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THE BANACH ALGEBRA INDUCED BY A DOUBLE CENTRALIZER

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ABSTRACT. Given a Banach algebra A, R. Larsen defined, in his book "*An introduction to the theory of multipliers*", a Banach algebra A_T by means of a *multiplier* T on A, and essentially used it in the case of a *commutative semisimple* Banach algebra A to prove a result on multiplications which preserve regular maximal ideals. Here, we consider the analogue Banach algebra $\mathcal{A}_R$ induced by a bounded *double centralizer* $\langle L, R\rangle$ of a Banach algebra A. Then, our main concern is devoted to the relationships between L, R, and the algebras of bounded double centralizers $\mathcal{W}(A)$ and $\mathcal{W}(\mathcal{A}_R)$ of A and $\mathcal{A}_R$, respectively. By removing the assumption of semisimplicity, we generalize some results proven by Larsen.

1. Preliminaries

Unless otherwise stated, we shall adopt throughout the sequel the following conventions: A will be an arbitrary Banach algebra, and the Banach algebra of all (resp. bounded) linear operators on it will be denoted by $\mathcal{L}(A)$ (resp. $B(A)$). The maps λ and μ, such that $a \mapsto \lambda_a$ and $a \mapsto \mu_a$ for $a \in A$, are the usual left and right regular representations of A. Composition of mappings will be denoted by simple juxtaposition.

Definition 1.1. A left (resp. right) *centralizer* of A is an element $L \in \mathcal{L}(A)$ (resp. $R \in \mathcal{L}(A)$), satisfying $L(ab) = L(a)\ b$ (resp. $R(ab) = a\ R(b)\ ,\ \forall\ a, b \in A$).

A *double centralizer* of A is a pair $\langle L, R\rangle$ where L (resp. R) is a left (resp. right) centralizer, and which together satisfy the following *Double Centralizer Property (DC-Property):*

$$a\ L(b)\ =\ R(a)\ b\ \ ,\ \ \forall\ a, b \in A\ . \tag{1}$$

The algebra $\mathcal{W}(A)$ of *bounded double centralizers* of A is the set of double centralizers with pointwise linear operations and with product and norm given by

$$\langle L, R\rangle\langle T, S\rangle = \langle LT, SR\rangle\ \ ,\ \ \|\langle L, R\rangle\| = Max\{\|L\|; \|R\|\},\quad \forall\ \langle L, R\rangle\ ,\ \langle T, S\rangle \in \mathcal{W}(A)\ . \tag{2}$$

Relation (2) above ensures the automatic continuity of the elements L and R of any double centralizer $\langle L, R\rangle$ of A. For the definitions of a *multiplier* and of the

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left annihilator $LAn(A)$, *right annihilator* $RAn(A)$ and *annihilator* $An(A)$ of A, the reader should refer to [3] and [4]. A is said to have *no annihilators* if $An(A) = LAn(A) = RAn(A) = \{0\}$.

2. The Banach algebra $\mathcal{A}_R$

Proposition 2.1. *Let* $\langle L, R\rangle \in \mathcal{W}(A)$ *such that* $\|R\| \leq 1$*, and define*

$$a \,.\, b \;=\; R(a)\, b \quad , \quad \forall\, a\,, b \in A\,. \tag{3}$$

(i) *Relation* (3) *endows* A *with a Banach algebra structure denoted by* $\mathcal{A}_R$*.*

(ii) R *(resp.* L*) is a homomorphism from* $\mathcal{A}_R$ *onto the image* $Im(R)$ *(resp.* $Im(L)$*) of* R *(resp.* L*).*

(iii) *If* A *has no annihilator, or if* A *admits a two-sided approximate identity, then the kernels of* L *and* R *satisfy* $Ker(R) = LAn(\mathcal{A}_R)$, $Ker(L) = RAn(\mathcal{A}_R)$*, and hence* $An(\mathcal{A}_R) = Ker(L) \cap Ker(R)$*.*

Proof. (i) and (ii) are obtained by straightforward computations. It is also easy to show the relations $R\Big[LAn(\mathcal{A}_R)\Big] \subset LAn(A)$ and $L\Big[RAn(\mathcal{A}_R)\Big] \subset RAn(A)$, from which ($iii$) follows, since A is without annihilators. □

Remark 2.2. It should be noticed that assuming $\|L\| \leq 1$, and defining the product in relation (3) as $a \,.\, b = a\, L(b)$, $\forall\, a\,, b \in A$, gives the same algebra $\mathcal{A}_R$. Therefore, when dealing in the sequel with the algebra $\mathcal{A}_R$, we shall say that $\mathcal{A}_R$ is *induced* or *defined* by $\langle L, R\rangle$. Considering $\mathcal{A}_R$ as a *left Banach module* over itself, the map R satisfying relation (3) is an example of a bounded module map from the module into the algebra (cf. [1]).

As an immediate consequence of Proposition 2.1 above, we have:

Corollary 2.3. *Let* A *have no annihilators. Then, for any* $\langle L, R\rangle \in \mathcal{W}(A)$*,* L *(resp.* R*) is one-to-one implies that* $An(\mathcal{A}_R) = \{0\}$*. Moreover,* $\mathcal{A}_R$ *has no annihilators, if and only if* L *and* R *are both one-to-one.* □

Remark 2.4. If A is commutative and without annihilators, then $\langle L, R\rangle$ is a multiplier $L = R = T$ and therefore $\mathcal{A}_R$ coincides with the algebra A_T of R. Larsen, and is also commutative.

For $\langle L\,,\, R\rangle$ defining $\mathcal{A}_R$, direct computations show that the condition $\langle T\,,\, S\rangle \in \mathcal{W}(\mathcal{A}_R)$ implies $\langle L\,T\,,\, R\,S\rangle \in \mathcal{W}(A)$.

