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SMOOTH NORMS ON CERTAIN $C(K)$ SPACES

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ABSTRACT. $C(K)$ spaces admit an equivalent C^∞-smooth renorming whenever $K^{(\omega_1)} = \emptyset$.

In this note we consider the problem of finding on a given $C(K)$ space an equivalent norm of the highest possible smoothness. It is a classical result (e.g. [DGZ]) that the existence of an equivalent C^1-smooth norm on a Banach space implies that the space is Asplund. On the other hand, deep examples of Haydon ([H], see also [DGZ]) show that not every $C(K)$ Asplund space admits an equivalent Gâteaux smooth renorming.

So far, an equivalent C^∞ renorming was constructed on $C(K)$ spaces where $K^{(\omega_0)} = \emptyset$ ([GPWZ]), and a C^1 norm is guaranteed when $K^{(\omega_1)} = \emptyset$ ([D]). Haydon's C^∞ renorming techniques work well for certain tree-like compact sets K, which may have nonempty derived sets of arbitrary large ordinal number, but their disadvantage is that they put very strong structural restrictions on K (apart from the obvious and necessary scatteredness). This is not accidental, because the above-mentioned example of $C(K)$ without a Gâteaux norm has $K^{(\omega_1)}$ a singleton.

In our note we show the existence of C^∞ renormings whenever $K^{(\omega_1)} = \emptyset$. This is the best possible result without additional structural assumptions on K. However, it is really only a small step towards a desired general theorem linking the existence of C^∞ renorming of $C(K)$ to some other properties of the space, such as the existence of a dual LUR renorming of $C(K)$. For background material and notation we refer to [DGZ].

Definition 1. Let $S \subset \ell_\infty(\Gamma)$, $\Phi : S \to \mathbb{R}$. We say that Φ locally depends on finitely many coordinates (LDF) if for every $f \in S$ there exist $\varepsilon > 0$, $\gamma_1, \dots, \gamma_n \in \Gamma$ and $F : \mathbb{R}^n \to \mathbb{R}$ such that:

$$\Phi(g) = F(g(\gamma_1), \dots, g(\gamma_n)) \quad \text{whenever } g \in B(f, \varepsilon) \cap S.$$

Given $1 > \delta > 0$, find $\phi_\delta : \mathbb{R} \to \mathbb{R}$ such that ϕ_δ is C^∞-smooth, even and convex, and $\phi_\delta([0, 1-\delta]) = 0$, $\phi_\delta(1) = 1$.

Definition 2. Let $f \in \ell_\infty(\Gamma)$. Put $f^\infty = \inf\{t, \operatorname{card}\{\gamma, |f(\gamma)| > t\} \text{ is finite}\}$.

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Lemma 3. *Let $1 > \delta > 0$, and let $\Phi : \ell_\infty(\Gamma) \to \mathbb{R} \cup \{+\infty\}$ be a convex function defined by*

$$\Phi(f) = \sum_{\gamma \in \Gamma} \phi_\delta(f(\gamma)).$$

Then Φ restricted to $\{f \in \ell_\infty(\Gamma), f^\infty < 1 - \delta\}$ is finite, LDF and C^∞-smooth.

Proof. Given $f, f^\infty < 1 - \delta$, the set $\Theta = \{\gamma \in \Gamma, |f(\gamma)| > f^\infty + \frac{1-\delta-f^\infty}{4}\}$ is finite. Thus for $g \in B(f, \frac{1-\delta-f^\infty}{2})$ we have $\Phi(g) = \sum_{\gamma \in \Theta} \phi_\delta(g(\gamma))$, which is a finite sum of C^∞ smooth convex functions. □

Theorem 4. *Let K be a scattered compact, $K^{(\omega_1)} = \emptyset$. Then $C(K)$ admits an equivalent LDF and C^∞-smooth norm.*

Proof. There is $\Lambda < \omega_1$ such that $K^{(\Lambda)} \neq \emptyset$ is finite and $K^{(\Lambda+1)} = \emptyset$. The space $C_0(K) = \{f \in C(K), f(K^{(\Lambda)}) = 0\}$ is isomorphic to $C(K)$.

