SMOOTH NORMS ON CERTAIN $C(K)$ SPACES

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Abstract. $C(K)$ spaces admit an equivalent $C^\infty$-smooth renorming whenever $K^{(\omega_1)} = \emptyset$.

In this note we consider the problem of finding on a given $C(K)$ space an equivalent norm of the highest possible smoothness. It is a classical result (e.g. [DGZ]) that the existence of an equivalent $C^1$-smooth norm on a Banach space implies that the space is Asplund. On the other hand, deep examples of Haydon ([H], see also [DGZ]) show that not every $C(K)$ Asplund space admits an equivalent Gâteaux smooth renorming.

So far, an equivalent $C^\infty$ renorming was constructed on $C(K)$ spaces where $K^{(\omega_0)} = \emptyset$ ([GPWZ]), and a $C^1$ norm is guaranteed when $K^{(\omega_1)} = \emptyset$ ([D]). Haydon’s $C^\infty$ renorming techniques work well for certain tree-like compact sets $K$, which may have nonempty derived sets of arbitrary large ordinal number, but their disadvantage is that they put very strong structural restrictions on $K$ (apart from the obvious and necessary scatteredness). This is not accidental, because the above-mentioned example of $C(K)$ without a Gâteaux norm has $K^{(\omega_1)}$ a singleton.

In our note we show the existence of $C^\infty$ renormings whenever $K^{(\omega_1)} = \emptyset$. This is the best possible result without additional structural assumptions on $K$.

However, it is really only a small step towards a desired general theorem linking the existence of $C^\infty$ renorming of $C(K)$ to some other properties of the space, such as the existence of a dual LUR renorming of $C(K)$. For background material and notation we refer to [DGZ].

Definition 1. Let $S \subset \ell_\infty(\Gamma)$, $\Phi : S \to \mathbb{R}$. We say that $\Phi$ locally depends on finitely many coordinates (LDF) if for every $f \in S$ there exist $\varepsilon > 0$, $\gamma_1, \ldots, \gamma_n \in \Gamma$ and $F : \mathbb{R}^n \to \mathbb{R}$ such that:

$$\Phi(g) = F(g(\gamma_1), \ldots, g(\gamma_n))$$

whenever $g \in B(f, \varepsilon) \cap S$.

Given $1 > \delta > 0$, find $\phi_\delta : \mathbb{R} \to \mathbb{R}$ such that $\phi_\delta$ is $C^\infty$-smooth, even and convex, and $\phi_\delta([0, 1 - \delta]) = 0$, $\phi_\delta(1) = 1$.

Definition 2. Let $f \in \ell_\infty(\Gamma)$. Put $f^\infty = \inf\{t, \text{card}\{\gamma, |f(\gamma)| > t\} \text{ is finite}\}$. 

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Lemma 3. Let $1 > \delta > 0$, and let $\Phi : \ell_\infty(\Gamma) \to \mathbb{R} \cup \{+\infty\}$ be a convex function defined by

$$\Phi(f) = \sum_{\gamma \in \Gamma} \phi_\delta(f(\gamma)).$$

Then $\Phi$ restricted to $\{f \in \ell_\infty(\Gamma), f^\infty < 1 - \delta\}$ is finite, LDF and $C^\infty$-smooth.

Proof. Given $f, f^\infty < 1 - \delta$, the set $\Theta = \{\gamma \in \Gamma, |f(\gamma)| > f^\infty + \frac{1 - \delta - f^\infty}{2}\}$ is finite. Thus for $g \in B(f, \frac{1 - \delta - f^\infty}{2})$ we have $\Phi(g) = \sum_{\gamma \in \Theta} \phi_\delta(g(\gamma))$, which is a finite sum of $C^\infty$ smooth convex functions. \qed

Theorem 4. Let $K$ be a scattered compact, $K^{(\omega_1)} = \emptyset$. Then $C(K)$ admits an equivalent LDF and $C^\infty$-smooth norm.