As noted by T. W. Palmer in [4], page 26, the set of left (resp. right) centralizers is contained in the commutant $(\mu_A)^c$ (resp. $(\lambda_A)^c$) of μ_A (resp. λ_A). Moreover, as pointed out in [2], page 775, if A has no annihilators, then given two arbitrary elements $\langle L, R\rangle$ and $\langle T, S\rangle$ of $\mathcal{W}(A)$, their components *cross-commute*, i.e. $T\,R = R\,T$ and $L\,S = S\,L$. As a consequence, any $\langle L, R\rangle \in \mathcal{W}(A)$ satisfies $L\,R = R\,L$.

For every $\langle L, R\rangle \in \mathcal{W}(A)$, let $Im(L)$ (resp. $Im(R)$) denote the image of L (resp. R), and define $I_{LR} = Im(L) \cap Im(R)$. Then we have:

Proposition 2.5. *Let* A *have no annihilators, and let* $\langle L, R\rangle \in \mathcal{W}(A)\backslash\{0\}$*. If* $\overline{Im(L)} \neq A$ *and* $\overline{Im(R)} \neq A$*, then each of* L *and* R *admits a nontrivial closed invariant subspace (the bar denotes the closure in the norm topology of* A*); moreover,* $I_{LR} = \{0\}$ *implies that* L *(resp.* R*) cannot be one-to-one.*

Proof. We shall write $L[Im(R)]$ and $R[Im(L)]$ to denote the restrictions of the maps L and R, respectively, on the subspaces $Im(R)$ and $Im(L)$ of A. According to the above comments, the components of $\langle L, R\rangle \in \mathcal{W}(A)$ commute. Hence the following hold: $L[Im(R)] \subset I_{LR}$ and $R[Im(L)] \subset I_{LR}$, from which we derive $L\left[\overline{Im(R)}\right] \subset \overline{Im(R)}$, $R\left[\overline{Im(L)}\right] \subset \overline{Im(L)}$. Then, if the images of L and R are not dense in A, these inclusions prove that each of L and$R$ admits a nontrivial closed invariant subspace. L and R being nonzero, the same inclusions imply that L (resp. R) cannot be one-to-one if $I_{LR} = \{0\}$. □

Theorem 2.6. *Let A and $\mathcal{A}_R$ be without annihilators and let $\langle L, R\rangle \in \mathcal{W}(A)$ be such that $Im(L) \subset R\left[Im(L)\right]$. Then, R is bijective, and R^{-1} is a right centralizer. The result remains clearly true if the roles of R and L are interchanged, and the right centralizer is replaced by the left centralizer.*

Proof. We have, according to Corollary 2.3 and the proof of Proposition 2.5, $R\left[Im(L)\right] \subset Im(L)$, which, together with the assumption on $Im(L)$ implies the equality $Im(L) = R\left[Im(L)\right]$. It then follows that R is onto A, according to the following result: $L(x_0) \in R\left[Im(L)\right] \implies x_0 \in Im(R)$. Indeed, let us assume by contradiction that $x_0 \notin Im(R)$. Then, L being one-to-one, we get $L(x_0) \neq LR(a)$, $\forall\, a \in A$. But since L commutes with R, it also follows that $L(x_0) \neq RL(a)$, $\forall\, a \in A$, which means that $L(x_0) \notin R\left[Im(L)\right]$. Hence, the desired result is established, and one easily checks that R^{-1} is a right centralizer. □

Let us define

$$\mathcal{C}_\ell(A) = \{L \in B(A) \;;\; \exists\, R \in B(A) \;;\; \langle L, R\rangle \in \mathcal{W}(A)\} \;; \tag{4}$$

$$\mathcal{C}_r(A) = \{S \in B(A) \;;\; \exists\, T \in B(A) \;;\; \langle T, S\rangle \in \mathcal{W}(A)\} \;; \tag{5}$$

$$\Phi(\langle L, R\rangle) = L \quad , \quad \Psi(\langle L, R\rangle) = R \quad , \quad \forall\, \langle L, R\rangle \in \mathcal{W}(A) \;. \tag{6}$$

Then we get:

Theorem 2.7. *Let A be a Banach algebra with an approximate identity bounded by one. Then $\mathcal{C}_\ell(A)$ and $\mathcal{C}_r(A)$ are Banach subalgebras of $B(A)$, and Φ is a faithful representation of $\mathcal{W}(A)$ on A, with $\Phi\Big(\mathcal{W}(A)\Big) = \mathcal{C}_\ell(A)$ (resp. Ψ is a one-to-one anti-homomorphism of $\mathcal{W}(A)$ into $B(A)$, with $\Psi\Big(\mathcal{W}(A)\Big) = \mathcal{C}_r(A)$).*

Proof. $\mathcal{C}_\ell(A)$ is clearly a linear subspace of $B(A)$. Moreover, if $L,\ T \in \mathcal{C}_\ell(A)$, then by definition, there exist $R,\ S \in B(A)$ such that $\langle L, R\rangle \in \mathcal{W}(A)$, $\langle T, S\rangle \in \mathcal{W}(A)$, with $L = \Phi(\langle L, R\rangle)$ and $T = \Phi(\langle T,\ S\rangle)$. Hence, $\langle L, R\rangle\ \langle T,\ S\rangle = \langle L\ T,\ S\ R\rangle \in \mathcal{W}(A)$, and therefore $L\ T = \Phi(\langle L\ T,\ S\ R\rangle) \in \mathcal{C}_\ell(A)$. So $\mathcal{C}_\ell(A)$ is a subalgebra of $B(A)$ and more precisely, a Banach algebra. Indeed, if $(L_n)_{n\geq 1}$ is a sequence in $\mathcal{C}_\ell(A)$ converging to some $L \in B(A)$, then L is a left centralizer. Moreover, the corresponding sequence $(R_n)_{n\geq 1}$ in $B(A)$ such that $L_n = \Phi(\langle L_n, R_n\rangle$, $\forall n \geq 1$, converges to some $R \in B(A)$. To see this, we proceed as follows: given $x \in A$, we let $x \mapsto \lambda_x$ be the left regular representation of A onto itself, and for fixed $n, m \in \mathbb{N}$ and $a \in A$, we consider the map

$$A \ni b \mapsto \Big(R_n - R_m\Big)(a)\ b = \lambda_{(R_n - R_m)(a)}\ (b) \;. \tag{7}$$

