ON ASYMMETRY OF THE FUTURE AND THE PAST FOR LIMIT SELF-JOININGS

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Abstract. Let $\triangle_T$ be an off-diagonal joining of a transformation $T$. We construct a non-typical transformation having asymmetry between limit sets of $\triangle_T^n$ for positive and negative powers of $T$. It follows from a correspondence between subpolymorphisms and positive operators, and from the structure of limit polynomial operators. We apply this technique to find all polynomial operators of degree 1 in the weak closure (in the space of positive operators on $L^2$) of powers of Chacon’s automorphism and its generalizations.

Introduction

In [1], D. Rudolph introduced the notion of joinings. This notion turned out to be very fruitful (see [2]-[6]). It is known that every automorphism or endomorphism can be characterized both as a measure on a graph of this map, i.e., a joining or, more generally, a polymorphism [7], and as an operator on $L^2$. This provides a way to study the structure of joinings by operator methods, and automorphisms by joinings.

It is not difficult to show that limit sets for positive and negative powers of a transformation $T$ are equal in the space of all automorphisms if $T$ is rigid. If $T$ is not rigid, then these sets are empty. Moreover, for rigid or mixing transformations (see Section 1) these limit sets are also equal in the space of all linear operators on $L^2$.

Nevertheless we prove (Theorem 2.4) that there exist rank-one transformations $T$ with different limit sets of off-diagonal joinings for positive and negative powers of $T$ or, in terms of operators, that the sets of limit operators are different.

We apply a new approach, via limit polynomials. This approach recently gave a solution to an old problem of Rokhlin (see [8], [9]) (see also references [10]-[13] about this problem) and answered some other well-known questions. See for example [8], which contains an answer to a question of Katok.

In Section 1 we introduce subpolymorphisms and a natural homeomorphism between the space of such measures and a subspace of positive operators on $L^2$. 

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Note that, for any transformation $T$, the space of its self-joinings lies in the space of subpolymorphisms.

Theorem 2.1 describes all possible limit linear polynomials in $T_{(k)}$ for powers of transformations $T_{(k)}$, where $T_{(k)}$ are constructed in Section 1. In particular, this implies that only one linear polynomial $1/2E + 1/2T$ is a limit of powers for the classical Chacon transformation $T$ (see [3, 14, 15]).

In Section 3 we show that the future and the past for limit self-joinings cannot be completely different for any automorphism. Moreover, their intersection contains an abelian semigroup that is trivial (i.e., $\{\mu \times \mu\}$) only for mixing transformations.

1. Basic definitions and notations

Let $T$ be a transformation defined on a non-atomic standard Borel probability space $(X, \mathcal{F}, \mu)$. A transformation and a unitary operator on $L_2(\mu) : Tf(x) = f(Tx)$ are often called automorphisms and denoted by the same symbol $T$. It is clear that $T$ is contained in the following space of positive operators on $L_2(\mu)$: $L^+ = \{U : (Uf \in L_2 \text{ if } f \in L_2) \& (Ug \geq 0 \text{ if } g \geq 0)\}$ equipped with the weak operator convergence. Everywhere below the identity automorphism will be denoted by $E$. The group of all automorphisms of $(X, \mathcal{F}, \mu)$, say $\text{Aut}(\mu)$, becomes a completely metrizable topological group when endowed with the weak operator convergence. Everywhere below the identity automorphism will be denoted by $E$. The group of all automorphisms of $(X, \mathcal{F}, \mu)$, say $\text{Aut}(\mu)$, becomes a completely metrizable topological group when endowed with the weak operator convergence of transformations $(T_n \to T)$ iff for any measurable $A$ we have $\mu(T_n(A) \Delta T(A)) \to 0$ as $n \to \infty$. Note that this topology is a restriction of the weak operator topology in $L^+$ to the non-closed $\text{Aut}(\mu)$. Denote by $C(T)$ the commutant of $T$, i.e., the set

