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ON ASYMMETRY OF THE FUTURE AND THE PAST FOR LIMIT SELF-JOININGS

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ABSTRACT. Let $\triangle_T$ be an off-diagonal joining of a transformation T. We construct a non-typical transformation having asymmetry between limit sets of $\triangle_{T^n}$ for positive and negative powers of T. It follows from a correspondence between subpolymorphisms and positive operators, and from the structure of limit polynomial operators. We apply this technique to find all polynomial operators of degree 1 in the weak closure (in the space of positive operators on L_2) of powers of Chacon's automorphism and its generalizations.

INTRODUCTION

In [1], D. Rudolph introduced the notion of joinings. This notion turned out to be very fruitful (see [2]-[6]). It is known that every automorphism or endomorphism can be characterized both as a measure on a graph of this map, i.e., a joining or, more generally, a polymorphism [7], and as an operator on L_2. This provides a way to study the structure of joinings by operator methods, and automorphisms by joinings.

It is not difficult to show that limit sets for positive and negative powers of a transformation T are equal in the space of all automorphisms if T is rigid. If T is not rigid, then these sets are empty. Moreover, for rigid or mixing transformations (see Section 1) these limit sets are also equal in the space of all linear operators on L_2.

Nevertheless we prove (Theorem 2.4) that there exist rank-one transformations T with different limit sets of off-diagonal joinings for positive and negative powers of T or, in terms of operators, that the sets of limit operators are different.

We apply a new approach, via limit polynomials. This approach recently gave a solution to an old problem of Rokhlin (see [8], [9]) (see also references [10]–[13] about this problem) and answered some other well-known questions. See for example [8], which contains an answer to a question of Katok.

In Section 1 we introduce subpolymorphisms and a natural homeomorphism between the space of such measures and a subspace of positive operators on L_2.

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Note that, for any transformation T, the space of its self-joinings lies in the space of subpolymorphisms.

Theorem 2.1 describes all possible limit linear polynomials in $T_{(k)}$ for powers of transformations $T_{(k)}$, where $T_{(k)}$ are constructed in Section 1. In particular, this implies that only one linear polynomial $1/2E + 1/2T$ is a limit of powers for the classical Chacon transformation T (see [3], [14], [15]).

In Section 3 we show that the future and the past for limit self-joinings cannot be completely different for any automorphism. Moreover, their intersection contains an abelian semigroup that is trivial (i.e., $\{\mu \times \mu\}$) only for mixing transformations.

1. Basic definitions and notations

Let T be a transformation defined on a non-atomic standard Borel probability space $(X, \mathcal{F}, \mu)$. A transformation and a unitary operator on $L_2(\mu) : Tf(x) = f(Tx)$ are often called automorphisms and denoted by the same symbol T. It is clear that T is contained in the following space of positive operators on $L_2(\mu)$: $\mathcal{L}^+ = \{U : (Uf \in L_2 \text{ if } f \in L_2) \ \&\ (Ug \geq 0 \text{ if } g \geq 0)\}$ equipped with the weak operator convergence. Everywhere below the identity automorphism will be denoted by E. The group of all automorphisms of $(X, \mathcal{F}, \mu)$, say $\mathbf{Aut}(\mu)$, becomes a completely metrizable topological group when endowed with the weak convergence of transformations ($T_n \to T$ iff for any measurable A we have $\mu(T_n(A) \triangle T(A)) \to 0$ as $n \to \infty$). Note that this topology is a restriction of the weak operator topology in $\mathcal{L}^+$ to the non-closed $\mathbf{Aut}(\mu)$. Denote by $C(T)$ the commutant of T, i.e., the set $\{S \in \mathbf{Aut}(\mu) : ST = TS\}$.

Let ν be a finite Borel measure on $X \times X$ with marginal measures, say $\pi_1\nu$ and $\pi_2\nu$, such that

$$\pi_i\nu(A) \leq \mu(A) \text{ for any } \mu\text{-measurable set } A. \tag{1.1}$$

Obviously, marginal measures are μ-absolutely continuous and $\|\nu\| = \nu(X \times X) \leq 1$. The set of all such measures, say $M(\mu)$, is a convex compact metrizable space with respect to the topology determined by

$$\mu_n \to \mu_0 \Leftrightarrow \mu_n(A \times B) \to \mu_0(A \times B)$$

for any μ-measurable sets A and B. Now we shall give the following definition.