Then we get

(8)
$$\|(R_n - R_m)\ (a)\| \ = \ \|\lambda_{(R_n-R_m)(a)}\| \ = \ Sup\ \{\|(R_n - R_m)(a)\ (b)\|\ ;\ \|b\| \le 1\}\ ,$$

so that the following holds:

(9)
$$\|R_n - R_m\| \ \le\ Sup\ \Big\{Sup\ \{\|(R_n - R_m)(a)\ (b)\|\ ;\ \|b\| \le 1\}\ ;\ \|a\| \le 1\Big\}\ .$$

Since $\langle L_n,\ R_n\rangle,\ \langle L_m,\ R_m\rangle \in\ \mathcal{W}(A)$, we have, for each pair of elements $a, b \in A$,

$$(10) \qquad R_n(a)\ b \ = \ a\ L_n(b) \quad \text{and} \quad R_m(a)\ b \ = \ a\ L_m(b)\ ,$$

and the equality (8) hence gives

$$\begin{aligned}\|R_n - R_m\| \quad &\le \quad Sup\ \Big\{\|a\|\ Sup\ \{\|L_n - L_m\|\ \|b\|\ ;\ \|b\| \le 1\}\ ;\ \ \|a\| \le 1\Big\}\\ &\le \quad \|L_n - L_m\|\ .\end{aligned}$$

$(R_n)_{n\ge1}$ is therefore a Cauchy sequence in $B(A)$, and hence converges to some $R \in B(A)$.

Now, for each $n \in \mathbb{N}$ and for all $a, b \in A$ the following relations are satisfied:

$$(11) \qquad (i) \quad R_n(a\ b) \ = \ a\ R_n(b) \quad \text{and} \quad (ii) \quad R_n(a)\ b \ = \ a\ L_n(b)\ ,$$

and we get, taking the limits in equalities (i) and (ii) when n tends to infinity: $R(a\ b) = a\ R(b)$ and $R(a)\ b = a\ L(b)$, that is, R is a right centralizer and $\langle L,\ R\rangle \in \mathcal{W}(A)$. Therefore $L = \Phi(\langle L, R\rangle) \in \mathcal{C}_\ell(A)$, and $\mathcal{C}_\ell(A)$ is hence norm-closed in $B(A)$. That $\mathcal{C}_r(A)$ is a Banach subalgebra of $B(A)$ is obtained by quite similar arguments as in the case of $\mathcal{C}_\ell(A)$. That Φ is a faithful representation of $\mathcal{W}(A)$ on A, and Ψ a one-to-one anti-homomorphism of $\mathcal{W}(A)$ into A, are obtained by routine computations. □

Given $R \in B(A)$, let $\{R\}^c \ = \ \{S \in B(A)\ ;\ \ RS = SR\}$ denote the *commutant* of $\{R\}$. According to the comments following Remark 2.4, if $\langle L, R\rangle \in \mathcal{W}(A)$ defines $\mathcal{A}_R$ and if $S \in \{R\}^c$ is such that $\langle T, S\rangle \in \mathcal{W}(\mathcal{A}_R)$, then $\langle L\ T, R\ S\rangle = \langle L\ T, S\ R\rangle \in \mathcal{W}(A)$, and it is "tempting" in this situation to write $\langle L\ T, S\ R\rangle = \langle L, R\rangle\ \langle T, S\rangle$. But, the product in the right-hand side of this equation would not make sense, unless $\mathcal{W}(A)$ and $\mathcal{W}(\mathcal{A}_R)$ are subalgebras of some larger Banach algebra. We construct in what follows such an algebra.

Consider the set $\mathcal{D}_{P0}(A)$ of all pairs $\langle L\ ,\ R\rangle$ with $L \in B(A)\ ,\ R \in B(A)$, made into a linear algebra under pointwise linear operations and with the same product as in $\mathcal{W}(A)$. Then $\mathcal{D}_{P0}(A)$ is a normed algebra when it is endowed with the same norm as in $\mathcal{W}(A)$. Now, since the elements of $\mathcal{D}_{P0}(A)$ are not assumed to satisfy the *DC-Property*, $\mathcal{D}_{P0}(A)$ may not be a Banach algebra. So we just have to consider the norm completion $\mathcal{D}_P(A)$ of $\mathcal{D}_{P0}(A)$ to have our needed Banach algebra. The notation $\mathcal{D}_P(A)$, for the algebra just constructed, is motivated by the fact that we may think of its elements as bounded *Double Pre-centralizers* of A.

Proposition 2.8 below follows from the above comments and the previous results.

Proposition 2.8. *$\mathcal{W}(A)$ and $\mathcal{W}(\mathcal{A}_R)$ are Banach subalgebras of the Banach algebra $\mathcal{D}_P(A)$, with nonvoid intersection, since $\langle L, R\rangle \in \mathcal{W}(A) \cap \mathcal{W}(\mathcal{A}_R)$. Moreover, $\langle T\ ,\ S\rangle \in \mathcal{W}(\mathcal{A}_R)$ if and only if $\langle L\ T\ ,\ R\ S\rangle \in \mathcal{W}(A)$.* □

N.B. Throughout the remainder of the sequel, and unless otherwise stated, A will always be a Banach algebra without annihilators and $\mathcal{A}_R$ will be the Banach algebra in Proposition 2.1, induced by some double centralizer $\langle L, R\rangle \in \mathcal{W}(A)$.