$$\{S \in \text{Aut}(\mu) : ST = TS\}.$$ 

Let $\nu$ be a finite Borel measure on $X \times X$ with marginal measures, say $\pi_1 \nu$ and $\pi_2 \nu$, such that

$$\pi_i \nu(A) \leq \mu(A) \text{ for any } \mu\text{-measurable set } A.$$ 

Obviously, marginal measures are $\mu$-absolutely continuous and $\|\nu\| = \nu(X \times X) \leq 1$. The set of all such measures, say $M(\mu)$, is a convex compact metrizable space with respect to the topology determined by

$$\mu_n \to \mu_0 \iff \mu_n(A \times B) \to \mu_0(A \times B)$$

for any $\mu$-measurable sets $A$ and $B$. Now we shall give the following definition.

**Definition 1.1.** Each measure from $M(\mu)$ is called a subpolymorphism.

Fix $T \in \text{Aut}(\mu)$. The set $J_\epsilon(T, T)$ of $c$-self-joinings, i.e., $T \times T$-invariant elements $\nu$ of $M(\mu)$, with $\|\nu\| = c$, is a closed subspace of $M(\mu)$ for any $c \in [0, 1]$. If $T$ is ergodic, then each $\nu$ from $J_\epsilon(T, T)$ has $c\mu$ as marginal measures. This gives that $J(T, T) = J_1(T, T)$ is exactly the set of well-known self-joinings. Let $S \in C(T)$. As usual, by an off-diagonal joining $\Delta_S$ we mean a measure from $J(T, T)$ completely defined by $\Delta_S(A \times B) = \mu(S^{-1}A \cap B)$, where $A, B$ are any $\mu$-measurable sets. Denote by $LJ_+(T)$ and $LJ_-(T)$ the limit sets of $\Delta_T^n$ in $M(\mu)$ for all positive $n$ and negative $n$ respectively. It is clear that $LJ_+(T) \subseteq J(T, T)$. If $T$ is mixing, then $LJ_+(T)$ consist of only one point $\mu \times \mu$. It is well known that $T$ is weakly mixing iff $LJ_+(T)$ contain at least $\mu \times \mu$. We say $T$ is rigid if $T^{n_k} \to E$ for some sequence $n_k \to \infty$.

**Proposition 1.2.** $LJ_-(T) = LJ_+(T)$ for rigid $T$. 

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**Proposition 1.3.**  \( LJ_+(T^*) = LJ_-(T) = \sigma LJ_+(T), \) where \( \sigma \) is the flip map (i.e., \( \sigma(x, y) = (y, x) \)), and \( S^* \) denotes the operator adjoint to \( S \).

**Corollary 1.4.**  \( LJ_-(T) = LJ_+(T) \) iff \( LJ_+(T) \) is invariant with respect to \( \sigma \).

We leave simple proofs of the above statements to the reader. It seems to be well known that the set of rigid transformations is a dense \( G_\delta \)-set in \( \text{Aut}(\mu) \) (see [16] regarding close results in different subclasses of the set of non-singular transformations). It turns out that \( LJ_-(T) = LJ_+(T) \) for a typical transformation \( T \).

1.1. **Construction of \( T_{(k)} \).** We consider the following “generalized” Chacon’s automorphisms. For each \( k \geq 3 \), let \( T_{(k)} \) be a rank-one transformation, where each column \( C_{n+1} \) is obtained by cutting \( C_n \) into \( k \) subcolumns, say \( C_n(i) \), of equal width, placing a spacer only on the subcolumn \( C_n(k-1) \), and then stacking the subcolumn \( C_n(i+1) \) on top of \( C_n(i) \) for \( 1 \leq i < k \). It is clear that \( T_{(3)} \) is exactly Chacon’s automorphism. For the column \( C_n \), let \( h_n \) be its height and let \( d_n \) be the measure of its one level, where \( n \geq 1 \).