Definition 1.1. Each measure from $M(\mu)$ is called a subpolymorphism.

Fix $T \in \mathbf{Aut}(\mu)$. The set $J_c(T, T)$ of c-self-joinings, i.e., $T \times T$-invariant elements ν of $M(\mu)$, with $\|\nu\| = c$, is a closed subspace of $M(\mu)$ for any $c \in [0, 1]$. If T is ergodic, then each ν from $J_c(T, T)$ has $c\mu$ as marginal measures. This gives that $J(T, T) = J_1(T, T)$ is exactly the set of well-known self-joinings. Let $S \in C(T)$. As usual, by an off-diagonal joining $\triangle_S$ we mean a measure from $J(T, T)$ completely defined by $\triangle_S(A \times B) = \mu(S^{-1}A \cap B)$, where A, B are any μ-measurable sets. Denote by $LJ_+(T)$ and $LJ_-(T)$ the limit sets of $\triangle_{T^n}$ in $M(\mu)$ for all positive n and negative n respectively. It is clear that $LJ_\pm(T) \subseteq J(T, T)$. If T is mixing, then $LJ_\pm(T)$ consist of only one point $\mu \times \mu$. It is well known that T is weakly mixing iff $LJ_\pm(T)$ contain at least $\mu \times \mu$. We say T is rigid if $T^{n_k} \to E$ for some sequence $n_k \to \infty$.

Proposition 1.2. *$LJ_-(T) = LJ_+(T)$ for rigid T.*

Proposition 1.3. *$LJ_+(T^*) = LJ_-(T) = \sigma LJ_+(T)$, where σ is the flip map (i.e., $\sigma(x,y) = (y,x)$), and S^* denotes the operator adjoint to S.*

Corollary 1.4. *$LJ_-(T) = LJ_+(T)$ iff $LJ_+(T)$ is invariant with respect to σ.*

We leave simple proofs of the above statements to the reader. It seems to be well known that the set of rigid transformations is a dense G_δ-set in $\mathbf{Aut}(\mu)$ (see [16] regarding close results in different subclasses of the set of non-singular transformations). It turns out that $LJ_-(T) = LJ_+(T)$ for a typical transformation T.

1.1. **Construction of $T_{(k)}$.** We consider the following "generalized" Chacon's automorphisms. For each $k \geq 3$, let $T_{(k)}$ be a rank-one transformation, where each column C_{n+1} is obtained by cutting C_n into k subcolumns, say $C_n(i)$, of equal width, placing a spacer only on the subcolumn $C_n(k-1)$, and then stacking the subcolumn $C_n(i+1)$ on top of $C_n(i)$ for $1 \leq i < k$. It is clear that $T_{(3)}$ is exactly Chacon's automorphism. For the column C_n, let h_n be its height and let d_n be the measure of its one level, where $n \geq 1$.

1.2. **Correspondence between positive operators and $M(\mu)$.** Consider

$$\mathcal{L}^+_\mu = \{U \in \mathcal{L}^+ : \int_A U(\mathbf{1})d\mu \leq \mu(A)$$
$$\& \quad \int_A U^*(\mathbf{1})d\mu \leq \mu(A) \text{ for any } \mu\text{-measurable set } A\}$$

with a restriction of the weak operator topology to this set. Obviously,

$$\mathcal{L}^+_\mu = \{U \in \mathcal{L}^+ : U(\mathbf{1}) \leq \mathbf{1}$$
$$\& \quad U^*(\mathbf{1}) \leq \mathbf{1} \text{ for a.e. } x \text{ with respect to } \mu\}.$$

Proposition 1.5. *The natural correspondence given by*

$$\langle U_\nu f, g\rangle = \int f \otimes \bar{g} d\nu, \tag{1.2}$$

for any $f, g \in L_2(\mu)$, defines a linear homeomorphism, say ϕ, between the topological spaces $M(\mu)$ and $\mathcal{L}^+_\mu$.

Note that in (1.2) f and $U_\nu f$ are from different spaces $L_2(\mu)$, but we naturally identify these spaces.

Proof. Indeed, the right part of (1.2) is a linear bounded functional for every $f \in L_2(\mu)$ because

$$|\int f \otimes \bar{g} d\nu| \leq \|f\|_\nu \|g\|_\nu = \|f\|_{\pi_1\nu}\|g\|_{\pi_2\nu} \leq \|f\|_\mu \|g\|_\mu, \tag{1.3}$$

where $\|h\|_l$ means the norm of $h \in L_2(l)$. Thus $U_\nu f \in L_2(\mu)$ for each $f \in L_2(\mu)$. The remaining properties of the map ϕ are obvious. □

Some properties of operators U_ν were considered in [17] due to Vershik for polymorphisms ν, i.e., for elements of $M(\mu)$ with exact equality in (1.1) for each A.

