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ON THE IRRATIONALITY OF A CERTAIN MULTIVARIATE q SERIES

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Abstract. We prove that for integers $q > 1, m \geq 1$ and positive rationals $r_1, r_2, \cdots, r_m \neq q^j, j = 1, 2, \cdots$, the series

$$\sum_{j=1}^{\infty} \frac{q^{-j}}{(1 - q^{-j}r_1)(1 - q^{-j}r_2) \cdots (1 - q^{-j}r_m)}$$

is irrational. Furthermore, if all the positive rationals $r_1, r_2, \cdots, r_m$ are less than q, then the series

$$\sum_{j_1, \cdots, j_m = 0}^{\infty} \frac{r_1^{j_1} \cdots r_m^{j_m}}{q^{j_1 + \cdots + j_m + 1} - 1}$$

is also irrational.

1. Introduction and results

The main result of this paper is the following theorem:

Theorem 1.1. *If q is an integer greater than one, m is a positive integer, and $r_1, r_2, \cdots, r_m$ are any positive rationals such that $r_1, r_2, \cdots, r_m \neq q^j$, $j = 1, 2, \cdots$, then the series*

$$\sum_{j=1}^{\infty} \frac{q^{-j}}{(1 - q^{-j}r_1)(1 - q^{-j}r_2) \cdots (1 - q^{-j}r_m)}$$

is irrational. Furthermore, if all the positive rationals $r_1, r_2, \cdots, r_m$ are less than q, then the series

$$\sum_{j_1, \cdots, j_m = 0}^{\infty} \frac{r_1^{j_1} \cdots r_m^{j_m}}{q^{j_1 + \cdots + j_m + 1} - 1}$$

is also irrational.

This generalizes the irrationality results of the single variable case proved in Borwein [3], Erdös [6], and Erdös and Graham [7]. The approach is via Padé approximants. These provide, when appropriately specialized, rational approximations that are "too good" to allow for rationality. These methods are also used in Borwein and Zhou [4], Mahler [9], Chudnovsky and Chudnovsky [5], Wallisser [10], and Zhou and Lubinsky [11]. Unfortunately the methods are not sufficiently general

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to allow a unified treatment and each new class of functions requires considerable additional work.

As in [4] we use the standard q analogues of factorials and binomial coefficients. The $q-$*factorial* is

$$[n]_q! := [n]! := \frac{(1-q^n)(1-q^{n-1})\cdots(1-q)}{(1-q)^n}, \tag{1.1}$$

where $[0]_q! := 1$. The $q-$*binomial coefficient* is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]! \cdot [n-k]!}. \tag{1.2}$$

As

$$q^i - 1 = (q-1)(q^{i-1} + q^{i-2} + \cdots + 1), \quad i \geq 1,$$

we have

$$\lim_{q \to 1} [n]_q! = n! \quad \text{and} \quad \lim_{q \to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}. \tag{1.3}$$

Note that (see Borwein [3])

$$[n]_{q^{-1}}! = q^{-n(n-1)/2}[n]!, \tag{1.4}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{-q} = q^{-k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}, \tag{1.5}$$

$$\prod_{\substack{h=0 \\ h \neq k}}^{n} (q^{-k} - q^{-h}) = (-1)^{n-k} q^{-k(k-1)/2 - n(n+1)/2} [n-k]![k]!(1-q)^n, \tag{1.6}$$

and (see Gasper and Rahman [8]) for $|t| < q^{-n}$,

$$\frac{1}{\prod_{k=0}^{n}(t - q^{-k})} = (-1)^{n+1} q^{n(n+1)/2} \sum_{l=0}^{\infty} \begin{bmatrix} n+l \\ l \end{bmatrix} t^l. \tag{1.7}$$

We prove some properties of approximants to a related function in section 2, and use those properties to prove Theorem 1.1 in section 3.