1.2. **Correspondence between positive operators and \( M(\mu) \).** Consider

\[
L_+^+ = \{ U \in L^+ : \int_A U(1)d\mu \leq \mu(A) \}
\]

with a restriction of the weak operator topology to this set. Obviously,

\[
L_+^+ = \{ U \in L^+ : U(1) \leq 1 \}
\]

**Proposition 1.5.**  The natural correspondence given by

\[
\langle U_\nu f, g \rangle = \int f \otimes \bar{g}d\nu,
\]

for any \( f, g \in L_2(\mu) \), defines a linear homeomorphism, say \( \phi \), between the topological spaces \( M(\mu) \) and \( L_+^+ \).

Note that in (1.2) \( f \) and \( U_\nu f \) are from different spaces \( L_2(\mu) \), but we naturally identify these spaces.

**Proof.**  Indeed, the right part of (1.2) is a bounded functional for every \( f \in L_2(\mu) \) because

\[
| \int f \otimes \bar{g}d\nu | \leq \| f \|_\nu \| g \|_\nu = \| f \|_{\pi_1, \nu} \| g \|_{\pi_2, \nu} \leq \| f \|_\mu \| g \|_\mu,
\]

where \( \| h \|_l \) means the norm of \( h \in L_2(l) \). Thus \( U_\nu f \in L_2(\mu) \) for each \( f \in L_2(\mu) \). The remaining properties of the map \( \phi \) are obvious.

Some properties of operators \( U_\nu \) were considered in [17] due to Vershik for polymorphisms \( \nu \), i.e., for elements of \( M(\mu) \) with exact equality in (1.1) for each \( A \).

**Remark 1.6.**  Clearly, \( \phi(\Delta_T) = T \), where \( T \in \text{Aut}(\mu) \). Also, \( U_\nu \) commutes with \( T \) if and only if \( \nu \in J_{\| \nu \|}(T, T) \).
Note that
\[ \int f \otimes g d\nu = \int_X g(y) d\pi_2 \nu \int_X f(x) d\nu_y(x) = \int_X g(y) \rho_2(y) d\mu(y) \int_X f(x) d\nu_y(x), \]
where \( \rho_2(y) \) is a density of \( \pi_2 \nu \) with respect to \( \mu \), and \( \nu_y(x) \) is a canonical system of conditional measures corresponding to \( \nu \). Hence, changing \( g \) to \( U_\nu f \) in (1.2) and (1.3), we get

**Corollary 1.7.** \( \mathcal{L}_\mu^+ \) is a compact convex metrizable space. For any subpolynomial \( \nu \in M(\mu) \), \( \| U_\nu \| \leq 1 \), and

\[ U_\nu(f) = \rho_2(y) \int_X f(x) d\nu_y(x). \]

**Remark 1.8.** The space \( \mathcal{L}_\mu^+ \) is a semigroup, and closed with respect to taking parts of operators, i.e., if \( 0 \leq V \leq U \) and \( U \in \mathcal{L}_\mu^+ \), then \( V \in \mathcal{L}_\mu^+ \).

**Remark 1.9.** It is readily seen that \( \phi(LJ_+ (T)) \) and \( \phi(LJ_- (T)) \) are \( T \)-invariant closed semigroups.

## 2. Limit Polynomials

The following theorem completely determines the simplest limit polynomial.

**Theorem 2.1.** Let \( \mathcal{P}_1[x] \) be the set of polynomials of degree at most 1. Then

\[ \phi(LJ_+ (T_{(k)})) \cap \mathcal{P}_1[T_{(k)}] = \left\{ \frac{1}{k-1}E + \frac{k-2}{k-1}T_{(k)} \right\}. \]

**Proof.** In order to prove Theorem 2.1 we need some definitions and a technical lemma. For clarity we restrict our attention to the case \( k = 4 \). The proof for \( k \neq 4 \) is analogous.