Remark 1.6. Clearly, $\phi(\triangle_T) = T$, where $T \in \mathbf{Aut}(\mu)$. Also, U_ν commutes with T if and only if $\nu \in J_{\|\nu\|}(T,T)$.

Note that

$$\int f\otimes\bar{g}d\nu=\int_X\bar{g}(y)d\pi_2\nu\int_X f(x)d\nu_y(x)=\int_X\bar{g}(y)\rho_2(y)d\mu(y)\int_X f(x)d\nu_y(x),$$

where $\rho_2(y)$ is a density of $\pi_2\nu$ with respect to μ, and $\nu_y(x)$ is a canonical system of conditional measures corresponding to ν. Hence, changing g to $U_\nu f$ in (1.2) and (1.3), we get

Corollary 1.7. *$\mathcal{L}_\mu^+$ is a compact convex metrizable space. For any subpolymorphism $\nu\in M(\mu)$, $\|U_\nu\|\leq 1$, and*

$$U_\nu(f)=\rho_2(y)\int_X f(x)d\nu_y(x).$$

Remark 1.8. The space $\mathcal{L}_\mu^+$ is a semigroup, and closed with respect to taking parts of operators, i.e., if $0\leq V\leq U$ and $U\in\mathcal{L}_\mu^+$, then $V\in\mathcal{L}_\mu^+$.

Remark 1.9. It is readily seen that $\phi(LJ_+(T))$ and $\phi(LJ_-(T))$ are T-invariant closed semigroups.

2. Limit polynomials

The following theorem completely determines the simplest limit polynomial.

Theorem 2.1. *Let $\mathcal{P}_1[x]$ be the set of polynomials of degree at most* 1. *Then*

$$\phi(LJ_+(T_{(k)}))\cap\mathcal{P}_1[T_{(k)}]=\{\frac{1}{k-1}E+\frac{k-2}{k-1}T_{(k)}\}.$$

Proof. In order to prove Theorem 2.1 we need some definitions and a technical lemma. For clarity we restrict our attention to the case $k=4$. The proof for $k\neq 4$ is analogous.

Let $m_i\to+\infty$ as $i\to+\infty$. Fix i and choose n such that $h_n\leq m_i<h_{n+1}$. Consider the $(n+1)^{\text{st}}$ column. Number its levels by $1,2,...,4h_n+1$ from the base consequently. There exists i_0 such that the $(3h_n-m_i)^{\text{th}}(\text{mod}h_{n+1})$ level belongs to $C_n(i_0)$ (we put $i_0=4$ in the remaining case $m_i=4h_n$). Let p_i be a number of higher levels in $C_n(i_0)$. Denote by $B_i(j)$ the set of the top p_i levels of $C_n(j)$ and $A_i(j)=C_n(j)\backslash B_i(j)$ ($A_i(j)$ or $B_i(j)$ can be empty). Define $O_i=\{x\in C_{n+1}: T_{(4)}^r x\in C_{n+1}$ for $r=1,2,...,h_{n+1}\}$. It is clear that on each level a measure of x from O_i is $2/3d_{n+1}$.

Define operators

$$Q_i=\chi_{A_i(i_0)}T_{(4)}^{m_i},\ R_i=\chi_{B_i(i_0)}T_{(4)}^{m_i},$$
$$Q_i'=\chi_{O_i}Q_i,\quad R_i'=\chi_{O_i}R_i.$$

Note that $Q_i,Q_i',\ R_i,R_i'$ are in $\mathcal{L}_\mu^+$ because they are less than $T_{(4)}^{m_i}$.

Lemma 2.2.

$$T_{(4)}^{m_i}-P_i\to 0\ as\ \ i\to+\infty,\tag{2.1}$$

where

(1) $P_i=(4/3T_{(4)}^{-1}+8/3E)Q_i+(1/3T_{(4)}^{-1}+2E+5/3T_{(4)})R_i$, *if* $h_n\leq m_i<2h_n$.
(2) $P_i=(1/3T_{(4)}^{-2}+2T_{(4)}^{-1}+5/3E)Q_i+(2/3T_{(4)}^{-1}+8/3E+2/3T_{(4)})R_i$, *if* $2h_n\leq m_i<3h_n$.