Given $a \in A$, let us set

$$\lambda_a^R(x) \;=\; R(a)\; x \;=\; \lambda_{R(a)}(x) \;=\; \Big[(\lambda\; R)(a)\Big]\,(x) \quad , \quad \forall\; x \in A\;, \tag{12}$$

$$\mu_a^R(x) \;=\; R(x)\; a \;=\; x\; L(a) \;=\; \Big[(\mu\; L)(a)\Big]\,(x) \quad , \quad \forall\; x \in A\;, \tag{13}$$

and let u_R be defined by $u_R(a) = \langle \lambda_a^R \;,\; \mu_a^R\rangle$, $\forall\; a \in \mathcal{A}_R$. Then we have:

Proposition 2.9. *Let $\langle L, R\rangle \in \mathcal{W}(A)$ define $\mathcal{W}(\mathcal{A}_R)$. Then*

(i) u_R is the regular homomorphism of $\mathcal{A}_R$ into $\mathcal{W}(\mathcal{A}_R)$, and if there exists $z \in A$ such that $L(z) = R(z)$, we have in the algebra $\mathcal{D}_P(A)$:

$$\Big\{\langle\lambda_z \;,\; \mu_z\rangle \;,\; \langle\lambda_z^R \;,\; \mu_z^R\rangle\Big\} \quad\subset\quad \mathcal{W}(A) \cap \mathcal{W}(\mathcal{A}_R). \tag{14}$$

(ii) If $\langle L, R\rangle$ is a multiplier $(T = L = R)$, then $\mathcal{W}(A)$ is a Banach subalgebra of $\mathcal{W}(\mathcal{A}_R)$.

Proof. (i) Short computations using the definition of the product in $\mathcal{W}(\mathcal{A}_R)$ show that $a \;\mapsto\; u_R(a) \;=\; \langle\lambda_a^R \;,\; \mu_a^R\rangle$ is a homomorphism into $\mathcal{W}(\mathcal{A}_R)$ and that λ_a^R and μ_a^R satisfy the DC-Property in $\mathcal{W}(\mathcal{A}_R)$. The regular homomorphism $u :\; a \mapsto \langle\lambda_a \;,\; \mu_a\rangle$, being from A into $\mathcal{W}(A)$, $\langle\lambda_z \;,\; \mu_z\rangle$ and $\langle\lambda_{R(z)} \;,\; \mu_{R(z)}\rangle$, belong to $\mathcal{W}(A)$, for each $z \in A$. Let us show that these two elements belong to $\mathcal{W}(\mathcal{A}_R)$, whenever $z \in A$ fulfills $L(z) = R(z)$. Indeed, for each $a \in A$, we have

$$x \;.\; \lambda_a(y) \;=\; R(x)\; a\; y \;=\; x\; L(a)\; y \quad , \quad \forall\; x,\; y \in A, \tag{15}$$

$$\mu_a(x) \;.\; y \;=\; R(x\; a)\; y \;=\; x\; R(a)\; y \quad , \quad \forall\; x,\; y \in A. \tag{16}$$

Hence, if $z \in A$ satisfies $L(z) = R(z)$, and since A is without annihilators, relations (15) and (16) above yield $\langle\lambda_z \;,\; \mu_z\rangle \in \mathcal{W}(\mathcal{A}_R)$. For z with the above property, we have

$$\begin{aligned} x \;.\; \lambda_{R(z)}(y) &= R(x)\; R(z)\; y = x\; L\; R(z)\; y = x\; R\; L(z)\; y \quad (L \text{ and } R \text{ commute}) \\ &= R\Big(x\; L(z)\Big)\; y \;=\; \Big(x\; L(z)\Big)\;.\; y \;=\; \Big(x\; R(z)\Big)\,.\; y \quad (L(z) = R(z)) \\ &= \mu_{R(z)}(x)\;.\; y\;, \end{aligned}$$

that is, $\langle\lambda_{R(z)} \;,\; \mu_{R(z)}\rangle \;=\; \langle\lambda_z^R \;,\; \mu_z^R\rangle \in \mathcal{W}(\mathcal{A}_R)$, and the desired result, $\langle\lambda_z^R \;, \mu_z^R\rangle \in \mathcal{W}(\mathcal{A}_R) \;\cap\; \mathcal{W}(A)$, hence follows.

(ii) Let $\langle L, R\rangle \in \mathcal{W}(A)$ be such that $L = R$, so that R satisfies $aR(b) = R(a)b$ for all $a, b \in A$. Then given any $\langle T, S\rangle \in \mathcal{W}(A)$, we have

$$\begin{aligned} a\;.\;T(b) \quad &= \quad R(a)\; T(b) \;=\; a\; R\;(T(b)) \;=\; R(a\; T(b)) \\ &= \quad R(S(a)\; b) \;=\; R(S(a))\; b \;=\; S(a)\;.\; b\;, \end{aligned}$$

which proves that $\langle T, S\rangle \;\in\; \mathcal{W}(\mathcal{A}_R)$. □

Remark 2.10. There exists a homomorphism φ from $\mathcal{W}(\mathcal{A}_R)$ into $\mathcal{W}(A)$ which makes the following diagram commutative:

$$\begin{array}{ccc} R:\mathcal{A}_R & \longrightarrow & Im(R)\subset A \\ \Big\downarrow{\scriptstyle u_R} & & \Big\downarrow{\scriptstyle u} \\ \varphi:\mathcal{W}(\mathcal{A}_R) & \longrightarrow & \mathcal{W}(A) \end{array}$$

where u and u_R denote respectively the regular homomorphisms of A into $\mathcal{W}(A)$ and of $\mathcal{A}_R$ into $\mathcal{W}(\mathcal{A}_R)$. Indeed, the homomorphisms uR and u_R are defined from $\mathcal{A}_R$, respectively into $\mathcal{W}(A)$ and $\mathcal{W}(\mathcal{A}_R)$, and satisfy $Ker(u_R) = \{0\} \subset Ker(uR)$. Then there exists a map $\varphi : u_R(\mathcal{A}_R) \longrightarrow \mathcal{W}(A)$ such that we have $\varphi\, u_R = u\, R$. Moreover, φ is clearly a homomorphism from $u_R(\mathcal{A}_R)$ into $\mathcal{W}(A)$. □

Under the hypotheses and notations of Remark 2.10, short computations lead to the following relations, for each $a \in A$, and each $\langle T, S\rangle \in \mathcal{W}(\mathcal{A}_R)$:

(17)
$$\langle T,S\rangle\; u_R(a) \;=\; \langle \lambda_{RT(a)}\;,\;\mu_{LT(a)}\rangle \;=\; \langle \lambda^R_{T(a)}\;,\;\mu^R_{T(a)}\rangle \;=\; u_R\;(T(a))\;,$$