Proof. There is $\Lambda < \omega_1$ such that $K^{(\Lambda)} \neq \emptyset$ is finite and $K^{(\Lambda+1)} = \emptyset$. The space $C_0(K) = \{f \in C(K), f(K^{(\Lambda)}) = 0\}$ is isomorphic to $C(K)$.

Put $L_\alpha = K^{(\alpha)} \setminus K^{(\alpha+1)}$, $\alpha \leq \Lambda$, and fix $\zeta_\alpha \in L_\alpha$, such that $\zeta_\alpha > 0$ and $\prod_{\alpha=0}^\Lambda (1 + \delta_\alpha)$. Let us define a convex function $\Psi : C_0(K) \to \mathbb{R} \cup \{+\infty\}$ by $\Psi(f) = \sum_{0 < \alpha \leq \Lambda} \sum_{\gamma \in L_\alpha} \rho(\zeta_\alpha, f(\gamma))$.

Our aim will be to show that $\Psi^{-1}([0, \frac{1}{2}])$ is the unit ball of an equivalent LDF (canonically, $C_0(K) \subset \ell_\infty(K)$) and $C^\infty$-smooth norm on $C_0(K)$.

For every $f \in \ell_\infty(K), 0 \leq \alpha \leq \Lambda$ put

$$a^f_\alpha = \langle f | L_\alpha \rangle^\infty, \quad b^f_\alpha = \|f | L_\alpha \|_\infty.$$  

Note that $\alpha \to D_\alpha$ is an increasing and thus upper semicontinuous (usc) function on $[0, \Lambda]$ (which is a compact space when considered with its natural interval topology). We have:

$$b^{f+}_{\alpha+1} = a^f_\alpha \quad \text{and} \quad \alpha \to b^f_\alpha \in C[0, \Lambda],$$

$$(*) \quad b^f_\alpha = \lim_{\tau / \alpha, \tau < \alpha} a^f_\alpha.$$

Therefore, for a fixed $f \neq 0$, $\alpha \to D_\alpha \cdot b^f_\alpha$ is usc on $[0, \Lambda]$, so it attains its maximum $M_f$ at some $\beta \in [0, \Lambda]$. Clearly, $M_f \geq D_{\alpha+1} b^f_{\alpha+1} = (1 + \delta_{\alpha+1}) D_\alpha a^f_\alpha$, and so $D_\alpha a^f_\alpha \leq \frac{M_f}{1 + \delta_{\alpha+1}} \leq M_f (1 - \frac{\delta_{\alpha+1}}{2})$, for all $\alpha \in [0, \Lambda]$.

First note that $D_{\beta} b^f_\beta > (D_{\alpha} a^f_\alpha)^\infty$. Indeed, otherwise there exist increasing sequences $\alpha_n \uparrow \beta$, $\alpha_n \in [0, \Lambda], D_{\alpha_n} a^f_{\alpha_n} \uparrow D_{\beta} b^f_\beta$. However, by $(*)$, $a^f_{\alpha_n} \to b^f_\gamma$, $\gamma > (1 + \delta_\gamma) D_{\alpha_n}$ for $n \in \mathbb{N}$, and consequently $D_{\beta} b^f_\beta \geq (1 + \delta_\gamma) D_{\beta} b^f_\beta$, a contradiction. So there exists $\varepsilon_f > 0$, such that $\text{card}\{\alpha, D_{\alpha} a^f_\alpha > D_{\beta} b^f_\beta - \varepsilon_f\}$ is finite. Next we claim that $\text{card}\{\alpha, D_{\beta} b^f_\beta > D_{\beta} b^f_\beta - \varepsilon_f\}$ is also finite.

