ON QUASINILPOTENT OPERATORS

IL BONG JUNG, EUNGIL KO, AND CARL PEARCY

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Abstract. In this note we modify a new technique of Enflo for producing hyperinvariant subspaces to obtain a much improved version of his “two sequences” theorem with a somewhat simpler proof. As a corollary we get a proof of the “best” theorem (due to V. Lomonosov) known about hyperinvariant subspaces for quasinilpotent operators that uses neither the Schauder-Tychonoff fixed point theorem nor the more recent techniques of Lomonosov.

1. Introduction

Let \( \mathcal{H} \) be a separable, infinite dimensional, complex Hilbert space, and denote by \( \mathcal{L}(\mathcal{H}) \) the algebra of all bounded linear operators on \( \mathcal{H} \) and by \( \mathbf{K} = \mathbf{K}(\mathcal{H}) \) the ideal of compact operators in \( \mathcal{L}(\mathcal{H}) \). Perhaps the first invariant-subspace theorem for operators in \( \mathcal{L}(\mathcal{H}) \), other than those provided by the spectral theorem for normal operators, was that every operator in \( \mathbf{K}(\mathcal{H}) \) has a nontrivial invariant subspace. According to Aronszajn-Smith [3], this was proved by John von Neumann (unpublished) about 1935. Thus there has now been over a half-century of work devoted to establishing that operators in \( \mathcal{L}(\mathcal{H}) \) that have a nice enough relation to some compact operator have nontrivial invariant subspaces. Without attempting to be exhaustive we mention the papers of Bernstein-Robinson [4], Halmos [9], [10], Arveson-Feldman [1], Deckard-Douglas-Pearcy [7], Pearcy-Salinas [14], Lomonosov [11], [12], [13], Pearcy-Shields [15], Scott Brown [5], and, more recently, Chevreau-Li-Pearcy [6], Simonic [16], Ansari-Enflo [2], and Enflo-Lomonosov [8]. Several of these works took something from previous ones, but many also added new techniques, some dramatically new (for example, the use by Lomonosov in [11] of the Schauder-Tychonoff fixed point theorem for nonlinear mappings).

In [2], a very recent new technique was introduced (and ascribed there to Enflo) for producing invariant subspaces for compact-related operators in \( \mathcal{L}(\mathcal{H}) \). The following old theorem of Lomonosov ([11]; cf. also [15]) was thus given in [2] a completely different proof (neither utilizing the Schauder-Tychonoff fixed point theorem nor the ideas of [12]).

**Theorem 1.1.** Every nonzero compact operator in \( \mathcal{L}(\mathcal{H}) \) has a nontrivial hyperinvariant subspace.
As another consequence of this technique, Enflo in \[8\] obtained the following interesting “two sequences” theorem.

**Theorem 1.2.** Let \( \mathcal{A} \subset \mathcal{L}(\mathcal{H}) \) be any commutative algebra that contains a nonzero quasinilpotent operator. Then there exist sequences \( \{s_k\}_{k=1}^{\infty} \) and \( \{t_k\}_{k=1}^{\infty} \) in \( \mathcal{H} \) that converge weakly to nonzero vectors \( s_0 \) and \( t_0 \), respectively, such that for every bounded sequence \( \{A_k\}_{k=1}^{\infty} \subset \mathcal{A} \),
\[
\lim_{k} (A_k s_k, t_k) = 0.
\]

This technique of proof (from \[2\] and \[8\]) uses some “extremal vectors” in a very clever way, and, as was mentioned in \[8\], is so new that most likely it will be some time before one knows whether the technique (or modifications thereof) will yield all the stronger theorems from \[11\] and \[12\] as well as perhaps some completely new results in the same direction.

The purpose of this note is to show that by modifying Enflo’s new technique, a considerably better version of Theorem 1.2, with a somewhat simpler proof, can be obtained as follows.

**Theorem 1.3.** Suppose \( Q \neq 0 \) is a quasinilpotent operator in \( \mathcal{L}(\mathcal{H}) \) and \( \{Q\}' \) denotes the commutant of \( Q \), i.e., \( \{Q\}' = \{A \in \mathcal{L}(\mathcal{H}) : AQ = QA\} \). Let \( B_0 \) be an arbitrary nonzero operator in \( \{Q\}' \) such that \( B_0 Q \neq 0 \). Then there exist sequences \( \{s_k\}_{k=1}^{\infty} \) and \( \{t_k\}_{k=1}^{\infty} \) in \( \mathcal{H} \) that converge weakly to nonzero vectors \( s_0 \) and \( t_0 \), respectively, with \( B_0 s_0 \neq 0 \), and a sequence \( \{\beta_k\} \) of positive numbers converging to zero, such that for every doubly indexed sequence \( \{A_{m,k}\}_{m,k \in \mathbb{N}} \) in the unit ball of \( \{Q\}' \), we have
\[
|(A_{m,k} s_k, t_k)| < \beta_k, \quad m, k \in \mathbb{N}.
\]

Also as a corollary of Theorem 1.3, the following better (than Theorem 1.1) but not so old theorem of Lomonosov \[12\] can be deduced.