Let \( m_i \to +\infty \) as \( i \to +\infty \). Fix \( i \) and choose \( n \) such that \( h_n \leq m_i < h_{n+1} \). Consider the \((n+1)\)st column. Number its levels by \( 1, 2, \ldots, 4h_n + 1 \) from the base consequently. There exists \( i_0 \) such that the \((3h_n - m_i)\)th level corresponds to \( C_n(i_0) \) (we put \( i_0 = 4 \) in the remaining case \( m_i = 4h_n \)). Let \( p_i \) be a number of higher levels in \( C_n(i_0) \). Denote by \( B_i(j) \) the set of the top \( p_i \) levels of \( C_n(j) \) and \( A_i(j) = C_n(j) \setminus B_i(j) \) (\( A_i(j) \) or \( B_i(j) \) can be empty). Define \( O_i = \{ x \in C_{n+1} : T_{(4)}^r x \in C_{n+1} \} \) for \( r = 1, 2, \ldots, h_{n+1} \). It is clear that on each level a measure of \( x \) from \( O_i \) is \( 2/3d_{n+1} \).

Define operators

\[ Q_i = \chi_{A_i(i_0)} T_{(4)}^{m_i}, \quad R_i = \chi_{B_i(i_0)} T_{(4)}^{m_i}, \]
\[ Q'_i = \chi_{O_i} Q_i, \quad R'_i = \chi_{O_i} R_i. \]

Note that \( Q_i, Q'_i, R_i, R'_i \) are in \( \mathcal{L}_\mu^+ \) because they are less than \( T_{(4)}^{m_i} \).

**Lemma 2.2.**

\[ T_{(4)}^{m_i} - P_i \to 0 \text{ as } i \to +\infty, \]

where

1. \( P_i = (4/3T_{(4)}^{-1} + 8/3E)Q_i + (1/3T_{(4)}^{-1} + 2E + 5/3T_{(4)})R_i \) if \( h_n \leq m_i < 2h_n \).
2. \( P_i = (1/3T_{(4)}^{-2} + 2T_{(4)}^{-1} + 5/3E)Q_i + (2/3T_{(4)}^{-1} + 8/3E + 2/3T_{(4)})R_i \) if \( 2h_n \leq m_i < 3h_n \).
Therefore we can assume that \( f \) and \( g \) are constant, say \( f_n(j) \) and \( g_n(j) \), on each \( j \)th level of \( C_\bullet \) for sufficiently large \( n \).

Obviously, if \( \mu(D_i) \to 0 \), then \( \chi_D, T^{m_i} \to 0 \). Thus

\[
(2.2) \quad T^{m_i}_{(4)} - \sum_j \chi_{A_i(j)} T^{m_i}_{(4)} - \sum_j \chi_{B_i(j)} T^{m_i}_{(4)} \to 0.
\]

Next we will calculate the connection between components of each sum in (2.2), using that \( T^{m_i}_{(4)} \) has a “regular” structure on sets \( A_i(j) \) and \( B_i(j) \) and \( g \) is independent of \( j \). Consider the \( (n+1) \)st column.

1. If \( m_i < 2h_n \), then \( \iota_0 = 2 \). Clearly,

\[
\langle \chi_{A_i(1)} T^{m_i}_{(4)} f, g \rangle = d_{n+1} \sum_{l=1}^{h_n-p_i} f_n(l + p_i)\tilde{g}_n(l) = \langle Q_1 f, g \rangle,
\]

\[
\langle \chi_{B_i(1)} T^{m_i}_{(4)} f, g \rangle = d_{n+1} \sum_{l=h_n-p_i+1}^{h_n} f_n(l + p_i - h_n)\tilde{g}_n(l) = \langle T^{(4)} R_1 f, g \rangle.
\]

