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ON STRONG CONVERGENCE TO COMMON FIXED POINTS OF NONEXPANSIVE SEMIGROUPS IN HILBERT SPACES

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ABSTRACT. In this paper, we prove the following strong convergence theorem: Let C be a closed convex subset of a Hilbert space H. Let $\{T(t) : t \geq 0\}$ be a strongly continuous semigroup of nonexpansive mappings on C such that $\bigcap_{t\geq 0} F\big(T(t)\big) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying $0 < \alpha_n < 1$, $t_n > 0$ and $\lim_n t_n = \lim_n \alpha_n/t_n = 0$. Fix $u \in C$ and define a sequence $\{u_n\}$ in C by $u_n = (1-\alpha_n)T(t_n)u_n + \alpha_n u$ for $n \in \mathbb{N}$. Then $\{u_n\}$ converges strongly to the element of $\bigcap_{t\geq 0} F\big(T(t)\big)$ nearest to u.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{R}_+$ the sets of positive integers, rational numbers, real numbers and nonnegative real numbers, respectively. Let C be a closed convex subset of a Hilbert space H. A mapping T on C is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. We know that $F(T)$ is nonempty if C is bounded; see [1]. Fix $u \in C$. Then for each α with $(0,1)$, there exists a unique point x_α of C satisfying $x_\alpha = (1-\alpha)Tx_\alpha + \alpha u$ because the mapping $x \mapsto (1-\alpha)Tx + \alpha u$ is contractive. In 1967, Browder [3] proved the following:

Theorem 1 (Browder [3])**.** *Let C be a closed convex subset of a Hilbert space H and let T be a nonexpansive mapping on C with a fixed point. Let $\{\alpha_n\}$ be a sequence of $(0,1)$ converging to 0. Fix $u \in C$ and define a sequence $\{u_n\}$ by*

$$u_n = (1-\alpha_n)Tu_n + \alpha_n u$$

for $n \in \mathbb{N}$. Then $\{u_n\}$ converges strongly to the element of $F(T)$ nearest to u.

Let $\{T(t) : t \in \mathbb{R}_+\}$ be a strongly continuous semigroup of nonexpansive mappings on a closed convex subset C of a Hilbert space H, i.e.,

(1) for each $t \in \mathbb{R}_+$, $T(t)$ is a nonexpansive mapping on C;
(2) $T(0)x = x$ for all $x \in C$;
(3) $T(s+t) = T(s) \circ T(t)$ for all $s, t \in \mathbb{R}_+$;
(4) for each $x \in X$, the mapping $T(\cdot)x$ from $\mathbb{R}_+$ into C is continuous.

We put $F(\mathcal{T}) = \bigcap_{t\in\mathbb{R}_+} F\big(T(t)\big)$. We know that $F(\mathcal{T})$ is nonempty if C is bounded; see [2]. The following theorem is a corollary of Theorem 8 in [9].

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Theorem 2 (Shioji and Takahashi [9]). *Let C be a closed convex subset of a Hilbert space H. Let $\{T(t) : t \in \mathbb{R}_+\}$ be a strongly continuous semigroup of nonexpansive mappings on C such that $F(\mathcal{T}) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying $0 < \alpha_n < 1$, $\lim_n \alpha_n = 0$, $t_n > 0$ and $\lim_n t_n = \infty$. Fix $u \in C$ and define a sequence $\{u_n\}$ in C by*

$$u_n = (1 - \alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds + \alpha_n u$$

for $n \in \mathbb{N}$. Then $\{u_n\}$ converges strongly to the element of $F(\mathcal{T})$ nearest to u.

In this paper, motivated by the above results, we prove another strong convergence theorem for a strongly continuous semigroup of nonexpansive mappings.

2. Main result

It is well known that all Hilbert spaces satisfy Opial's condition.

Proposition (Opial [5]). *Let H be a Hilbert space. If $\{x_n\}$ is a sequence in H and converges weakly to $z_0 \in H$, then $\liminf_n \|x_n - z_0\| < \liminf_n \|x_n - z\|$ for all $z \in H$ with $z \neq z_0$.*

Now we prove our main result.