(18)
$$u_R(a)\;\langle T,S\rangle \;=\; \langle \lambda_{RS(a)}\;,\;\mu_{LS(a)}\rangle \;=\; \langle \lambda^R_{S(a)}\;,\;\mu^R_{S(a)}\rangle \;=\; u_R\;(S(a))\;.$$

Relations (17) and (18) show that for each $a \in A$, φ is still defined on $\mathcal{W}(\mathcal{A}_R)\, u_R(a)$ and on $u_R(a)\; \mathcal{W}(\mathcal{A}_R)$ as follows:

(19)
$$\varphi\Big(\langle T,S\rangle\; u_R(a)\Big) \;=\; \varphi(u_R(T(a))) \;=\; u\;[RT(a)] \;=\; \langle \lambda_{RT(a)},\;\mu_{RT(a)}\rangle \quad ,$$

(20)
$$\varphi\Big(u_R(a)\;\langle T,S\rangle\Big) \;=\; \varphi(u_R(S(a))) \;=\; u\;[RS(a)] \;=\; \langle \lambda_{RS(a)},\;\mu_{RS(a)}\rangle\;.$$

The next result gives the conditions under which φ extends to the whole space $\mathcal{W}(\mathcal{A}_R)$.

Theorem 2.11. *If for each $\langle T,S\rangle \in \mathcal{W}(\mathcal{A}_R)$ there exists $\langle P,Q\rangle \in \mathcal{W}(A)$ such that the system below is satisfied:*

$$(*)\qquad \begin{cases} P\;R &= R\;T, \\ Q\;R &= R\;S. \end{cases}$$

Then the map $\Theta : \mathcal{W}(\mathcal{A}_R) \longrightarrow \mathcal{W}(A)$, defined by $\Theta(\langle T,S\rangle) = \langle P,\;Q\rangle$, extends the homomorphism $\varphi :\; u_R(A) \longrightarrow \mathcal{W}(A)$. Furthermore, Θ is an isomorphism, provided R is one-to-one.

Proof. If φ extends to the whole of $\mathcal{W}(\mathcal{A}_R)$, then for each $a \in A$ and each $\langle T,\;S\rangle \in \mathcal{W}(\mathcal{A}_R)$, the element $\langle P,Q\rangle = \varphi(\langle T,S\rangle)$ would satisfy, according to (19) and (20),

$$\begin{aligned} \varphi\Big(\langle T,S\rangle\; u_R(a)\Big) &= \langle P\;,\;Q\rangle\;\varphi(u_R(a)) \;=\; \langle P\;\;,\;Q\rangle\;\langle \lambda_{R(a)}\;,\;\mu_{R(a)}\rangle \\ &= \langle \lambda_{RT(a)}\;,\;\mu_{RT(a)}\rangle\;, \\ \varphi\Big(u_R(a)\;\langle T,S\rangle\Big) &= \varphi(u_R(a))\;\langle P\;,\;Q\rangle \;=\; \langle \lambda_{R(a)}\;,\;\mu_{R(a)}\rangle\;\langle P\;,\;Q\rangle \\ &= \langle \lambda_{RS(a)}\;,\;\mu_{RS(a)}\rangle\;. \end{aligned}$$

But we have

(21)
$$\langle P\ ,\ Q\rangle\ \langle\lambda_{R(a)}\ ,\ \mu_{R(a)}\rangle\ =\ \langle P\ \lambda_{R(a)}\ ,\ \mu_{R(a)}\ Q\rangle\ =\ \langle\lambda_{PR(a)}\ ,\ \mu_{PR(a)}\rangle\ ,$$

(22)
$$\langle\lambda_{R(a)}\ ,\ \mu_{R(a)}\rangle\ \langle P\ ,\ Q\rangle\ =\ \langle\lambda_{R(a)}\ P\ ,\ Q\ \mu_{R(a)}\rangle\ =\ \langle\lambda_{QR(a)}\ ,\ \mu_{QR(a)}\rangle\ .$$

Hence, if φ extends to $\mathcal{W}(\mathcal{A}_R)$, then we must have

$$\begin{cases}\langle\lambda_{RT(a)}\ ,\ \mu_{RT(a)}\rangle\ =\ \langle\lambda_{PR(a)}\ ,\ \mu_{PR(a)}\rangle,\\ \langle\lambda_{RS(a)}\ ,\ \mu_{RS(a)}\rangle\ =\ \langle\lambda_{QR(a)}\ ,\ \mu_{QR(a)}\rangle,\end{cases}$$

from which the system $(*)$ follows. So, if for each $\langle T,S\rangle\in\mathcal{W}(\mathcal{A}_R)$ there exists $\langle P,\ Q\rangle\in\mathcal{W}(A)$ such that the system $(*)$ is satisfied, we define the map $\Theta:\mathcal{W}(\mathcal{A}_R)\longrightarrow\mathcal{W}(A)$ by

$$\Theta(\langle T,S\rangle)\ =\ \langle P,\ Q\rangle\ . \tag{23}$$

The restriction of Θ coincides with φ on $u_R(A_R)$ since for each $u_R(a)=\langle\lambda_{R(a)},\ \mu_{R(a)}\rangle$, the element $\langle P,\ Q\rangle=\langle\lambda_{R(a)},\ \mu_{R(a)}\rangle\in\mathcal{W}(A)$ clearly fulfills the system $(*)$.