2. Some results on a related function

Let $q > 1$, $|x_1|, \cdots, |x_m| < q$, and integer $m \geq 1$, and let

$$L^*(x_1, \cdots, x_m) := \sum_{j_1, \cdots, j_m = 0}^{\infty} \frac{x_1^{j_1} \cdots x_m^{j_m}}{q^{j_1 + \cdots + j_m + 1} - 1}. \tag{2.1}$$

For $m = 1$, and $|x| < 1$,

$$\begin{aligned} \lim_{q \to 1} (q-1) L^*(x) &= \lim_{q \to 1} \sum_{j=0}^{\infty} \frac{(q-1)x^j}{q^{j+1} - 1} \\ &= \sum_{j=0}^{\infty} \frac{x^j}{j+1} \\ &= \frac{1}{x} \ln(1-x). \end{aligned}$$

So we call $L^*(x_1,\cdots,x_m)$ a multivariate q analogue of log. Now for $k\geq 1$ integer and $|x_1|,\cdots,|x_m|<q$, as

$$\begin{aligned} L^*(q^{-1}x_1,\cdots,q^{-1}x_m) &= \sum_{j_1,\cdots,j_m=0}^{\infty}\frac{q^{-(j_1+\cdots+j_m)}x_1^{j_1}\cdots x_m^{j_m}}{q^{j_1+\cdots+j_m+1}-1}\\ &= \sum_{j_1,\cdots,j_m=0}^{\infty}\frac{(1-q^{j_1+\cdots+j_m+1}+q^{j_1+\cdots+j_m+1})x_1^{j_1}\cdots x_m^{j_m}}{q^{j_1+\cdots+j_m}(q^{j_1+\cdots+j_m+1}-1)}\\ &= \sum_{j_1,\cdots,j_m=0}^{\infty}\frac{qx_1^{j_1}\cdots x_m^{j_m}}{q^{j_1+\cdots+j_m+1}-1}-\sum_{j_1,\cdots,j_m=0}^{\infty}\frac{x_1^{j_1}\cdots x_m^{j_m}}{q^{j_1+\cdots+j_m}}\\ &= qL^*(x_1,\cdots,x_m)-\frac{1}{(1-q^{-1}x_1)\cdots(1-q^{-1}x_m)}, \end{aligned}$$

we have

$$\begin{aligned} L^*(q^{-k}x_1,\cdots,q^{-k}x_m) &= q^kL^*(x_1,\cdots,x_m)-\sum_{j=1}^{k}\frac{q^{k-j}}{(1-q^{-j}x_1)\cdots(1-q^{-j}x_m)}\\ &=: q^kL^*(x_1,\cdots,x_m)-S_k(x_1,\cdots,x_m), \qquad (2.2) \end{aligned}$$

where

$$S_k(x_1,\cdots,x_m):=\sum_{j=1}^{k}\frac{q^{k-j}}{(1-q^{-j}x_1)\cdots(1-q^{-j}x_m)}. \qquad (2.3)$$

From (2.2), we have

$$L^*(x_1,\cdots,x_m)=q^{-k}L^*(q^{-k}x_1,\cdots,q^{-k}x_m)+\sum_{j=1}^{k}\frac{q^{-j}}{(1-q^{-j}x_1)\cdots(1-q^{-j}x_m)},$$

and then

$$\begin{aligned} L^*(x_1,\cdots,x_m) &= \lim_{k\to\infty}q^{-k}L^*(q^{-k}x_1,\cdots,q^{-k}x_m)\\ &\quad+\lim_{k\to\infty}\sum_{j=1}^{k}\frac{q^{-j}}{(1-q^{-j}x_1)\cdots(1-q^{-j}x_m)}\\ &= \sum_{j=1}^{\infty}\frac{q^{-j}}{(1-q^{-j}x_1)\cdots(1-q^{-j}x_m)}. \qquad (2.4) \end{aligned}$$

Now let $q>1,\ x_1,\cdots,x_m\neq q^j,\ j=1,2,\cdots,$ and integer $m\geq 1$, and let

$$L(x_1,\cdots,x_m):=\sum_{j=1}^{\infty}\frac{q^{-j}}{(1-q^{-j}x_1)\cdots(1-q^{-j}x_m)}. \qquad (2.5)$$

Then $L(x_1,\cdots,x_m)$ is an extension of $L^*(x_1,\cdots,x_m)$, i.e.

$$L(x_1,\cdots,x_m)=L^*(x_1,\cdots,x_m),\ \text{for}\ |x_1|,\cdots,|x_m|<q. \qquad (2.6)$$

It is easy to see that we also have the following functional equation for $L(x_1,\cdots,x_m)$:

$$L(q^{-k}x_1,\cdots,q^{-k}x_m)=q^kL(x_1,\cdots,x_m)-S_k(x_1,\cdots,x_m), \qquad (2.7)$$

where $k\geq 1$ is an integer and $S_k(x_1,\cdots,x_m)$ is defined by (2.3). Now we prove some properties of the function $L(x_1,\cdots,x_m)$.