**Corollary 1.4** (\[12\]). Suppose that \( Q \neq 0 \) is a quasinilpotent operator in \( \mathcal{L}(\mathcal{H}) \) and there exist a sequence \( \{D_m\}_{m \in \mathbb{N}} \subset \{Q\}' \) converging in the weak operator topology to a nonzero \( (C \in \{Q\}') \) and a sequence \( \{K_m\}_{m \in \mathbb{N}} \) of compact operators such that
\[
(1) \quad \lim_m \|D_m - K_m\| = 0.
\]
(In other words, in the language of \[12\], we suppose that \( \{Q\}' \) has the Pearcy-Salinas property.) Then \( Q \) has a nontrivial hyperinvariant subspace.

In other words, this note may be considered as a first step in the direction of determining what are the best theorems that can be obtained by (modifications of) this new Enflo technique from \[2\] and \[8\]. We remark that Corollary 1.4 is the “strongest” theorem known which produces hyperinvariant subspaces for a quasinilpotent operator, so, at least in this direction, Enflo’s new technique produces the “best” theorem known.

2. **Some lemmas**

Our proof of Theorem 1.3 depends on several lemmas (essentially) from \[2\].

**Lemma 2.1.** Suppose \( u \) and \( v \) are nonzero vectors in \( \mathcal{H} \) such that for every \( z \in \mathcal{H} \), \( \text{Re}(u, z) < 0 \) implies that \( \text{Re}(v, z) \geq 0 \). Then there exists a negative number \( r_0 \) such that \( v = r_0 u \).
Thus suppose that $H = \delta > 0$ then for $t$ function $$Re(u, z) = Re(u, \gamma u + x) = \|u\|^2 Re(\gamma) < 0.$$ Thus, for all $x$ orthogonal to $u$ and for all $\gamma$ with $Re(\gamma) < 0$, we have, by hypothesis, $$Re(v, z(\gamma, x)) = Re(\alpha_0 u + w, \gamma u + x) = \|u\|^2 Re(\alpha_0 \gamma) + Re(w, x) \geq 0.$$ Upon fixing $\gamma$ and taking $x$ to be a large enough negative scalar multiple of $w$, we see that necessarily $w = 0$ and that $Re(\alpha_0 \gamma) \geq 0$. Upon writing $\alpha_0 = r_0 + is_0$ and $\gamma = t + iq$ where $r_0$, $s_0$, $t$, $q$ are real, we get that $r_0 t - s_0 q \geq 0$ for all $q \in \mathbb{R}$ and all $t < 0$. Fixing $t$ and letting $q$ run we get $s_0 = 0$ and then $r_0 \leq 0$. Since $v = r_0 u$ and $v \neq 0$, we must have $r_0 < 0$, so the proof is complete. 

**Lemma 2.2.** Suppose $T \in \mathcal{L}(\mathcal{H})$ and has dense range. Suppose also that $x_0$ is a nonzero vector in $\mathcal{H}$ and that $\varepsilon$ satisfies $0 < \varepsilon < \|x_0\|$. Then there exists a unique nonzero vector $y_0 = y_0(x_0, \varepsilon)$ such that

a) $\|y_0\| = \inf\{\|y\| : \|Ty - x_0\| \leq \varepsilon\}$ and

b) $\|Ty_0 - x_0\| = \varepsilon$.

**Proof.** Let $\mathcal{F} = \{y \in \mathcal{H} : \|Ty - x_0\| \leq \varepsilon\}$. Since $T$ has dense range, clearly $\mathcal{F}$ is nonempty, and since $T$ is continuous and $\mathcal{F}$ is the inverse image under $T$ of the norm-closed ball centered at $x_0$ with radius $\varepsilon$, $\mathcal{F}$ is a norm-closed set. Moreover an easy calculation shows that $\mathcal{F}$ is a convex set. But, as is well-known, such a set has a unique vector $y_0$ of minimal norm. Thus a) is satisfied, and if $\|Ty_0 - x_0\| < \varepsilon$, then for $\delta > 0$ sufficiently small, $(1 - \delta)y_0$ would belong to $\mathcal{F}$ and have smaller norm, so b) is satisfied. 