We need to show that Θ as defined above is a homomorphism. So let $\langle T,S\rangle$ and $\langle T',S'\rangle$ belong to $\mathcal{W}(\mathcal{A}_R)$, and let $\langle P,Q\rangle$ and $\langle P',S'\rangle$ be elements of $\mathcal{W}(A)$ satisfying

$$\Theta(\langle T,S\rangle)\ =\ \langle P,\ Q\rangle\quad\text{ and }\quad\Theta(\langle T',S'\rangle)\ =\ \langle P',\ Q'\rangle\ . \tag{24}$$

Then $\langle T\ T',\ S'\ S\rangle\ \in\ \mathcal{W}(\mathcal{A}_R)$, $\langle P\ P',\ Q'\ Q\rangle\ \in\ \mathcal{W}(\mathcal{A}_R)$, and we have

$$(P\ P')\ R\ =\ P\ (P'\ R)\ =\ P\ (R\ T')\ =\ (P\ R)\ T'\ =\ R\ (T\ T')\ ,$$
$$(Q'\ Q)\ R\ =\ Q'\ (Q\ R)\ =\ Q'\ (R\ S)\ =\ (Q'\ R)\ S\ =\ R\ (S'\ S)\ .$$

So, $\langle PP',\ QQ'\rangle\ \in\ \mathcal{W}(A)$ fulfills the system $(*)$. Therefore, by definition of Θ, we get

$$\begin{aligned}\Theta\Big(\langle T\ T',\ S'\ S\rangle\Big)\ &=\ \langle P\ P',\ Q'\ Q\rangle\ =\ \langle P,\ Q\rangle\ \langle P',\ Q'\rangle\\ &=\ \Theta(\langle T,\ S\rangle)\ \Theta(\langle T',\ S'\rangle)\end{aligned}$$

and Θ is hence a homomorphism from $\mathcal{W}(\mathcal{A}_R)$ into $\mathcal{W}(A)$, as desired.

Now, if $\langle T,S\rangle\in\mathcal{W}(\mathcal{A}_R)$ is such that $\langle P,\ Q\rangle=\Theta(\langle T,S\rangle)=0$, then by virtue of the system $(*)$, the following holds: $Im(T)\ \cup\ Im(S)\ \subset\ Ker(R)$, and if R is a one-to-one map, then T and S must be identically null. □

We next deal with the uniqueness of the above extension of the homomorphism φ.

Theorem 2.12. *Let us assume that there exists another extension Σ of the homomorphism φ to the entire $\mathcal{W}(\mathcal{A}_R)$. Then, Σ "fulfills" the system $(*)$, and in the particular case where R is bijective, there exists a unique way of extending φ to $\mathcal{W}(\mathcal{A}_R)$.*

Proof. Σ must coincide with φ on the image $u_R(A_R)$ of the regular homomorphism from $\mathcal{A}_R$ into $\mathcal{W}(\mathcal{A}_R)$, that is,

$$\Sigma\ (u_R(a))\ =\ \Sigma\ (\langle\lambda_{R(a)},\ \mu_{L(a)}\rangle)\ =\ \langle\lambda_{R(a)},\ \mu_{R(a)}\rangle\ , \tag{25}$$

and since $u_R(A)$ is a two-sided ideal in $\mathcal{W}(\mathcal{A}_R)$, Σ must also satisfy

$$\Sigma\,(\langle T,S\rangle\; u_R(a)) \;=\; \Sigma\,(\langle \lambda_{RT(a)},\ \mu_{LT(a)}\rangle) \;=\; \langle \lambda_{RT(a)},\ \mu_{RT(a)}\rangle\ , \tag{26}$$

$$\Sigma\,(u_R(a)\;\langle T,S\rangle) \;=\; \Sigma\,(\langle \lambda_{RS(a)},\ \mu_{LS(a)}\rangle) \;=\; \langle \lambda_{RS(a)},\ \mu_{RS(a)}\rangle\ . \tag{27}$$

So, if $\Sigma(\langle T,S\rangle) \;=\; \langle U,\ V\rangle$, then we must have

$$\begin{array}{rcl}
\Sigma\,(\langle T,S\rangle\; u_R(a)) & = & \langle U,\ V\rangle\; \langle \lambda_{R(a)},\ \mu_{R(a)}\rangle \;=\; \langle \lambda_{UR(a)},\ \mu_{UR(a)}\rangle \\
& = & \langle \lambda_{RT(a)},\ \mu_{RT(a)}\rangle\ , \\
\Sigma\,(u_R(a)\;\langle T,S\rangle) & = & \langle \lambda_{R(a)},\ \mu_{R(a)}\rangle\; \langle U,\ V\rangle \;=\; \langle \lambda_{VR(a)},\ \mu_{VR(a)}\rangle \\
& = & \langle \lambda_{RS(a)},\ \mu_{RS(a)}\rangle\ .
\end{array}$$

The above relations show that $\Sigma(\langle T,S\rangle) = \langle U,V\rangle$ satisfies the system $(*)$ as well, and therefore the following system $(**)$ is also fulfilled by $\langle P,Q\rangle$ and $\langle U,V\rangle$:

$$\left\{ \begin{array}{rcl} P\ R & = & U\ R, \\ Q\ R & = & V\ R. \end{array} \right. \tag{$**$}$$

But the conditions in $(**)$ above may also be interpreted in the following way, in terms of restrictions of maps: $P|_{Im(R)} \;=\; U|_{Im(R)}$ and $Q|_{Im(R)} \;=\; V|_{Im(R)}$, which, in the case where R is bijective (and hence onto A), gives

$$\Theta\,(\langle T,S\rangle) \;=\; \langle P,Q\rangle \;=\; \langle U,V\rangle \;=\; \Sigma\,(\langle T,S\rangle)\ , \tag{28}$$

that is, $\Theta = \Sigma$, which proves the uniqueness of the extension of φ. In such a case where the map R is bijective, R admits a well-defined inverse $R^{-1} \in B(A)$ which is a right centralizer, and therefore, the system $(*)$ gives, for $\langle P,Q\rangle = \Theta(\langle T,S\rangle)$,

$$P \;=\; R\ T\ R^{-1} \quad \text{and} \quad Q \;=\; R\ S\ R^{-1}\ , \tag{29}$$

which completes the proof. $\square$

Let us denote by $W^R(A)$, the normed algebra of bounded double centralizers of $Im(R) \subseteq A$. Then it is clear that $\mathcal{W}(A) \subseteq W^R(A)$, and that equality holds whenever R is onto A. The next result links $W^R(A)$ with the space of elements $P,Q \in B(A)$ which satisfy the system $(*)$ with the elements of $\mathcal{W}(\mathcal{A}_R)$.