Here and next values of \( f \) and \( g \) can be written incorrectly on bases and tops of \( A_i(j) \) and \( B_i(j) \), but this fact is not essential for the convergence of such operators. It is clear that

\[
\chi_{A_i(1)} T^{m_i}_{(4)} - Q_1 \to 0,
\]

\[
\chi_{B_i(1)} T^{m_i}_{(4)} - T^{(4)} R_1 \to 0.
\]

Analogously,

\[
\chi_{A_i(3)} T^{m_i}_{(4)} - T^{-(4)} Q_1 \to 0.
\]

By construction of \( T^{(4)} \), the function \( f(T^{m_i}_{(4)} x) \) has at most two values on each level from \( B_i(3), A_i(4), B_i(4) \) up to two base levels of \( B_i(3) \). The first one is exactly at \( x \) from \( O_i \), and the second one is at the remaining part of this level. Indeed, fix some \( x \) from such a level. It is clear that \( N(T^{m_i}_{(4)} x) = N(x) + m_i \mod h_{n+1} \) for \( x \in O_i \), where \( N(y) \) means a number of the level having \( y \). The set \( B_i(3) \) starts from the \((h_{n+1} - m_i)\)th level. Thus \( \{x, T^{(4)} x, ..., T^{m_i}_{(4)} x\} \cap C_{n+1} \neq \emptyset \), if \( x \) is not in \( O_i \). Therefore \( N(T^{m_i}_{(4)} x) = N(x) + m_i - 1 \mod h_{n+1} \).

Thus

\[
\langle \chi_{B_i(3)} T^{m_i}_{(4)} f, g \rangle = \langle \chi_{B_i(3)} \cap O_i T^{m_i}_{(4)} f, g \rangle + \langle \chi_{B_i(3)} \setminus O_i T^{m_i}_{(4)} f, g \rangle
\]

\[
= \frac{2}{3} d_{n+1} \sum_{l=h_n-p_i+2}^{h_n} f_n(l + p_i - h_n - 1)\tilde{g}_n(l)
\]

\[
+ \frac{1}{3} d_{n+1} \sum_{l=h_n-p_i+3}^{h_n} f_n(l + p_i - h_n - 2)\tilde{g}_n(l)
\]

\[
= \frac{2}{3} \langle R_1 f, g \rangle + \frac{1}{3} \langle T^{-(4)} R_1 f, g \rangle.
\]
Hence

\[ \chi_{B_i(3)} T_{(4)}^{m_i} - \left( \frac{2}{3} E + \frac{1}{3} T_{(4)}^{-1} \right) R_i \rightarrow 0. \]

In the same way, we get

\[ \chi_{A_i(4)} T_{(4)}^{m_i} - \left( \frac{2}{3} E + \frac{1}{3} T_{(4)}^{-1} \right) Q_i \rightarrow 0, \]

\[ \chi_{B_i(4)} T_{(4)}^{m_i} - \left( \frac{2}{3} T_{(4)} + \frac{1}{3} E \right) R_i \rightarrow 0. \]

This completes the calculation in the case 1.

2. Here \( i_0 = 1 \). As above, we have

\[ \chi_{A_i(2)} T_{(4)}^{m_i} - T_{(4)}^{-1} Q_i \rightarrow 0, \quad \chi_{B_i(2)} T_{(4)}^{m_i} - \left( \frac{2}{3} E + \frac{1}{3} T_{(4)}^{-1} \right) R_i \rightarrow 0, \]

\[ \chi_{A_i(3)} T_{(4)}^{m_i} - \left( \frac{2}{3} T_{(4)}^{-1} + \frac{1}{3} T_{(4)}^{-2} \right) Q_i \rightarrow 0, \quad \chi_{B_i(3)} T_{(4)}^{m_i} - \left( \frac{2}{3} E + \frac{1}{3} T_{(4)}^{-1} \right) R_i \rightarrow 0, \]

\[ \chi_{A_i(4)} T_{(4)}^{m_i} - \left( \frac{2}{3} E + \frac{1}{3} T_{(4)}^{-1} \right) Q_i \rightarrow 0, \quad \chi_{B_i(4)} T_{(4)}^{m_i} - \left( \frac{2}{3} T_{(4)} + \frac{1}{3} E \right) R_i \rightarrow 0. \]