Put $L_\alpha = K^{(\alpha)} \backslash K^{(\alpha+1)}$, $\alpha \leq \Lambda$, and fix $\{\delta_\alpha\}_{\alpha \leq \Lambda}$, such that $\delta_\alpha > 0$ and $\prod_{\alpha=0}^{\Lambda}(1 + \delta_\alpha) < 2$. Put $\psi_\alpha = \phi_{\frac{\delta_{\alpha+1}}{8}}$, $D_\alpha = \prod_{\beta=0}^{\alpha}(1 + \delta_\beta)$. Let us define a convex function $\Psi : C_0(K) \to \mathbb{R} \cup \{+\infty\}$ by $\Psi(f) = \sum_{0 \leq \alpha \leq \Lambda} \sum_{\gamma \in L_\alpha} \psi_\alpha(D_\alpha \cdot f(\gamma))$.

Our aim will be to show that $\Psi^{-1}([0, \frac{1}{2}])$ is the unit ball of an equivalent LDF (canonically, $C_0(K) \subset \ell_\infty(K)$) and C^∞ smooth norm on $C_0(K)$.

For every $f \in C_\infty(K)$, $0 \leq \alpha \leq \Lambda$ put

$$a_\alpha^f = (f|_{L_\alpha})^\infty, \qquad b_\alpha^f = \|f|_{L_\alpha}\|_\infty.$$

Note that $\alpha \to D_\alpha$ is an increasing and thus upper semicontinuous (usc) function on $[0, \Lambda]$ (which is a compact space when considered with its natural interval topology). We have:

$$(*) \qquad \begin{aligned} b_{\alpha+1}^f = a_\alpha^f \quad &\text{and} \quad \alpha \to b_\alpha^f \in C[0, \Lambda], \\ b_\alpha^f = &\lim_{\tau \nearrow \alpha, \tau < \alpha} a_\tau^f. \end{aligned}$$

Therefore, for a fixed $f \neq 0$, $\alpha \to D_\alpha \cdot b_\alpha^f$ is usc on $[0, \Lambda]$, so it attains its maximum M_f at some $\beta \in [0, \Lambda)$. Clearly, $M_f \geq D_{\alpha+1} b_{\alpha+1}^f = (1 + \delta_{\alpha+1}) D_\alpha a_\alpha^f$, and so $D_\alpha a_\alpha^f \leq \frac{M_f}{1+\delta_{\alpha+1}} \leq M_f(1 - \frac{\delta_{\alpha+1}}{2})$, for all $\alpha \in [0, \Lambda]$.

First note that $D_\beta b_\beta^f > (D_\alpha a_\alpha^f)^\infty$. Indeed, otherwise there exist increasing sequences $\alpha_n \nearrow \gamma$, $\alpha_n \in [0, \Lambda)$, $D_{\alpha_n} a_{\alpha_n}^f \nearrow D_\beta b_\beta^f$. However, by $(*)$, $a_{\alpha_n}^f \to b_\gamma^f$, $D_\gamma > (1 + \delta_\gamma) D_{\alpha_n}$ for $n \in \mathbb{N}$, and consequently $D_\gamma b_\gamma^f \geq (1 + \delta_\gamma) D_\beta b_\beta^f$, a contradiction. So there exists $\varepsilon_f > 0$, such that $\operatorname{card}\{\alpha, D_\alpha a_\alpha^f > D_\beta b_\beta^f - 2\varepsilon_f\}$ is finite. Next we claim that $\operatorname{card}\{\alpha, D_\alpha b_\alpha^f > D_\beta b_\beta^f - \varepsilon_f\}$ is also finite.