**Lemma 2.3.** Suppose $T$, $x_0$, $y_0$, and $\varepsilon$ are as in Lemma 2.2. Then there exists a negative number $r$ such that $T^*(Ty_0 - x_0) = ry_0$. 

**Proof.** We apply Lemma 2.1 to $u = T^*(Ty_0 - x_0)$ and $v = y_0$. Clearly it suffices to show that $u$ and $v$ satisfy the hypotheses of Lemma 2.1 (and then set $r = 1/r_0$). Thus suppose that $z_0$ is any vector in $\mathcal{H}$ satisfying $$Re(u, z) = Re(T^*(Ty_0 - x_0), z_0) = Re(Ty_0 - x_0, Tz_0) < 0.$$ It follows easily that there exists a sufficiently small interval $[0, t_0]$, on which the function $t \rightarrow \|(Ty_0 - x_0) + tTz_0\|^2$ is strictly monotone decreasing (its derivative is continuous and negative at the origin). Thus for $t \in (0, t_0]$ we have $$\|T(y_0 + tz_0) - x_0\| < \|Ty_0 - x_0\| = \varepsilon.$$ Thus for $t \in (0, t_0]$, $y_0 + tz_0 \in \mathcal{F}$, and by the minimality of $y_0$, we must have $$\|y_0 + tz_0\|^2 \geq \|y_0\|^2, \quad t \in (0, t_0].$$ But the derivative of the function $$t \rightarrow \|y_0 + tz_0\|^2$$ is continuous and its value at the origin is $2Re(y_0, z_0)$, which must therefore satisfy $Re(y_0, z_0) \geq 0$, and the lemma is proved. 

**Lemma 2.4.** Suppose $T \in \mathcal{L}(\mathcal{H})$ is quasinilpotent with dense range, let $x_0$ be a nonzero vector in $\mathcal{H}$, let $\varepsilon$ satisfy $0 < \varepsilon < \|x_0\|$, and (via Lemma 2.2) let, for each $n \in \mathbb{N}$, $y_n = y_n(\varepsilon, x_0)$ be a (nonzero) vector satisfying

a) $\|y_n\| = \inf\{\|y\| : \|T^n y - x_0\| \leq \varepsilon\}$ and
b) \[\|T^n y_n - x_0\| = \varepsilon.\]

Then there exists a subsequence \(\{y_{n_k}\}\) of the sequence \(\{y_n\}\) satisfying

\[\lim_{k} \frac{\|y_{n_k}\|}{\|y_{n_k+1}\|} = 0.\]

Proof. Suppose, to the contrary, that there exist \(t > 0\) and \(N_i \in \mathbb{N}\) such that

\[\inf_{n \geq N_i} \frac{\|y_n\|}{\|y_{n+1}\|} = t.\]

Then

\[\|y_{N_i}\| \geq t\|y_{N_i+1}\| \geq t^2\|y_{N_i+2}\| \geq \cdots \geq t^n\|y_{N_i+n}\|, \quad n \in \mathbb{N}.\]

By the minimality of \(\|y_{N_i}\|\) from a), we have (since \(\|T^{N_i+n} y_{N_i+n} - x_0\| = \varepsilon\))

\[\|T^n y_{N_i+n}\| \geq \|y_{N_i}\|, \quad n \in \mathbb{N}.\]

Thus

\[\|T^n\| \geq \|y_{N_i+n}\| \geq t^n\|y_{N_i+n}\|, \quad n \in \mathbb{N},\]

and hence \(\|T^n\| \geq t^n, n \in \mathbb{N}\), which contradicts the fact that \(\sigma(T) = \{0\}\). The result follows. \(\square\)

3. PROOFS OF THE RESULTS

On the basis of these lemmas, we now prove Theorem 1.3 and Corollary 1.4.

Proof of Theorem 1.3. Let \(B_0 \in \{Q\}^\perp\) be such that \(B_0 Q \neq 0\). If \(\mathcal{M} = \text{(range } Q)^\perp \neq \mathcal{H}\), then \(\mathcal{M}\) is a nontrivial hyperinvariant subspace for \(Q\) and the result follows by choosing nonzero vectors \(s_0 \in \mathcal{M}\) and \(t_0 \in \mathcal{M}^\perp\) such that \(B_0 s_0 \neq 0\) and defining \(s_k = s_0, t_k = t_0, k \in \mathbb{N}\), and \(\{\beta_k\}\) to be an arbitrary sequence of positive numbers tending to zero. Thus we may suppose that \(Q\) has dense range (which implies that each \(Q^n\) also has dense range). Let \(x_0\) be a nonzero vector in \(\mathcal{H}\) such that \(B_0 Q x_0 (= Q B_0 x_0) \neq 0\), and let \(\varepsilon\) satisfy

\[0 < \varepsilon < \min\{\|x_0\|, \|Q x_0\|, 1/\|B_0\|\} \|B_0 x_0\|\}.\]