Theorem 2.1. *Let $n \geq 0$ be an integer, and let $L(x_1, \cdots, x_m), S_k(x_1, \cdots, x_m)$ be defined by (2.5) and (2.3) respectively. Let*

$$R_n(x_1, \cdots, x_m) := \prod_{j=1}^{n} \left((1-q^{-j}x_1) \cdots (1-q^{-j}x_m)\right) \tag{2.8}$$

and

$$I(x_1, \cdots, x_m) := \frac{R_n(x_1, \cdots, x_m)}{2\pi i} \int_\Gamma \frac{L(tx_1, \cdots, tx_m)dt}{\left(\prod_{k=0}^{n}(t-q^{-k})\right) t^{n+1}}, \tag{2.9}$$

where Γ is a circular contour containing $0, q^{-n}, \cdots, q^0$, and let

(2.10)

$$Q(x_1, \cdots, x_m) := \frac{q^{n(n+1)/2}}{(1-q)^n[n]!} \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} q^{nk+k(k+1)/2} R_n(x_1, \cdots, x_m),$$

$$\begin{aligned} P(x_1, \cdots, x_m) := {} & \frac{q^{n(n+1)/2}}{(1-q)^n[n]!} \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} q^{nk+k(k+1)/2} \\ & \cdot R_n(x_1, \cdots, x_m) S_k(x_1, \cdots, x_m) \\ & + \frac{R_n(x_1, \cdots, x_m)}{n!} \frac{d^n}{dt^n} \left\{ \frac{L(tx_1, \cdots, tx_m)}{\prod_{k=0}^{n}(t-q^k)} \right\}_{t=0}. \end{aligned} \tag{2.11}$$

Then
(i)

$$I(x_1, \cdots, x_m) = Q(x_1, \cdots, x_m) L(x_1, \cdots, x_m) + P(x_1, \cdots, x_m); \tag{2.12}$$

(ii)

$$q^{(m-1)n(n+1)/2} \left(\prod_{j=1}^{n} (q^j - 1) \right) Q(x_1, \cdots, x_m) \in \mathbb{Z}[q, x_1, \cdots, x_m], \tag{2.13}$$

where $\mathbb{Z}[q, x_1, \cdots, x_m]$ is the set of polynomials in $q, x_1, \cdots, x_m$ with integer coefficients;
(iii)

$$q^{(m-1)n(n+1)/2} \left(\prod_{j=1}^{n+1} (q^j - 1) \right) P(x_1, \cdots, x_m) \in \mathbb{Z}[q, x_1, \cdots, x_m]; \tag{2.14}$$

(iv) *for $n \in \mathbb{N}$ fixed, and $0 < |x_1|, \cdots, |x_m| < q$,*

$$|I(x_1, \cdots, x_m)| \leq \frac{c_q}{q^{2mn(n+1)}}, \tag{2.15}$$

where c_q is a constant depending only on q, m, and $x_1, \cdots, x_m$.