(3) $P_i = (2/3T_{(4)}^{-2} + 8/3T_{(4)}^{-1} + 2/3E)3/2Q'_i + (4/3T_{(4)}^{-1} + 8/3E)3/2R'_i$, *if* $3h_n \leq m_i < h_{n+1}$.

Proof. It is enough to show (2.1) on pairs of functions from some dense set in $L_2(\mu)$. Therefore we can assume that f and g are constant, say $f_n(j)$ and $g_n(j)$, on each j^{th} level of C_n for sufficiently large n.

Obviously, if $\mu(D_i) \to 0$, then $\chi_{D_i} T^{m_i} \to 0$. Thus

$$T_{(4)}^{m_i} - \sum_j \chi_{A_i(j)} T_{(4)}^{m_i} - \sum_j \chi_{B_i(j)} T_{(4)}^{m_i} \to 0. \tag{2.2}$$

Next we will calculate the connection between components of each sum in (2.2), using that $T_{(4)}^{m_i} f$ has a "regular" structure on sets $A_i(j)$ and $B_i(j)$ and g is independent of j. Consider the $(n+1)^{\text{st}}$ column.

1. If $m_i < 2h_n$, then $i_0 = 2$. Clearly,

$$\langle \chi_{A_i(1)} T_{(4)}^{m_i} f, g\rangle = d_{n+1} \sum_{l=1}^{h_n - p_i} f_n(l + p_i)\bar{g}_n(l) = \langle Q_i f, g\rangle, \tag{2.3}$$

$$\langle \chi_{B_i(1)} T_{(4)}^{m_i} f, g\rangle = d_{n+1} \sum_{l=h_n - p_i + 1}^{h_n} f_n(l + p_i - h_n)\bar{g}_n(l) = \langle T_{(4)} R_i f, g\rangle.$$

Here and next values of f and g can be written incorrectly on bases and tops of $A_i(j)$ and $B_i(j)$, but this fact is not essential for the convergence of such operators. It is clear that

$$\chi_{A_i(1)} T_{(4)}^{m_i} - Q_i \to 0,$$
$$\chi_{B_i(1)} T_{(4)}^{m_i} - T_{(4)} R_i \to 0.$$

Analogously,

$$\chi_{A_i(3)} T_{(4)}^{m_i} - T_{(4)}^{-1} Q_i \to 0.$$

By construction of $T_{(4)}$, the function $f(T_{(4)}^{m_i} x)$ has at most two values on each level from $B_i(3), A_i(4), B_i(4)$ up to two base levels of $B_i(3)$. The first one is exactly at x from O_i, and the second one is at the remaining part of this level. Indeed, fix some x from such a level. It is clear that $N(T_{(4)}^{m_i} x) = N(x) + m_i \pmod{h_{n+1}}$ for $x \in O_i$, where $N(y)$ means a number of the level having y. The set $B_i(3)$ starts from the $(h_{n+1} - m_i)^{\text{th}}$ level. Thus $\{x, T_{(4)} x, ..., T_{(4)}^{m_i} x\} \backslash C_{n+1} \neq \emptyset$, if x is not in O_i. Therefore $N(T_{(4)}^{m_i} x) = N(x) + m_i - 1 \pmod{h_{n+1}}$.

Thus

$$\begin{aligned}
\langle \chi_{B_i(3)} T_{(4)}^{m_i} f, g\rangle &= \langle \chi_{B_i(3)\cap O_i} T_{(4)}^{m_i} f, g\rangle + \langle \chi_{B_i(3)\backslash O_i} T_{(4)}^{m_i} f, g\rangle \\
&= \frac{2}{3} d_{n+1} \sum_{l=h_n - p_i + 2}^{h_n} f_n(l + p_i - h_n - 1)\bar{g}_n(l) \\
&\quad + \frac{1}{3} d_{n+1} \sum_{l=h_n - p_i + 3}^{h_n} f_n(l + p_i - h_n - 2)\bar{g}_n(l) \\
&= \frac{2}{3}\langle R_i f, g\rangle + \frac{1}{3}\langle T_{(4)}^{-1} R_i f, g\rangle.
\end{aligned}$$

Hence

$$\chi_{B_i(3)}T_{(4)}^{m_i} - (\frac{2}{3}E + \frac{1}{3}T_{(4)}^{-1})R_i \to 0.$$

In the same way, we get

$$\chi_{A_i(4)}T_{(4)}^{m_i} - (\frac{2}{3}E + \frac{1}{3}T_{(4)}^{-1})Q_i \to 0,$$

$$\chi_{B_i(4)}T_{(4)}^{m_i} - (\frac{2}{3}T_{(4)} + \frac{1}{3}E)R_i \to 0.$$

This completes the calculation in the case 1.