Again, otherwise there exists $\alpha_n \nearrow \gamma$, $D_{\alpha_n} b_{\alpha_n}^f > D_\beta b_\beta^f - \varepsilon_f$. Using $(*)$, and passing to a suitable subsequence of $\{\alpha_n\}$ we find $\beta_n \nearrow \gamma$, $\alpha_n \leq \beta_n < \alpha_{n+1}$ such that

$$D_{\beta_n} \alpha_{\beta_n}^f \geq D_{\alpha_{n+1}} b_{\alpha_{n+1}}^f - \frac{\varepsilon_f}{2}.$$

This is a contradiction with the definition of ε_f, since

$$\{\beta_n\} \subset \{\alpha, D_\alpha a_\alpha^f > D_\beta \alpha_\beta^f - 2\varepsilon_f\}.$$

Put $O = \{f \in C_0(K), M_f < 1\} \subset 2B_{C_0(K)}$. We claim that $\Psi|_O$ is finite, C^∞-smooth and LDF. Moreover, $\Psi^{-1}([0, \frac{1}{2}]) \subset \operatorname{int} O$, which by the implicit function theorem ([Di]) finishes the proof.

Choose any $f \in O$. Consider the finite set $A = \{\alpha \in [0, \Lambda]$, either $D_\alpha b_\alpha^f > M_f - \varepsilon_f$ or $\delta_{\alpha+1} \geq \frac{\varepsilon_f}{4}\}$, and put $\delta_f = \min\{\frac{\delta_{\alpha+1}}{16}, \alpha \in A\}$. Let us check that $\Psi|_{B(f,\delta_f)}$ depends on finitely many coordinates (therefore it is necessarily C^∞-smooth). If $\gamma \in L_\alpha$, $\alpha \notin A$, then $4\delta_{\alpha+1} < \varepsilon_f$ and $D_\alpha b_\alpha^f \leq M_f - \varepsilon_f \leq 1 - \varepsilon_f \leq 1 - 4\delta_{\alpha+1}$. For $g \in B(f, \delta_f)$, $|g(\gamma)| \leq |f(\gamma)| + \delta_f$, so

$$|D_\alpha g(\gamma)| \leq D_\alpha b_\alpha^f + 2\delta_f \leq 1 - \varepsilon_f + 2\delta_f \leq 1 - 2\delta_{\alpha+1}.$$

Consequently,

$$\psi_\alpha(D_\alpha g(\gamma)) = 0 \quad \text{and} \quad \Psi(g) = \sum_{\alpha \in A} \sum_{\gamma \in L_\alpha} \psi_\alpha(D_\alpha g(\gamma)).$$

If $\alpha \in A$, $\gamma \in L_\alpha$, and $|f(\gamma)| < a_\alpha^f(1 + \frac{\delta_{\alpha+1}}{8})$, then

$$|D_\alpha f(\gamma)| \leq D_\alpha a_\alpha^f \left(1 + \frac{\delta_{\alpha+1}}{8}\right) \leq M_f \left(1 - \frac{\delta_{\alpha+1}}{2}\right)\left(1 + \frac{\delta_{\alpha+1}}{8}\right) \leq 1 - \frac{\delta_{\alpha+1}}{4}.$$

If $g \in B(f, \delta_f)$, we then have

$$|D_\alpha g(\gamma)| \leq 1 - \frac{\delta_{\alpha+1}}{4} + 2\delta_f \leq 1 - \frac{\delta_{\alpha+1}}{8}.$$

Consequently, in this case also $\psi_\alpha(D_\alpha g(\gamma)) = 0$. The remaining set $S = \{\gamma, \gamma \in L_\alpha$ for $\alpha \in A$ and $f(\gamma) \geq a_\alpha^f(1 + \frac{\delta_{\alpha+1}}{8})\}$ is clearly finite, and we have

$$\Psi(g) = \sum_{\alpha \in A} \sum_{\gamma \in S \cap L_\alpha} (D_\alpha g(\gamma))$$

whenever $g \in B(f, \delta_f)$. This proves (Lemma 3) that $\Psi|_O$ is C^∞-smooth and LDF. It is obvious that $M_f \leq \frac{1}{4}$ implies $\Psi(f) = 0$ and $M_f = 1$ implies $\Psi(f) \geq 1$. Thus $B = \Psi^{-1}([0, \frac{1}{2}])$ is an equivalent unit ball of $C_0(K)$. By the implicit function theorem, its Minkowski functional is C^∞-smooth and LFD. □

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