Proposition 2.13. *Let* $P,Q \in B(A)$ *satisfy the system* $(*)$ *with some* $\langle T,S\rangle \in \mathcal{W}(\mathcal{A}_R)$. *Then* $\langle P,Q\rangle \in W^R(A)$.

Proof. For all $a,b \in Im(R)$, we have $\alpha,\beta \in A$, such that $a = R(\alpha)$; $b = R(\beta)$, and then, for each $\langle P,\ Q\rangle \in B(A)$ satisfying the system $(*)$ with some $\langle T,S\rangle \in \mathcal{W}(\mathcal{A}_R)$, the following holds:

$$\begin{array}{rcl}
a\ P(b) & = & R(\alpha)\ (P\ R)(\beta) \;=\; R(\alpha)\ (R\ T)(\beta) \quad \text{(since } (*) \text{ holds)} \\
& = & R\Big(R(\alpha)\ T(\beta)\Big) \\
& = & R\Big(\alpha\ .\ T(\beta)\Big) \;=\; R\Big(S(\alpha)\ .\ \beta\Big) \quad \text{(since } \langle T,S\rangle \in \mathcal{W}(\mathcal{A}_R)\text{)} \\
& = & R\Big(R\ S(\alpha)\ \beta\Big) \\
& = & R\ S(\alpha)\ R(\beta) \;=\; Q\ R(\alpha)\ R(\beta) \quad \text{(since } (*) \text{ holds)} \\
& = & Q(a)\ b\ ,
\end{array}$$

which is the *DC-Property* in $W^R(A)$, and the proof is complete. $\square$

As is well known, the theory of double centralizers is a helpful device in the study of extensions of algebras. Let us recall the following definition (cf. [4], pp. 33 - 34).

Definition 2.14. Let A and B be Banach algebras. An *extension of* A *by* C is a *short exact sequence* $0 \longrightarrow A \overset{\rho}{\to} B \overset{\psi}{\to} C \longrightarrow 0$ of Banach algebras. The extension is called a *semidirect product* if there is a continuous homomorphism $\chi : C \longrightarrow B$, such that $\psi\,\chi$ is the identity map on C. The short exact sequence is then said to *split*, with splitting homomorphism χ.

Theorem 2.15. *Let* A *be a Banach algebra with a left approximate identity bounded by one. Let* $\mathcal{A}_R$ *be induced by* $\langle L, R\rangle \in \mathcal{W}(A)$ *with* R *bijective. Then there exists an extension of* A *by* $\mathcal{W}(\mathcal{A}_R)$*:*

$$0 \longrightarrow A \overset{\rho}{\to} B \overset{\psi}{\to} \mathcal{W}(\mathcal{A}_R) \longrightarrow 0\ , \tag{30}$$

and a continuous homomorphism $\Theta : B \longrightarrow \mathcal{W}(A)$*, such that the following diagram commutes:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{u} & \mathcal{W}(A) & \xrightarrow{\pi} & \mathcal{W}(A)/A & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle I_A} & & \uparrow{\scriptstyle \Theta} & & \uparrow{\scriptstyle \tau} & & \\ 0 & \longrightarrow & A & \xrightarrow{\rho} & B & \xrightarrow{\psi} & \mathcal{W}(\mathcal{A}_R) & \longrightarrow & 0. \end{array}$$

This extension, which is unique up to equivalence, is moreover a semidirect product.

Proof. Identifying A with $u(A)$ under the regular homomorphism u of A into $\mathcal{W}(A)$, consider the short exact sequence $0 \longrightarrow A \overset{u}{\to} \mathcal{W}(A) \overset{\pi}{\to} C \longrightarrow 0$, where π is the natural map. Since R is bijective, Theorem 2.12 asserts that Θ is a homeomorphic isomorphism of $\mathcal{W}(\mathcal{A}_R)$ onto $\mathcal{W}(A)$. The map $\tau : \mathcal{W}(\mathcal{A}_R) \longrightarrow \mathcal{W}(A)/A$, defined by

$$\tau\,(\langle T,\ S\rangle) \ = \ \Theta\,(\langle T,\ S\rangle) \ + \ A \quad , \quad \forall\ \langle T, S\rangle \in \mathcal{W}(\mathcal{A}_R)\ , \tag{31}$$

is hence a continuous homomorphism satisfying: $\tau \ = \ \pi\,\Theta$. Then, the hypotheses of Theorem 1.2.10 in [4], page 34, are satisfied, and its conclusion provides the existence of the extension of A given by relation (30), and unique up to equivalence. Furthermore, in virtue of Theorem 1.2.11 in [4], page 35, the existence of τ satisfying $\tau = \pi\,\Theta$ shows that the extension is a semidirect product. □

Now, let us assume that A is *commutative*, and let us denote by $\mathcal{M}(A)$ the algebra of multipliers of A, which necessarily coincides with $\mathcal{W}(A)$. In [3], Corollary 1.3.1, page 23, R. Larsen proved that for a *commutative semisimple* Banach algebra A, the following equality holds: $\mathcal{M}(A) \ = \ \mathcal{M}(\mathcal{A}_R)$. In what follows, we still obtain the same result when the assumption of semisimplicity on A is removed. We shall need the following:

Lemma 2.16. *Let* A *be a commutative Banach algebra and let* $R \in \mathcal{M}(A)$*. Then the following conditions are equivalent:*

(i) $\mathcal{A}_R$ *has no annnihilators;*

(ii) A *has no annihilators and* R *is one-to-one.*

Proof. Assume that (i) is fulfilled, and let $a \in A$ be given. Then, A being commutative and R being a multiplier of A, we get for all $x \in A$:

(32)
$$R(x\ a) \ = \ R(a\ x) \ = \ R(a)\ x \ = \ a\ .\ x \ = \ x\ .\ a \quad (\mathcal{A}_R \text{ is commutative})\ ,$$

which, since $\mathcal{A}_R$ is without annihilators, implies that A has no annihilators. Moreover, due to Corollary 2.3, R is one-to-one, if $\mathcal{A}_R$ has no annihilators. Hence (i) implies (ii).