2. Here $i_0 = 1$. As above, we have

$$\chi_{A_i(2)}T_{(4)}^{m_i} - T_{(4)}^{-1}Q_i \to 0, \quad \chi_{B_i(2)}T_{(4)}^{m_i} - (\frac{2}{3}E + \frac{1}{3}T_{(4)}^{-1})R_i \to 0,$$

$$\chi_{A_i(3)}T_{(4)}^{m_i} - (\frac{2}{3}T_{(4)}^{-1} + \frac{1}{3}T_{(4)}^{-2})Q_i \to 0, \; \chi_{B_i(3)}T_{(4)}^{m_i} - (\frac{2}{3}E + \frac{1}{3}T_{(4)}^{-1})R_i \to 0,$$

$$\chi_{A_i(4)}T_{(4)}^{m_i} - (\frac{2}{3}E + \frac{1}{3}T_{(4)}^{-1})Q_i \to 0, \; \chi_{B_i(4)}T_{(4)}^{m_i} - (\frac{2}{3}T_{(4)} + \frac{1}{3}E)R_i \to 0.$$

3. In this case, $i_0 = 4$, and $f(T_{(4)}^{m_i}x)$ has two values on each level, except for from $A_i(1)$. Using

$$\langle Q_i' f, g\rangle = \frac{2}{3}d_{n+1}\sum_{l=1}^{h_n-p_i} f_n(l+p_i)\bar{g}_n(l),$$

calculate

$$\langle \chi_{A_i(1)}T_{(4)}^{m_i}f, g\rangle = d_{n+1}\sum_{l=1}^{h_n-p_i} f_n(l+p_i-1)\bar{g}_n(l) = \frac{3}{2}\langle T_{(4)}^{-1}Q_i' f, g\rangle,$$

$$\begin{aligned}\langle \chi_{A_i(4)}T_{(4)}^{m_i}f, g\rangle &= \langle \chi_{A_i(4)\cap O_i}T_{(4)}^{m_i}f, g\rangle + \langle \chi_{A_i(4)\setminus O_i}T_{(4)}^{m_i}f, g\rangle \\ &= \langle Q_i' f, g\rangle + \frac{1}{3}d_{n+1}\sum_{l=1}^{h_n-p_i} f_n(l+p_i-1)\bar{g}_n(l) \\ &= \langle Q_i' f, g\rangle + \frac{1}{2}\langle T_{(4)}^{-1}Q_i' f, g\rangle.\end{aligned}$$

For $j = 2, 3$, we get

$$\begin{aligned}\langle \chi_{A_i(j)}T_{(4)}^{m_i}f, g\rangle &= \frac{2}{3}d_{n+1}\sum_{l=1}^{h_n-p_i} f_n(l+p_i-1)\bar{g}_n(l) \\ &\quad + \frac{1}{3}d_{n+1}\sum_{l=1}^{h_n-p_i} f_n(l+p_i-2)\bar{g}_n(l) \\ &= \langle T_{(4)}^{-1}Q_i' f, g\rangle + \frac{1}{2}\langle T_{(4)}^{-2}Q_i' f, g\rangle.\end{aligned}$$

As before, for any j

$$\begin{aligned}\langle \chi_{B_i(j)} T_{(4)}^{m_i} f, g\rangle &= \frac{2}{3} d_{n+1} \sum_{l=h_n-p_i+2}^{h_n} f_n(l+p_i-h_n-1)\bar{g}_n(l) \\ &\quad + \frac{1}{3} d_{n+1} \sum_{l=h_n-p_i+3}^{h_n} f_n(l+p_i-h_n-2)\bar{g}_n(l) \\ &= \langle R_i' f, g\rangle + \frac{1}{2}\langle T_{(4)}^{-1} R_i' f, g\rangle.\end{aligned}$$