Proof of Theorem 2.1. *Proof of* (i). We can see that the integrand in (2.9) has simple poles at $t = q^0, q^{-1}, \cdots, q^{-n}$, and a pole of order $n+1$ at $t = 0$, inside the contour Γ. By the residue theorem and the functional equation (2.7), and (1.6), we have

$$\begin{aligned}
I(x_1, \cdots, x_m) &= \frac{R_n(x_1, \cdots, x_m)}{2\pi i} \int_\Gamma \frac{L(tx_1, \cdots, tx_m)dt}{\left(\prod_{k=0}^{n}(t - q^{-k})\right) t^{n+1}} \\
&= R_n(x_1, \cdots, x_m) \sum_{k=0}^{n} \frac{L(q^{-k}x_1, \cdots, q^{-k}x_m)}{\left(\prod_{\substack{h=0\\h\neq k}}^{n}(q^{-k} - q^{-h})\right) q^{-k(n+1)}} \\
&\quad + \frac{R_n(x_1, \cdots, x_m)}{n!} \frac{d^n}{dt^n} \left\{ \frac{L(tx_1, \cdots, tx_m)}{\prod_{k=0}^{n}(t - q^{-k})} \right\}_{t=0} \\
&= \frac{q^{n(n+1)/2} R_n(x_1, \cdots, x_m)}{(1-q)^n [n]!} \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} q^{nk + k(k+1)/2} q^k L(x_1, \cdots, x_m) \\
&\quad - \frac{q^{n(n+1)/2} R_n(x_1, \cdots, x_m)}{(1-q)^n [n]!} \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} q^{nk + k(k+1)/2} S_k(x_1, \cdots, x_m) \\
&\quad + \frac{R_n(x_1, \cdots, x_m)}{n!} \frac{d^n}{dt^n} \left\{ \frac{L(tx_1, \cdots, tx_m)}{\prod_{k=0}^{n}(t - q^{-k})} \right\}_{t=0} \\
&= Q(x_1, \cdots, x_m) L(x_1, \cdots, x_m) + P(x_1, \cdots, x_m).
\end{aligned}$$

Proof of (ii). As $\left[{n \atop k} \right]$ is a polynomial in q with integer coefficients, and

$$\begin{aligned}
R_n(x_1, \cdots, x_m) &= \prod_{j=1}^{n} \left((1 - q^{-j}x_1) \cdots (1 - q^{-j}x_m) \right) \\
&= q^{-mn(n+1)/2} \prod_{j=1}^{n} \left((q^j - x_1) \cdots (q^j - x_m) \right), \qquad (2.16)
\end{aligned}$$

we have (2.13).

Proof of (iii). From (2.3) and (2.8), for $1 \le k \le n$,

$$R_n(x_1, \cdots, x_m) S_k(x_1, \cdots, x_m) = \sum_{h=1}^{k} q^{k-h} \prod_{\substack{j=1\\j\neq h}}^{n} \left((1 - q^{-j}x_1) \cdots (1 - q^{-j}x_m) \right),$$

so from (2.16),

$$q^{mn(n+1)/2} R_n(x_1, \cdots, x_m) S_k(x_1, \cdots, x_m) \in \mathbb{Z}[q, x_1, \cdots, x_m]. \qquad (2.17)$$

Now for $t < q^{-\ell}$, where $\ell > 0$ is an integer such that $\left|q^{-\ell}x_i\right| < q$, for all $i = 1, \cdots, m$,

$$L(tx_1, \cdots, tx_m) = L^*(tx_1, \cdots, tx_m) = \sum_{j_1, \cdots, j_m = 0}^{\infty} \frac{x_1^{j_1} \cdots x_m^{j_m} t^{j_1 + \cdots + j_m}}{q^{j_1 + \cdots + j_m + 1} - 1}, \qquad (2.18)$$

then from (1.7) and (2.18), for $t < \min\{q^{-n}, q^{-\ell}\}$,

$$\frac{L(tx_1, \cdots, tx_m)}{\prod_{k=0}^{n}(t-q^{-k})} = (-1)^{n+1} q^{n(n+1)/2} \sum_{j_1,\cdots,j_m,l=0}^{\infty} \begin{bmatrix} n+l \\ l \end{bmatrix} \frac{x_1^{j_1}\cdots x_m^{j_m} t^{j_1+\cdots+j_m+l}}{q^{j_1+\cdots+j_m+1}-1}.$$

So

$$\begin{aligned} & \frac{1}{n!}\frac{d^n}{dt^n}\left\{\frac{L(tx_1,\cdots,tx_m)}{\prod_{k=0}^{n}(t-q^{-k})}\right\}_{t=0} \\ (2.19) \qquad = \; & (-1)^{n+1} q^{n(n+1)/2} \sum_{\substack{j_1+\cdots+j_m+l=n \\ 0\le j_1,\cdots,j_m,l\le n}} \begin{bmatrix} n+l \\ l \end{bmatrix} \frac{x_1^{j_1}\cdots x_m^{j_m}}{q^{j_1+\cdots+j_m+1}-1}, \end{aligned}$$

and (2.14) follows from (2.11), (2.17) and (2.19).