3. In this case, \( i_0 = 4 \), and \( f(T_{(4)}^{m_i} x) \) has two values on each level, except for from \( A_i(1) \). Using

\[ \langle Q'_i f, g \rangle = \frac{2}{3} d_{n+1} \sum_{l=1}^{h_n - p_i} f_n(l + p_i) \bar{g}_n(l), \]

calculate

\[ \langle \chi_{A_i(1)} T_{(4)}^{m_i} f, g \rangle = d_{n+1} \sum_{l=1}^{h_n - p_i} f_n(l + p_i - 1) \bar{g}_n(l) = \frac{3}{2} \langle T_{(4)}^{-1} Q'_i f, g \rangle, \]

\[ \langle \chi_{A_i(4)} T_{(4)}^{m_i} f, g \rangle = \langle \chi_{A_i(4)\cap O} T_{(4)}^{m_i} f, g \rangle + \langle \chi_{A_i(4)\setminus O} T_{(4)}^{m_i} f, g \rangle \]

\[ = \langle Q'_i f, g \rangle + \frac{1}{3} d_{n+1} \sum_{l=1}^{h_n - p_i} f_n(l + p_i - 1) \bar{g}_n(l) \]

\[ = \langle Q'_i f, g \rangle + \frac{1}{2} \langle T_{(4)}^{-1} Q'_i f, g \rangle. \]

For \( j = 2, 3 \), we get

\[ \langle \chi_{A_i(j)} T_{(4)}^{m_i} f, g \rangle = \frac{2}{3} d_{n+1} \sum_{l=1}^{h_n - p_i} f_n(l + p_i - 1) \bar{g}_n(l) \]

\[ + \frac{1}{3} d_{n+1} \sum_{l=1}^{h_n - p_i} f_n(l + p_i - 2) \bar{g}_n(l) \]

\[ = \langle T_{(4)}^{-1} Q'_i f, g \rangle + \frac{1}{2} \langle T_{(4)}^{-2} Q'_i f, g \rangle. \]
As before, for any $j$

\[
\langle \chi B_i(\mu) T_{\mu}^{m_i} f, g \rangle = \frac{2}{3} d_{n+1} \sum_{l=h_n-p_i+2}^{h_n} f_n(l+p_i-h_n-1) \bar{g}_n(l) + \frac{1}{3} d_{n+1} \sum_{l=h_n-p_i+3}^{h_n} f_n(l+p_i-h_n-2) \bar{g}_n(l)
\]

\[
= \langle R_i f, g \rangle + \frac{1}{2} \langle T_{\mu}^{-1} R_i f, g \rangle.
\]

1. Fix $P = aE + bT_{\mu}$. Let $T_{\mu}^{m_i} \rightarrow P$ for some $m_i \rightarrow +\infty$. It is clear that $a, b \geq 0$, because $P \in L^+_\mu$. Choose a subsequence of $m_i$ (if necessary) such that

\[
Q_{i}^{(x_i)} \rightarrow Q, \quad R_{i}^{(x_i)} \rightarrow R,
\]

where $S_{i}^{(x_i)}$ means $S_i$ or $S'_i$ for each $i$, and our choice is completely determined by $m_i$ as in Lemma 2.2. Obviously, $Q, R \in L^+_\mu$. By construction of $T_{\mu}$,

\[
\|T_{\mu}(\chi D_i - \chi D_i)\|_\mu^2 \leq 2d_{n+1} \rightarrow 0,
\]

where $D_i$ is $A_i(i0)$ or $A_i(i0) \cap O_i$. Thus

\[
\|T_{\mu}Q_{i}^{(x_i)} - Q_{i}^{(x_i)}T_{\mu}f\|_\mu^2 \leq \|T_{\mu}(\chi D_i - \chi D_i)\|_\mu^2 \rightarrow 0,
\]

for any $f \in L_2(\mu)$. Therefore $Q$ commutes with $T_{\mu}$. Analogously, we have that $R$ commutes with $T_{\mu}$. Denote by $P^+_T[x]$ the subset of $P_1[x]$ with non-negative coefficients.