Conversely if (ii) is satisfied, then Corollary 2.3 again implies that $\mathcal{A}_R$ has no annihilators, and the equivalence of (i) and (ii) is therefore established. □

We can now prove:

Theorem 2.17. *Let A be a commutative Banach algebra and let $R \in \mathcal{M}(A)$. Then $\mathcal{M}(A) = \mathcal{M}(\mathcal{A}_R)$ if and only if one of the equivalent conditions in Lemma* 2.16 *is satisfied.*

Proof. We assume that condition (i) of Lemma 2.16 holds. Let $T \in \mathcal{M}(A)$; then for all $a, b \in A$, we get $T(a\ .\ b) = T[R(a)\ b] = R(a)\ T(b) = a\ .\ T(b)$and

$$\begin{aligned} T(a\ .\ b) &= R(a)\ T(b) = T(b)\ R(a) = R[T(b)\ a] = R[b\ T(a)] \\ &= R(T(a)\ b) = R[T(a)]\ b = T(a)\ .\ b\ , \end{aligned}$$

which means that $T \in \mathcal{M}(A)$, and consequently, $\mathcal{M}(A) \subseteq \mathcal{M}(\mathcal{A}_R)$.

Conversely, let $T \in \mathcal{M}(\mathcal{A}_R)$. Then, for all $a, b, x \in A$, we get on one hand

$$\begin{aligned} R(x)\ T(a\ b) &= x\ .\ T(a\ b) = T(x)\ .\ a\ b = a\ b\ T(x) = R(ab)\ T(x) \\ &= [R(a)\ b]\ T(x) = R(a)\ T(x)\ b = [a\ .\ T(x)]\ b = [T(a)\ .\ x]\ b \\ &= (R[T(a)]\ x) = T(a)\ R(x)\ b = R(x)[T(a)\ b]\ , \end{aligned}$$

which implies, since $\mathcal{A}_R$ is without annihilators, $T(a\ b) = T(a)\ b$.

On the other hand, let $a, b, x \in A$; then

$$R(x)\ T(a\ b) = R(a\ b)\ T(x) = a\ [b\ .\ T(x)] = a\ [T(b)\ R(x)] = R(x)\ [a\ T(b)]\ ,$$

and once again, the fact that $\mathcal{A}_R$ is without annihilators, leads to $T(a\ b) = a\,T(b)$. Therefore, $T \in \mathcal{M}(A)$, and this provides the reverse inclusion $\mathcal{M}(\mathcal{A}_R) \subseteq \mathcal{M}(A)$, and hence the desired equality $\mathcal{M}(\mathcal{A}_R) = \mathcal{M}(A)$. □

The following result also appears in [3], Corollary 1.3.2, page 23:

Proposition 2.18. *Let A be a commutative semisimple Banach algebra which admits factorization, i.e.: For every $a \in A$, there exist $x, y \in A$, such that $a = xy$. Then the following conditions are equivalent:*

(i) *A_T admits factorization;*

(ii) *T is invertible.*

Dropping the assumption of semisimplicity on A, we can prove the following:

Theorem 2.19. *Let A be a commutative Banach algebra which admits factorization, and let $R \in \mathcal{M}(A)$ be such that $\mathcal{A}_R$ is without annihilators. Then $\mathcal{A}_R$ admits factorization, if and only if R is invertible.*

Proof. Assume that $\mathcal{A}_R$ admits factorization. Then, since $\mathcal{A}_R$ is without annihilators, R is one-to-one, according to Lemma 2.16. Moreover, since $\mathcal{A}_R$ admits factorization, for each $a \in \mathcal{A}_R$ there exist $\alpha, \beta \in \mathcal{A}_R$, such that $a = \alpha\ .\ \beta = R(\alpha)\ \beta = R(\alpha\ \beta)$, which means that R is onto A, whence R is bijective and R^{-1} exists.

Conversely, let R be invertible; then R is one-to-one, and since it is also onto A, we get for each $a \in A$ some $\alpha \in \mathcal{A}_R$ satisfying $a = R(\alpha)$. But, A admits factorization, and therefore α has the following decomposition: $\alpha = u\,v$, $u, v \in A$.

Hence we get $a = R(\alpha) = R(u\,v) = R(u)\,v = u\,.\,v$, so that $\mathcal{A}_R$ also admits factorization, and the proof is complete. □

Coming back to the general case where A is neither commutative nor semisimple, we give the analogous version of Theorem 2.17, when the multiplier algebra is replaced by the double centralizer algebra, as follows:

Theorem 2.20. *Let A be without annihilators, and let $\langle L, R\rangle \in \mathcal{W}(A)$ define $\mathcal{A}_R$ and be such that R is bijective. Then $\mathcal{W}(\mathcal{A}_R)$ and $\mathcal{W}(A)$ are homeomorphically isomorphic.*

Proof. According to Theorems 2.11 and 2.12, there exists a homomorphism Θ which extends in a unique way the homomorphism φ constructed in Remark 2.10 to the algebra $\mathcal{W}(\mathcal{A}_R)$. Moreover, according to Theorem 2.11, each element $\langle T, S\rangle \in Ker(\Theta)$ satisfies $Im(T) \cup Im(S) \subset Ker(R)$. So, if R is bijective and hence one-to-one, $Ker(R) = \{0\}$, and Θ is also one-to-one. Therefore, Θ is a bijection of $\mathcal{W}(\mathcal{A}_R)$ onto $\mathcal{W}(A)$. But, according to relation (29), the positive constant $K_R = \|R^{-1}\|\|R\|$ (depending only on R) satisfies $\|\Theta\ (\langle T, S\rangle)\| \leq K_R\ \|\langle T, S\rangle\|$; $\forall\ \langle T, S\rangle \in \mathcal{W}(\mathcal{A}_R)$, so that Θ is a bounded linear map from $\mathcal{W}(\mathcal{A}_R)$ onto the Banach space $\mathcal{W}(A)$, that is, a homeomorphism, and the proof is complete. □

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