Theorem 3. *Let C be a closed convex subset of a Hilbert space H. Let $\{T(t) : t \in \mathbb{R}_+\}$ be a strongly continuous semigroup of nonexpansive mappings on C such that $F(\mathcal{T}) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying $0 < \alpha_n < 1$, $t_n > 0$ and $\lim_n t_n = \lim_n \alpha_n/t_n = 0$. Fix $u \in C$ and define a sequence $\{u_n\}$ in C by*

$$u_n = (1 - \alpha_n)T(t_n)u_n + \alpha_n u$$

for $n \in \mathbb{N}$. Then $\{u_n\}$ converges strongly to the element of $F(\mathcal{T})$ nearest to u.

Proof. Let v be the element of $F(\mathcal{T})$ nearest to u. From

$$\begin{aligned}\|u_n - v\| &= \|(1 - \alpha_n)T(t_n)u_n + \alpha_n u - v\| \\ &\leq (1 - \alpha_n)\|T(t_n)u_n - v\| + \alpha_n\|u - v\| \\ &\leq (1 - \alpha_n)\|u_n - v\| + \alpha_n\|u - v\|\end{aligned}$$

we have $\|T(t_n)u_n - v\| \leq \|u_n - v\| \leq \|u - v\|$ for $n \in \mathbb{N}$. Therefore $\{u_n\}$ and $\{T(t_n)u_n\}$ are bounded. Let $\{u_{n_i}\}$ be an arbitrary subsequence of $\{u_n\}$. Then there exists a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ which converges weakly to x. We claim that $x \in F(\mathcal{T})$. Put $x_j = u_{n_{i_j}}$, $\beta_j = \alpha_{n_{i_j}}$ and $s_j = t_{n_{i_j}}$ for $j \in \mathbb{N}$. Fix $t > 0$. From

$$\begin{aligned}\|x_j - T(t)x\| \leq& \sum_{k=0}^{[t/s_j]-1} \|T((k+1)s_j)x_j - T(ks_j)x_j\| \\ &+ \|T([t/s_j]s_j)x_j - T([t/s_j]s_j)x\| + \|T([t/s_j]s_j)x - T(t)x\| \\ \leq& [t/s_j]\|T(s_j)x_j - x_j\| + \|x_j - x\| + \|T(t - [t/s_j]s_j)x - x\| \\ =& [t/s_j]\beta_j\|T(s_j)x_j - u\| + \|x_j - x\| + \|T(t - [t/s_j]s_j)x - x\| \\ \leq& t\beta_j/s_j\|T(s_j)x_j - u\| + \|x_j - x\| + \max\{\|T(s)x - x\| : 0 \leq s \leq s_j\}\end{aligned}$$

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for $j \in \mathbb{N}$, we have

$$\liminf_{j\to\infty} \|x_j - T(t)x\| \le \liminf_{j\to\infty} \|x_j - x\|.$$

By the Proposition, this implies $T(t)x = x$. Therefore $x \in F(\mathcal{T})$. We next prove $\{x_j\}$ converges strongly to v. From

$$\begin{aligned}&\beta_j\|x_j - v\|^2 + (1-\beta_j)\langle (x_j - T(s_j)x_j) - (v - T(s_j)v), x_j - v\rangle \\ &= \beta_j\langle u - v, x_j - v\rangle\end{aligned}$$

and

$$\begin{aligned}&\langle (x_j - T(s_j)x_j) - (v - T(s_j)v), x_j - v\rangle \\ &\ge \|x_j - v\|^2 - \|T(s_j)x_j - T(s_j)v\| \cdot \|x_j - v\| \\ &\ge 0,\end{aligned}$$

we obtain $\|x_j - v\|^2 \le \langle u - v, x_j - v\rangle$ for $j \in \mathbb{N}$. Since $\langle u - v, x - v\rangle \le 0$, we have

$$\begin{aligned}\|x_j - v\|^2 &\le \langle u - v, x_j - v\rangle \\ &= \langle u - v, x_j - x\rangle + \langle u - v, x - v\rangle \\ &\le \langle u - v, x_j - x\rangle\end{aligned}$$

for $j \in \mathbb{N}$ and hence $\{x_j\}$ converges strongly to v. Since $\{u_{n_i}\}$ is arbitrary, we obtain that $\{u_n\}$ converges strongly to v. □

We have some remarks about Theorem 3.

Remark. (1) By the proof of theorem 5.1 in [7], we can prove the following statement: Let E be a smooth uniformly convex Banach space with a duality mapping which is weakly sequentially continuous at zero, and let C be a closed convex subset of E. Let $\{T(t) : t \in \mathbb{R}_+\}$, $\{\alpha_n\}$, $\{t_n\}$, u and $\{u_n\}$ be as in Theorem 3. Then $\{u_n\}$ converges strongly to Pu, where P is the sunny nonexpansive retract from C onto $F(\mathcal{T})$.