Proof of (iv). For $R > 1$ and $\Gamma := \{z : |z| = R\}$, we have from (2.9),

$$\begin{aligned} |I(x_1,\cdots,x_m)| & \le R \cdot \frac{|R_n(x_1,\cdots,x_m)| \max_{|t|=R} |L(tx_1,\cdots,tx_m)|}{R^{n+1}\prod_{k=0}^{n}(R-|q|^{-k})} \\ (2.20) \qquad & \le \frac{f_q \max_{|t|=R}|L(tx_1,\cdots,tx_m)|}{R^n \prod_{k=0}^{n}(R-q^{-k})}. \end{aligned}$$

Now for $0 < |x_1|, \cdots, |x_m| < q$,

$$\begin{aligned} |R_n(x_1,\cdots,x_m)| & = \prod_{j=1}^{n} \left|(1-q^{-j}x_1)\cdots(1-q^{-j}x_m)\right| \\ (2.21) \qquad & \le \prod_{j=0}^{\infty}(1+q^{-j})^m := f_q, \end{aligned}$$

where f_q is a constant depending only on q and m.

Let $R = q^{mn}$. As

$$\max_{|t|=R}\left|1-q^{-j}tx_i\right| \ge \max_{|t|=R}\left|1-q^{-j}|t||x_i|\right| \ge \left|1-q^{mn-j+1}\right|,$$

for $1 \le i \le m,\ j = 1, 2, \cdots$, and

$$q^j - 1 = q^j\left(1-q^{-j}\right) \ge \frac{1}{2}q^j,$$

as q is an integer greater than 1, then

$$\begin{aligned} (2.22) \qquad & \max_{|t|=R}|L(tx_1,\cdots,tx_m)| \\ \le \; & \max_{|t|=R}\sum_{j=1}^{\infty}\left|\frac{q^{-j}}{(1-q^{-j}x_1t)\cdots(1-q^{-j}x_mt)}\right| \\ \le \; & \left(\sum_{j=1}^{mn}\frac{q^{j-mn-1}}{(q^j-1)^m}\right) + \frac{q^{-mn-1}}{(1-x_1/q)\cdots(1-x_m/q)} + \left(\sum_{j=1}^{\infty}\frac{q^{-j-mn-1}}{(1-q^{-j})^m}\right) \end{aligned}$$

$$\begin{aligned}
&\leq \; q^{-mn-1}\left(\sum_{j=1}^{mn-1}\frac{2^m}{q^{(m-1)j}}+\frac{1}{(1-x_1/q)\cdots(1-x_m/q)}+L(1,\cdots,1)\right)\\
&\leq \; q^{-mn-1}\left(2^m+\frac{1}{(1-x_1/q)\cdots(1-x_m/q)}+L(1,\cdots,1)\right)\\
&=: \; C_1q^{-mn},
\end{aligned}$$

where $C_1 := 2^m q + \frac{q}{(1-x_1/q)\cdots(1-x_m/q)} + qL(1,\cdots,1)$ is a constant depending only on q, m, and $x_1,\cdots,x_m$. Now

$$\begin{aligned}
R^n\prod_{k=0}^{n}(R-q^{-k}) &= R^{2n+1}\prod_{k=0}^{n}(1-q^{-n-k})\\
&\geq R^{2n+1}\prod_{j=0}^{\infty}(1-q^{-j})\\
&\geq C_2q^{mn(2n+1)}, \qquad (2.23)
\end{aligned}$$

where $C_2 := \prod_{j=0}^{\infty}(1-q^{-j})$ is a constant depending only on q. Putting (2.22) and (2.23) into (2.20), we have

$$|I(x_1,\cdots,x_m)| \leq c_q q^{-2mn(n+1)},$$

where

$$c_q := f_q C_1/C_1.$$

This completes the proof of Theorem 2.1.