□

1. Fix $P = aE + bT_{(4)}$. Let $T_{(4)}^{m_i} \to P$ for some $m_i \to +\infty$. It is clear that $a, b \geq 0$, because $P \in \mathcal{L}_\mu^+$. Choose a subsequence of m_i (if necessary) such that

$$\begin{aligned} Q_i^{(x_i)} &\to Q, \\ R_i^{(x_i)} &\to R, \end{aligned} \tag{2.4}$$

where $S_i^{(x_i)}$ means S_i or S_i' for each i, and our choice is completely determined by m_i as in Lemma 2.2. Obviously, $Q, R \in \mathcal{L}_\mu^+$. By construction of $T_{(4)}$,

$$\|T_{(4)}\chi_{D_i} - \chi_{D_i}\|_\mu^2 \leq 2d_{n+1} \to 0,$$

where D_i is $A_i(i_0)$ or $A_i(i_0) \cap O_i$. Thus

$$\|(T_{(4)}Q_i^{(x_i)} - Q_i^{(x_i)}T_{(4)})f\|_\mu^2 \leq \|T_{(4)}\chi_{D_i} - \chi_{D_i}\|_\mu \|T_{(4)}^2 f\|_\mu \to 0,$$

for any $f \in L_2(\mu)$. Therefore Q commutes with $T_{(4)}$. Analogously, we have that R commutes with $T_{(4)}$. Denote by $\mathcal{P}_1^+[x]$ the subset of $\mathcal{P}_1[x]$ with non-negative coefficients.

Next we will show that if $P = \sum_{i=1}^l S_i$, where $S_i \in \mathcal{L}_\mu^+$, and S_i commute with $T_{(4)}$, then $S_i \in \mathcal{P}_1^+[T_{(4)}]$. Indeed, measures $\phi^{-1}S_i$ are $T_{(4)} \times T_{(4)}$-invariant and absolutely continuous with respect to the subpolymorphism $\phi^{-1}P = a\triangle_E + b\triangle_{T_{(4)}}$. The transformation $T_{(4)} \times T_{(4)}$ is ergodic for measures $\triangle_E$ and $\triangle_{T_{(4)}}$. Hence every $T_{(4)} \times T_{(4)}$-invariant part of the measure $\triangle_E$ ($\triangle_{T_{(4)}}$) is $c\triangle_E$ ($c\triangle_{T_{(4)}}$) for some $c > 0$. This gives $\phi^{-1}S_i = a_i\triangle_E + b_i\triangle_{T_{(4)}}$ for some $a_i, b_i \geq 0$.

From Lemma 2.2 and (2.4) we have that

$$P = UQ + VR, \tag{2.5}$$

where (U, V) is at least one of the following pairs:

$$\begin{gathered}(4/3T_{(4)}^{-1} + 8/3E, 1/3T_{(4)}^{-1} + 2E + 5/3T_{(4)}), \\ (1/3T_{(4)}^{-2} + 2T_{(4)}^{-1} + 5/3E, 2/3T_{(4)}^{-1} + 8/3E + 2/3T_{(4)}), \\ (T_{(4)}^{-2} + 4T_{(4)}^{-1} + E, 2T_{(4)}^{-1} + 4E).\end{gathered}$$

In any case P contains c_1Q and c_2R as parts. Thus $Q, R \in \mathcal{P}_1^+[T_{(4)}]$. Obviously,

$$1 = \langle T_{(4)}^{m_i}\mathbf{1}, \mathbf{1}\rangle \to \langle P\mathbf{1}, \mathbf{1}\rangle. \tag{2.6}$$

This implies that $P \neq 0$. Therefore equality (2.5) is possible only when $P = (4/3T_{(4)}^{-1} + 8/3E)cT_{(4)}$. It remains to mention, using (2.6), that $c = 1/4$.

2. The proof that $1/3E + 2/3T_{(4)} \in \phi(LJ_+(T_{(4)}))$ is almost obvious. Namely, consider $m_i = h_n$. We obtain $i_0 = 2, p_i = 0, R_i = 0$. Thus (2.3) gives

$$\langle Q_i f, g\rangle = d_{n+1}\sum_{l=1}^{h_n} f_n(l)\bar{g}_n(l) = \frac{d_n}{4}\sum_{l=1}^{h_n} f_n(l)\bar{g}_n(l) = \frac{1}{4}\langle f, g\rangle.$$

This implies that

$$\langle P_i f, g\rangle = \langle(\frac{1}{3}T_{(4)}^{-1} + \frac{2}{3}E)f, g\rangle.$$

Thus Theorem 2.1 follows from Lemma 2.2 and Remark 1.9. □

Remark 2.3. By the same argument as in [3], it is not difficult to show that $T_{(k)}$ have minimal self-joinings. Then Q, R can be written in the following form:

$$\alpha\int + \sum_j a_j T_{(k)}^j,$$

where $\int$ is the orthogonal projection onto the space of constants, and $0 \le \alpha,\ 0 \le a_i$. This gives that the first part of Theorem 2.1 also follows directly from (2.5).

