho}$. □

Now we fix a σ_0 and an invertible ρ_0. All endomorphisms of the C_M-module $\mathrm{H}^1_{MC_M}(C_M)$ are of the form $\Phi^{-1}_{\sigma_0,\rho_0} \circ \Phi_{\sigma,\rho}$. Since different pairs of σ and ρ determine different C_M-linear maps $\Phi_{\sigma,\rho}$, the completion $\widehat{C_M}$ of C_M can be described as the set $\{\Phi_{\sigma,\rho}\}_{\sigma,\rho\in\hat{A}}$ with addition

$$\Phi_{\sigma_1,\rho_1} + \Phi_{\sigma_2,\rho_2} = \Phi_{\sigma_1+\sigma_2,\rho_1+\rho_2},$$

unit Φ_{σ_0,ρ_0}, and multiplication

$$\Phi_{\sigma_1,\rho_1} * \Phi_{\sigma_2,\rho_2} = \Phi_{\sigma_1,\rho_1} \circ \Phi^{-1}_{\sigma_0,\rho_0} \circ \Phi_{\sigma_2,\rho_2}$$

given by composition of endomorphisms. If $\sigma_0 = 0$ and $\rho_0 = 1$, then

$$\Phi_{\sigma_1,\rho_1} * \Phi_{\sigma_2,\rho_2} = \Phi_{\sigma_1\rho_2+\sigma_2\rho_1-2\sigma_1\sigma_2 t(z_0-a_0),\rho_1\rho_2-\sigma_1\sigma_2 t^2(z_0-a_0)^2}.$$

Identifying $\Phi_{\sigma,\rho}$ with $\rho+\sigma X$ in $\hat{A}[X]/(X+t(z-a_0))^2$, and comparing their additions and multiplications, we get the following description of $\widehat{C_M}$.

Corollary 7. $\widehat{C_M} \simeq \hat{A}[X]/(X+t(z-a_0))^2$.

For $\rho \in C_M$, the endomorphism $\Phi_{0,1}^{-1} \circ \Phi_{0,\rho}$ of $\mathrm{H}^1_{MC_M}(C_M)$ is multiplication by ρ. Therefore, with respect to the isomorphism in Corollary 7, the embedding $C_M \to \hat{A}[X]/(X+t(z-a_0))^2$ of completion is the composition

$$C_M \xrightarrow{\text{inclusion}} \hat{A} \xrightarrow{\text{canonical}} \hat{A}[X]/(X+t(z-a_0))^2.$$

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Institute of Mathematics, Academia Sinica, Nankang, Taipei 11529, Taiwan, Republic of China

E-mail address: ichuang@math.sinica.edu.tw

Institute of Information Science, Academia Sinica, Nankang, Taipei 11529, Taiwan, Republic of China

E-mail address: ljl@iis.sinica.edu.tw