Again, otherwise there exists $\alpha_n \uparrow \gamma$, $D_{\alpha_n} b^f_{\alpha_n} > D_{\beta} b^f_\beta - \varepsilon_f$. Using $(*)$, and passing to a suitable subsequence of $\{\alpha_n\}$ we find $\beta_n \uparrow \gamma$, $\alpha_n \leq \beta_n < \alpha_{n+1}$ such that

$$D_{\beta_n} a^f_{\beta_n} \geq D_{\alpha_{n+1}} b^f_{\alpha_{n+1}} - \frac{\varepsilon_f}{2}.$$  

This is a contradiction with the definition of $\varepsilon_f$, since

$$\{\beta_n\} \subset \{\alpha, D_{\alpha} a^f_\alpha > D_{\beta} a^f_\beta - 2\varepsilon_f\}.$$
Put \( O = \{ f \in C_0(K), M_f < 1 \} \subset 2B_{C_0(K)} \). We claim that \( \Psi|_O \) is finite, \( C^\infty \)-smooth and LDF. Moreover, \( \Psi^{-1}(0,\frac{1}{2}] \subset \text{int} \, O \), which by the implicit function theorem \([D]\) finishes the proof.

Choose any \( f \in O \). Consider the finite set \( A = \{ \alpha \in [0,1], \text{either } D_\alpha b_\alpha > M_f - \varepsilon_f \text{ or } \delta_{\alpha+1} \geq \frac{\varepsilon_f}{4} \} \), and put \( \delta_f = \min\{\varepsilon_f, \alpha \in A\} \). Let us check that \( \Psi|_{B(f,\delta_f)} \) depends on finitely many coordinates (therefore it is necessarily \( C^\infty \)-smooth). If \( \gamma \in L_\alpha, \alpha \notin A \), then \( 4\delta_{\alpha+1} < \varepsilon_f \) and \( D_\alpha b_\alpha \leq M_f - \varepsilon_f \leq 1 - \varepsilon_f \leq 1 - 4\delta_{\alpha+1} \). For \( g \in B(f,\delta_f) \), we then have

\[
|D_\alpha g(\gamma)| \leq D_\alpha b_\alpha + 2\delta_f \leq 1 - \varepsilon_f + 2\delta_f \leq 1 - 2\delta_{\alpha+1}.
\]

Consequently,

\[
\psi_\alpha(D_\alpha g(\gamma)) = 0 \quad \text{and} \quad \Psi(g) = \sum_{\alpha \in A} \sum_{\gamma \in L_\alpha} \psi_\alpha(D_\alpha g(\gamma)).
\]

If \( \alpha \in A, \gamma \in L_\alpha \), and \( |f(\gamma)| < a_\alpha(1 + \frac{\delta_{\alpha+1}}{8}) \), then

\[
|D_\alpha f(\gamma)| \leq D_\alpha b_\alpha \left( 1 + \frac{\delta_{\alpha+1}}{8} \right) \leq M_f \left( 1 - \frac{\delta_{\alpha+1}}{2} \right) \left( 1 + \frac{\delta_{\alpha+1}}{8} \right) \leq 1 - \frac{\delta_{\alpha+1}}{4}.
\]

If \( g \in B(f,\delta_f) \), we then have

\[
|D_\alpha g(\gamma)| \leq 1 - \frac{\delta_{\alpha+1}}{4} + 2\delta_f \leq 1 - \frac{\delta_{\alpha+1}}{8}.
\]

Consequently, in this case also \( \psi_\alpha(D_\alpha g(\gamma)) = 0 \). The remaining set \( S = \{ \gamma, \gamma \in L_\alpha \text{ for } \alpha \in A \text{ and } f(\gamma) \geq a_\alpha(1 + \frac{\delta_{\alpha+1}}{8}) \} \) is clearly finite, and we have

\[
\Psi(g) = \sum_{\alpha \in A} \sum_{\gamma \in S \setminus L_\alpha} (D_\alpha g(\gamma))
\]

whenever \( g \in B(f,\delta_f) \). This proves (Lemma 3) that \( \Psi|_O \) is \( C^\infty \)-smooth and LDF.

It is obvious that \( M_f \leq \frac{1}{4} \) implies \( \Psi(f) = 0 \) and \( M_f = 1 \) implies \( \Psi(f) \geq 1 \). Thus \( B = \Psi^{-1}(0,\frac{1}{2}] \) is an equivalent unit ball of \( C_0(K) \). By the implicit function theorem, its Minkowski functional is \( C^\infty \)-smooth and LDF.

\[\square\]

**References**


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