Next we will show that if $P = \sum_{i=1}^l S_i$, where $S_i \in L^+_\mu$, and $S_i$ commute with $T_{\mu}$, then $S_i \in P^+_T[T_{\mu}]$. Indeed, measures $\phi^{-1} S_i$ are $T_{\mu}$-invariant and absolutely continuous with respect to the subpolymorphism $\phi^{-1} P = a\Delta_E + b\Delta_T(\phi)$. The transformation $T_{\mu} \times T_{\mu}$ is ergodic for measures $\Delta_E$ and $\Delta_T$. Hence every $T_{\mu} \times T_{\mu}$-invariant part of the measure $\Delta_E$ ($\Delta_T$) is $c\Delta_E$ ($\Omega \Delta_T$) for some $c > 0$. This gives $\phi^{-1} S_i = a_i \Delta_E + b_i \Delta_T$ for some $a_i, b_i \geq 0$.

From Lemma 2.2 and (2.4) we have that

\[
P = UQ + VR,
\]

where $(U, V)$ is at least one of the following pairs:

\[
(4/3T_{\mu}^{-1} + 8/3E, 1/3T_{\mu}^{-1} + 2E + 5/3T_{\mu}),
\]

\[
(1/3T_{\mu}^{-2} + 2T_{\mu}^{-1} + 5/3E, 2/3T_{\mu}^{-1} + 8/3E + 2/3T_{\mu}),
\]

\[
(T_{\mu}^{-2} + 4T_{\mu}^{-1} + E, 2T_{\mu}^{-1} + 4E).
\]

In any case $P$ contains $c_1 Q$ and $c_2 R$ as parts. Thus $Q, R \in P^+_T[T_{\mu}]$. Obviously,

\[
1 = \langle T_{\mu}^{m_i} \mathbf{1}, \mathbf{1} \rangle \rightarrow \langle P \mathbf{1}, \mathbf{1} \rangle.
\]

This implies that $P \neq 0$. Therefore equality (2.5) is possible only when $P = (4/3T_{\mu}^{-1} + 8/3E)cT_{\mu}$. It remains to mention, using (2.6), that $c = 1/4$. 


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2. The proof that $1/3E + 2/3T_{(4)} \in \phi(LJ_+(T_{(4)}))$ is almost obvious. Namely, consider $m_i = h_n$. We obtain $i_0 = 2, p_i = 0, R_i = 0$. Thus (2.3) gives 

$$\langle Q, f, g \rangle = d_{n+1} \sum_{l=1}^{h_n} f_n(l)\tilde{g}_n(l) = \frac{d_n}{4} \sum_{l=1}^{h_n} f_n(l)\tilde{g}_n(l) = \frac{1}{4} (f, g).$$

This implies that

$$\langle P_i f, g \rangle = \langle (\frac{1}{3}T^{-1}_{(4)} + \frac{2}{3}E)f, g \rangle.$$ 

Thus Theorem 2.1 follows from Lemma 2.2 and Remark 1.9. \hfill \Box

Remark 2.3. By the same argument as in [3], it is not difficult to show that $T_{(k)}$ have minimal self-joinings. Then $Q, R$ can be written in the following form:

$$\alpha \int + \sum_j a_j T_{(k)}^j,$$

where $\int$ is the orthogonal projection onto the space of constants, and $0 \leq \alpha, 0 \leq a_i$. This gives that the first part of Theorem 2.1 also follows directly from (2.5).

Our main result is the following.