For each \(n \in \mathbb{N}\), let \(y_n = y_n(\varepsilon, x_0)\) satisfy a) and b) of Lemma 2.4 (with \(T = Q\)). By Lemma 2.4, we can choose a subsequence \(\{y_{n_k}\}\) of \(\{y_n\}\) such that

\[\lim_{k} \frac{\|y_{n_k}\|}{\|y_{n_k+1}\|} = 0.\]

By dropping down to successive subsequences of \(\{y_{n_k}\}\) we may suppose (without changing the notation accordingly), since all of the vectors \(Q^n y_n, n \in \mathbb{N}\), belong to the norm-closed ball of radius \(\varepsilon\) centered at \(x_0\), that the sequence \(\{Q^n y_{n_k}\}\) converges weakly to a vector \(z_0\) and, similarly, that the sequence \(\{Q^{n+1} y_{n_k+1}\}\) converges weakly to a vector \(v_0\). Since norm-closed balls in \(\mathcal{H}\) are weakly closed, we have \(\|v_0 - x_0\| \leq \varepsilon, \|z_0 - x_0\| \leq \varepsilon\) (so, in particular, \(v_0 \neq 0 \neq z_0\), \(\|B_0 z_0 - B_0 x_0\| \leq \|B_0\| \varepsilon,\) and \(\|B_0 z_0 - B_0 x_0\| \neq 0\).

We next show that \(v_0 - x_0 \neq 0\), which shows also (since \(\mathcal{M} = \mathcal{H}\)) that \(Q^*(v_0 - x_0) \neq 0\). By the definition of the \(y_n\) and Lemma 2.3, we have

\[\varepsilon^2 = \|Q^{n+1} y_{n+1} - x_0\|^2 = (y_{n+1}, (Q^{n+1})^*(Q^{n+1} y_{n+1} - x_0)) - (x_0, Q^{n+1} y_{n+1} - x_0) = r_{n+1} \|y_{n+1}\|^2 - (x_0, Q^{n+1} y_{n+1} - x_0), \quad k \in \mathbb{N},\]
where \( r_{n+1} < 0 \) for all \( k \in \mathbb{N} \). Thus

\[-\varepsilon^2 \geq (x_0, Q^{n+1}y_{n+1} - x_0), \quad k \in \mathbb{N},\]

and, taking limits as \( k \to \infty \), we get \(-\varepsilon^2 \geq (x_0, \nu_0 - x_0)\), so \( \nu_0 - x_0 \neq 0 \) and \( Q^*(\nu_0 - x_0) \neq 0 \).

Now define

\[ s_k = Q^n y_{nk}, \]

\[ t_k = Q^*(Q^{n+1}y_{nk+1} - x_0), \]

\[ \beta_k = \|y_{nk}\| \varepsilon (\|x_0\| + \varepsilon) / \|y_{nk+1}\|, \quad k \in \mathbb{N}, \]

and note that the sequences \( \{s_k\}_{k=1}^\infty \) and \( \{t_k\}_{k=1}^\infty \) converge weakly to the nonzero vectors \( s_0 := z_0 \) and \( t_0 := Q^*(\nu_0 - x_0) \), respectively, and that the sequence \( \{\beta_k\} \) converges to zero. Next we let \( \{A_{m,k}\}_{m,k \in \mathbb{N}} \) be an arbitrary doubly indexed sequence in the unit ball of \( \{Q^{\prime}\} \), and we write

\[ A_{m,k} y_{nk} = a^{(m)}_{nk} y_{nk+1} + w^{(m)}_{nk+1}, \quad m, k \in \mathbb{N}, \]

where \( a^{(m)}_{nk} \in \mathbb{C} \) and \( w^{(m)}_{nk+1} \) is orthogonal to \( y_{nk+1} \) for all \( m, k \in \mathbb{N} \). Note that

\[ \|y_{nk}\|^2 \geq \|A_{m,k} y_{nk}\|^2 \]

\[ = |a^{(m)}_{nk}|^2 \|y_{nk+1}\|^2 + \|w^{(m)}_{nk+1}\|^2, \quad m, k \in \mathbb{N}, \]

and thus

\[ \|y_{nk}\| \geq |a^{(m)}_{nk}|, \quad m, k \in \mathbb{N}. \]