Our main result is the following.

Theorem 2.4. *$LJ_+(T_{(k)}) \ne LJ_-(T_{(k)})$ for $k > 3$.*

Proof. Indeed, it is clear that

$$\phi(LJ_+(T^*)) = \{U_\nu^* : \nu \in LJ_+(T)\}.$$

Therefore, using Proposition 1.3, Remark 1.9, and Theorem 2.1, we have

$$\phi(LJ_-(T_{(k)})) \cap \mathcal{P}_1[T_{(k)}] = \{\frac{k-2}{k-1}E + \frac{1}{k-1}T_{(k)}\},$$

and Theorem 2.4 is proved. □

3. Closing remarks

Proposition 3.1. *For any $T \in \mathbf{Aut}(\mu)$*

$$LJ_+(T) \cap LJ_-(T) \ne \emptyset.$$

This is an immediate consequence of the next proposition.

Proposition 3.2.

$$\phi(LJ_+(T)) \cap \phi(LJ_-(T)) \supseteq \{T_+T_- : T_\pm \in \phi(LJ_\pm(T))\}, \tag{3.1}$$

and

$$\{T_+T_- : T_\pm \in \phi(LJ_\pm(T))\} = \{\int\} \Leftrightarrow T \textit{ is mixing},$$

where $\int$ is defined as in Remark 2.3.

Proof. Fix $T_\pm \in \phi(LJ_\pm(T))$, and $n_i, k_i \to +\infty$ such that $T^{k_i} \to T_+,\ T^{-n_i} \to T_-$. Consider also a dense set of functions from $L_2(\mu)$, say f_l. For each $\epsilon > 0$ and $m \in \mathbb{N}$, choose i such that

$$|\langle T^{-n_i} f_{l_1}, T_+^* f_{l_2}\rangle - \langle T_- f_{l_1}, T_+^* f_{l_2}\rangle| < \epsilon,$$

for all $l_1, l_2 \le m$. Finally choose $j = j(i)$ such that $k_{j(i)} - n_i > m$ and

$$|\langle T^{k_{j(i)}} T^{-n_i} f_{l_1}, f_{l_2}\rangle - \langle T_+ T^{-n_i} f_{l_1}, f_{l_2}\rangle| < \epsilon,$$

for all $l_1, l_2 \leq m$. Thus

$$|\langle T^{k_{j(i)}-n_i} f_{l_1}, f_{l_2}\rangle - \langle T_- f_{l_1}, T_+^* f_{l_2}\rangle| < 2\epsilon.$$

Therefore $T^{k_{j(i)}-n_i} \to T_+T_-$, where $k_{j(i)} - n_i \to +\infty$.

Operators T_+ and T_- belong to the von Neumann algebra generated by T. Thus $T_+T_- = T_-T_+$. This implies that arguing as above, we see that $T^{-n_{j'(i)}+k_i} \to T_+T_-$, where $-n_{j'(i)} + k_i \to -\infty$.

The second part of the proof is more or less standard. Indeed, if T is not mixing, then there exists T_+ from $\phi(LJ_+(T))\backslash\{\int\}$. Thus $T_- = T_+^* \in \phi(LJ_-(T))\backslash\{\int\}$. Next for $S^*S = \int$, where $S \in \phi(J(T,T))$, we have

$$\int f d\mu = 0 \Rightarrow \langle Sf, Sf\rangle = \langle S^*Sf, f\rangle = \langle \int f, f\rangle = 0.$$

This means that $Sf = 0$, and finally $S = \int$. Therefore the operator $T_+T_- = T_-T_+ = T_+^*T_+$ is not $\int$. □

Remark 3.3. Obviously, in (3.1) we have an exact equality if T is rigid or mixing. However, taking into account Remark 2.3, the operator $1/2E + 1/2T$ cannot be represented as T_+T_- for Chacon's transformation T. This yields that, in general, the left part of (3.1) is different from the right part.

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