3. Proof of Theorem 1.1

We first prove that for $0 < x_1,\cdots,x_m < q$, and $q > 1$,

$$|I(x_1,\cdots,x_m)| \neq 0, \qquad (3.1)$$

where $I(x_1,\cdots,x_m)$ is defined by (2.9). Note that if we choose the contour in (2.9) to be $\Gamma = \{z \in \mathbb{C} : |z| = 1+\epsilon\}$, where $\epsilon > 0$ is small enough such that $0 < |tx_1|,\cdots,|tx_m| < q$, for $t \in \Gamma$, then

$$L(tx_1,\cdots,tx_m) = L^*(tx_1,\cdots,tx_m), \quad t \in \Gamma.$$

Now

$$R_n(x_1,\cdots,x_m) = \prod_{j=1}^{n}\big((1-q^{-j}x_1)\cdots(1-q^{-j}x_m)\big) > 0$$

for $0 < x_1, \cdots, x_m < q$, and

$$\begin{aligned}
I(x_1, \cdots, x_m) &= \frac{R_n(x_1, \cdots, x_m)}{2\pi i} \int_\Gamma \frac{L(tx_1, \cdots, tx_m)dt}{t^{2n+2}\left(\prod_{k=0}^{n}(1-1/(q^k t))\right.} \\
&= \frac{R_n(x_1, \cdots, x_m)}{2\pi i} \int_\Gamma \frac{L(tx_1, \cdots, tx_m)}{t^{2n+2}} \left(\sum_{j_0, \cdots, j_n \geq 0} \prod_{k=0}^{n} \left(\frac{1}{q^k t}\right)^{j_k} \right) dt \\
&= R_n(x_1, \cdots, x_m) \sum_{j_0, \cdots, j_n \geq 0} q^{-\sum_{k=0}^{n} k j_k} \cdot \frac{1}{2\pi i} \int_\Gamma \{ \frac{1}{t^{2n+2+(j_0+\cdots j_n)}} \\
&\quad \cdot \sum_{i_1, \cdots i_m = 0}^{\infty} \frac{x_1^{i_1} \cdots x_m^{i_m} t^{i_1+\cdots+i_m}}{q^{i_1+\cdots+i_m+1}-1} \} dt \\
&= R_n(x_1, \cdots, x_m) \sum_{j_0, \cdots, j_n \geq 0} q^{-\sum_{k=0}^{n} k j_k} \\
&\quad \cdot \sum_{i_1+\cdots+i_m-(2n+j_0+\cdots j_n+2)=-1} \frac{x_1^{i_1} \cdots x_m^{i_m}}{q^{i_1+\cdots+i_m+1}-1} \\
&= R_n(x_1, \cdots, x_m) \sum_{\substack{i_1+\cdots+i_m=2n+j_0+\cdots j_n+1 \\ qquad;j_0, \cdots, j_n \geq 0}} q^{-\sum_{k=0}^{n} k j_k} \frac{x_1^{i_1} \cdots x_m^{i_m}}{q^{i_1+\cdots+i_m+1}-1} \\
&> 0,
\end{aligned} \tag{3.2}$$

as $x_1, \cdots, x_m \geq 0,\ q > 1$, and as infinitely many terms above are positive, so (3.1) holds.

Now let $r_1, r_2, \cdots, r_m$ be any fixed positive rational numbers such that $r_1, r_2, \cdots, r_m \neq q^j$ for all $j = 1, 2, \cdots$. From (2.7), we can see that the irrationality of $L(r_1, r_2, \cdots, r_m)$ is equivalent to the irrationality of $L(q^{-k}r_1, q^{-k}r_2, \cdots, q^{-k}r_m)$ for any integer $k \geq 1$, so we can assume that $0 < r_1, r_2, \cdots, r_m < q$, and then

$$L(r_1, r_2, \cdots, r_m) = \sum_{j_1, \cdots, j_m = 0}^{\infty} \frac{r_1^{j_1} \cdots r_m^{j_m}}{q^{j_1+\cdots+j_m+1}-1} > 0.$$