(2) Halpern [4] proved the strong convergence theorem for a nonexpansive mapping by the explicit iteration. So, we have one problem of whether there is an explicit iteration concerning Theorem 3.

3. Appendix

In Theorem 3, it is needed that $T(\cdot)x$ is continuous for all $x \in C$. In this section, we give an example. By Axiom of Choice, there exist a subset $\mathbb{A}$ of $\mathbb{R}_+$ and a mapping θ from $\mathbb{R}_+$ onto $\mathbb{A}$ such that $\theta^{-1}(a) = (a + \mathbb{Q}) \cap \mathbb{R}_+$ for all $a \in \mathbb{A}$. Note that the following hold:

(1) $\theta(a) = a$ for all $a \in \mathbb{A}$;
(2) $\theta(t) - t \in \mathbb{Q}$ for all $t \in \mathbb{R}_+$;
(3) $\theta(s) = \theta(t)$ if and only if $s - t \in \mathbb{Q}$ for all $s, t \in \mathbb{R}_+$;
(4) $\theta(t + q) = \theta(t)$ and $\theta(\theta(s) + t) = \theta(s + t)$ for all $q \in \mathbb{Q} \cap \mathbb{R}_+$ and $s, t \in \mathbb{R}_+$.

Example. Let H be a Hilbert space consisting of all the functions x from $\mathbb{A}$ into $\mathbb{R}$ satisfying $\sum_{a\in\mathbb{A}} |x(a)|^2 < \infty$ with inner product $\langle x, y\rangle = \sum_{a\in\mathbb{A}} x(a) \cdot y(a)$ for all $x, y \in H$. Define a semigroup $\{T(t) : t \in \mathbb{R}_+\}$ of linear nonexpansive mappings on H by $(T(t)x)(a) = x(\theta(a + t))$ for all $x \in H$ and $a \in \mathbb{A}$. Fix $u \in H$ satisfying

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$u(\theta(0)) = 1$ and $u(a) = 0$ for all $a \in \mathbb{A}$ with $a \neq \theta(0)$. Define a sequence $\{u_n\}$ in H by

$$u_n = (1 - 1/n^2)T(1/n)u_n + (1/n^2)u$$

for $n \in \mathbb{N}$. Then $u_n = u$ for $n \in \mathbb{N}$ and u is not a common fixed point of $\{T(t) : t \in \mathbb{R}_+\}$.

Proof. We first show that the mapping τ_t on $\mathbb{A}$ defined by $\tau_t(a) = \theta(a+t)$ is bijective for each $t \in \mathbb{R}_+$. If $\theta(a + t) = \theta(b + t)$ for some $a, b \in \mathbb{A}$, then $(a + t) - (b + t) = a - b \in \mathbb{Q}$. So, we obtain $a = \theta(a) = \theta(b) = b$. For each $a \in \mathbb{A}$, we have

$$\tau_t(\theta(a - t + [t] + 1)) = \theta(\theta(a - t + [t] + 1) + t) = \theta(a + [t] + 1) = \theta(a) = a.$$

These imply that τ_t is bijective for each $t \in \mathbb{R}_+$. Hence, $T(t)$ is well-defined and isometric for each $t \in \mathbb{R}_+$. Note that $F(\mathcal{T}) = \{0\}$. Since

$$\begin{aligned}(T(s) \circ T(t)x)(a) &= (T(t)x)(\theta(a + s)) = x(\theta(\theta(a + s) + t)) \\ &= x(\theta(a + s + t)) = (T(s + t)x)(a)\end{aligned}$$

for $x \in H$ and $a \in \mathbb{A}$, we have $T(s + t) = T(s) \circ T(t)$ for all $s, t \in \mathbb{R}_+$. This shows that $\{T(t) : t \in \mathbb{R}_+\}$ is a semigroup. From $(T(q)x)(a) = x(\theta(a + q)) = x(a)$ for $a \in \mathbb{A}$, we have $T(q)x = x$ for all $q \in \mathbb{Q} \cap \mathbb{R}_+$ and $x \in H$. Especially, $T(0)x = x$ for all $x \in H$. However, for each $x \in H$ with $x \neq 0$, $T(\cdot)x$ is not continuous everywhere. Since $T(1/n)u_n = u_n$, we get $u_n = u$ for $n \in \mathbb{N}$. This completes the proof. □

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