**Theorem 2.4.** $LJ_+(T_{(k)}) \neq LJ_-(T_{(k)})$ for $k > 3$.

**Proof.** Indeed, it is clear that

$$\phi(LJ_+(T^*)) = \{U^*_\nu : \nu \in LJ_+(T)\}.$$

Therefore, using Proposition 1.3, Remark 1.9, and Theorem 2.1, we have

$$\phi(LJ_-(T_{(k)})) \cap \mathcal{P}_1[T_{(k)}] = \{\frac{k-2}{k-1}E + \frac{1}{k-1}T_{(k)}\},$$

and Theorem 2.4 is proved. \hfill \Box

3. Closing remarks

**Proposition 3.1.** For any $T \in \text{Aut}(\mu)$

$$LJ_+(T) \cap LJ_-(T) \neq \emptyset.$$ 

This is an immediate consequence of the next proposition.

**Proposition 3.2.**

(3.1) \quad $\phi(LJ_+(T)) \cap \phi(LJ_-(T)) \supseteq \{T_+T_- : T_\pm \in \phi(LJ_\pm(T))\},$ 

and

$$\{T_+T_- : T_\pm \in \phi(LJ_\pm(T))\} = \{\int\} \Leftrightarrow T \text{ is mixing},$$

where $\int$ is defined as in Remark 2.3.

**Proof.** Fix $T_\pm \in \phi(LJ_\pm(T))$, and $n_i, k_i \to +\infty$ such that $T^{k_i} \to T_+, T^{-n_i} \to T_-$. Consider also a dense set of functions from $L_2(\mu)$, say $f_1$. For each $\epsilon > 0$ and $m \in \mathbb{N}$, choose $i$ such that

$$|\langle T^{-n_i}f_1, T_+^* f_2 \rangle - \langle T_- f_1, T_+^* f_2 \rangle| < \epsilon,$$

for all $l_1, l_2 \leq m$. Finally choose $j = j(i)$ such that $k_{j(i)} - n_i > m$ and

$$|\langle T^{k_{j(i)}} T^{-n_i}f_1, f_2 \rangle - \langle T_+ T^{-n_i}f_1, f_2 \rangle| < \epsilon,$$
for all $l_1, l_2 \leq m$. Thus
\[
|\langle T^k_{i(j)} f_{l_1}, f_{l_2} \rangle - \langle T^{-n_i} f_{l_1}, T^* f_{l_2} \rangle| < 2\varepsilon.
\]
Therefore $T^k_{i(j)} - n_i \to T_+T_-$, where $k_{i(j)} - n_i \to +\infty$.

Operators $T_+$ and $T_-$ belong to the von Neumann algebra generated by $T$. Thus $T_+T_- = T_-T_+$. This implies that arguing as above, we see that $T^{-n_i}T^k_{i(j)} + k_i \to T^k_{i(j)} + k_i \to -\infty$.

The second part of the proof is more or less standard. Indeed, if $T$ is not mixing, then there exists $T_+ \phi \in \phi(LJ_+(T)) \setminus \{f\}$. Thus $T_- = T^*_+ \in \phi(LJ_-(T)) \setminus \{f\}$. Next for $S^*S = f$, where $S \in \phi(J(T, T))$, we have
\[
\int f \, d\mu = 0 \Rightarrow \langle Sf, Sf \rangle = \langle S^*Sf, f \rangle = \langle f, f \rangle = 0.
\]
This means that $Sf = 0$, and finally $S = f$. Therefore the operator $T_+T_- = T_-T_+ = T^*_+T^*_+$ is not $f$.

**Remark 3.3.** Obviously, in (3.1) we have an exact equality if $T$ is rigid or mixing. However, taking into account Remark 2.3, the operator $1/2E + 1/2T$ cannot be represented as $T_+T_-$ for Chacon’s transformation $T$. This yields that, in general, the left part of (3.1) is different from the right part.

References


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