Now let

$$H_{m,n}(q) := q^{(m-1)n(n+1)/2} \left(\prod_{j=1}^{n+1} (q^j - 1) \right). \tag{3.3}$$

Then

$$0 < |H_{m,n}(q)| \leq q^{(mn+2)(n+1)/2} \tag{3.4}$$

and

$$H_{m,n}(q) \cdot \{Q(r_1, \cdots, r_m), P(r_1, \cdots, r_m)\} \subset \mathbb{Z}[q, r_1, \cdots, r_m]. \tag{3.5}$$

Now as

$$\begin{aligned}
\Delta_{m,n} &:= |H_{m,n}(q)Q(r_1, \cdots, r_m)L(r_1, \cdots, r_m) + H_{m,n}(q)P(r_1, \cdots, r_m)| \\
&= |H_{m,n}(q)|\,|I(r_1, \cdots, r_m)| \\
&> 0,
\end{aligned} \tag{3.6}$$

and from (2.15) and (3.3), we have

$$\begin{aligned}\Delta_{m,n} &\leq q^{(mn+2)(n+1)/2}\frac{c_q}{q^{2mn(n+1)}}\\ &= \frac{c_q}{q^{3mn(n+1)/2-1}}\\ &\leq \frac{c_q}{q^{mn^2}}. \qquad (3.7)\end{aligned}$$

Finally, if

$$r_1 := \frac{i_1}{l_1},\ r_2 := \frac{i_2}{l_2},\ \cdots,\ r_m := \frac{i_m}{l_m}, \tag{3.8}$$

with $i_1, \cdots, i_m$ and $l_1, \cdots, l_m$ positive integers, then

$$Q^*(r_1, \cdots, r_m) := (l_1 \cdots l_m)^{2n} H_{m,n}(q) Q(r_1, \cdots, r_m) \tag{3.9}$$

and

$$P^*(r_1, \cdots, r_m) := (l_1 \cdots l_m)^{2n} H_{m,n}(q) P(r_1, \cdots, r_m) \tag{3.10}$$

are integers, and by (3.6) to (3.10),

$$\begin{aligned}0 &< |Q^*(r_1, \cdots, r_m)L(r_1, \cdots, r_m) + P^*(r_1, \cdots, r_m)|\\ &= (l_1 \cdots l_m)^{2n}\ |H_{m,n}(q)|\,|Q(r_1, \cdots, r_m)L(r_1, \cdots, r_m) + P(r_1, \cdots, r_m)|\\ &\leq (l_1 \cdots l_m)^{2n}\frac{c_q}{q^{mn^2}},\end{aligned}$$

which tends to zero as $n \to \infty$. This shows that $L(r_1, \cdots, r_m)$ is irrational, that is,

$$\sum_{j_1, \cdots, j_m=0}^{\infty} \frac{r_1^{j_1} \cdots r_m^{j_m}}{q^{j_1+\cdots+j_m+1} - 1}$$

is irrational for $q > 1$ integer, $r_1, r_2, \cdots, r_m$ positive rationals less than q and integer $m \geq 1$, and

$$\sum_{j=1}^{\infty} \frac{q^{-j}}{(1-q^{-j}r_1)(1-q^{-j}r_2)\cdots(1-q^{-j}r_m)}$$

is irrational for $q > 1$ integer, $r_1, r_2, \cdots, r_m$ positive rationals such that $r_1, r_2, \cdots, r_m \neq q^j$ for all $j = 1, 2, \cdots$, and integer $m \geq 1$.

This completes the proof of Theorem 1.1.

Now by the standard methods (as in chapter 11 of Borwein and Borwein [1]), the estimates in the proof of Theorem 1.1 gives that, under the assumption of the theorem,

$$\left|L(r_1, \cdots, r_m) - \frac{s}{t}\right| > \frac{1}{t^{\alpha}},$$

for some constant α and all integers s and t, and hence

$$\sum_{j_1, \cdots, j_m=0}^{\infty} \frac{r_1^{j_1} \cdots r_m^{j_m}}{q^{j_1+\cdots+j_m+1} - 1}$$

is not a Liouville number.

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