An application of \( Q^{n+1} \) to each side of (2) gives

\[ QA_{m,k} Q^n y_{nk} = Q^{n+1} A_{m,k} y_{nk} \]

\[ = a^{(m)}_{nk} Q^{n+1} y_{nk+1} + Q^{n+1} w^{(m)}_{nk+1}, \quad m, k \in \mathbb{N}. \]

Upon taking the inner product of each side of (4) with \( Q^{n+1}y_{nk+1} - x_0 \), we obtain

\[ (A_{m,k} s_k, t_k) = a^{(m)}_{nk} (Q^{n+1} y_{nk+1}, Q^{n+1} y_{nk+1} - x_0), \quad m, k \in \mathbb{N}, \]

since, by Lemma 2.3,

\[ (w^{(m)}_{nk+1}, (Q^{n+1})^*(Q^{n+1} y_{nk+1} - x_0)) = (w^{(m)}_{nk+1}, r_{nk+1} y_{nk+1}) = 0, \quad m, k \in \mathbb{N}. \]

Moreover, since

\[ |(Q^{n+1} y_{nk+1}, Q^{n+1} y_{nk+1} - x_0)| \leq (\|x_0\| + \varepsilon) \varepsilon, \quad k \in \mathbb{N}, \]

we have from the definition of \( \beta_k \), (3), and (5) that

\[ |(A_{m,k} s_k, t_k)| \leq \beta_k, \quad m, k \in \mathbb{N}. \]

Since we saw earlier that \( B_0 s_0 = B_0 z_0 \neq 0 \), the theorem is proved.

\( \square \)

**Proof of Corollary 1.4.** We may suppose, without loss of generality, that \( Q \) is a quasiaffinity (otherwise \( \ker Q \) or \((\text{range } Q)^-\) is a nontrivial hyperinvariant subspace for \( Q \)). Thus \( CQ \neq \emptyset \), and we set \( B_0 \) of Theorem 1.3 equal to \( C \). Now let the sequences \( \{s_k\}, \{t_k\}, \) and \( \{\beta_k\} \) be as in Theorem 1.3, with \( \{s_k\} \) and \( \{t_k\} \) having nonzero weak limits \( s_0 \) and \( t_0 \), respectively. Also let \( A_0 \) be an arbitrary operator in the unit ball of \( \{Q^{\prime}\} \) such that \( A_0 C \neq 0 \). We will show that \( (A_0 C s_0, t_0) = 0 \), and therefore that \( \mathcal{M} = (\{Q^{\prime}\} C s_0)^- \) is the desired nontrivial hyperinvariant subspace.
for $Q$. (Note that $0 \neq C_{s_0} \in \mathcal{M}$ and that $t_0$ is orthogonal to $M_t$.) We may suppose, without loss of generality, that the sequence $\{D_m\}$ lies in the unit ball of $\{Q\}$.

Define the doubly indexed sequence $\{A_{m,k}\}_{m,k\in\mathbb{N}}$ by $A_{m,k} = A_0D_m$, $m, k \in \mathbb{N}$. Then, from Theorem 1.3, we know that

$$
\|D_m - K_m\| < \eta/\{2 \|A_0\| (\sup_k \|s_k\| \|t_k\|)\}, \quad m \geq M_\eta.
$$

Then, via (6) and (8),

$$
\|A_0D_m s_k, t_k\| \leq \|A_0D_m s_k, t_k\| + \|A_0(K_m - D_m)s_k, t_k\| < \eta, \quad m \geq M_\eta, \quad k \geq K.
$$

Fix an arbitrary $m_0 \geq M_\eta$, and note that since $\{s_k\}$ tends weakly to $s_0$ and $A_0K_{m_0}$ is compact, we obtain

$$
\lim_k \|A_0K_{m_0}s_k - A_0K_{m_0}s_0\| = 0.
$$

Moreover, since $\{t_k\}$ tends weakly to $t_0$, we get from (9), (10), and a short calculation, that

$$
\|A_0K_{m_0} s_0, t_0\| = \lim_k \|A_0K_{m_0} s_k, t_k\| \leq \eta, \quad m_0 \geq M_\eta,
$$

which establishes (7) and completes the proof. \hfill \Box

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References


Department of Mathematics, Kyungpook National University, Taegu 702-701, Korea
E-mail address: ibjung@kyungpook.ac.kr

Department of Mathematics, Ewha Women’s University, Seoul 120-750, Korea
E-mail address: eiko@mm.ewha.ac.kr

Department of Mathematics, Texas A&M University, College Station, Texas 77843
E-mail address: pearcy@math.tamu.edu