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H^∞-CALCULUS FOR SUBMARKOVIAN GENERATORS

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Abstract. Let $-A$ be the generator of a symmetric submarkovian semigroup in $L_2(\Omega)$. In this note we show that on $L_p(\Omega), 1 < p < \infty$, the operator A admits a bounded H^∞ functional calculus on the sector $\Sigma(\phi) = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \phi\}$ for each $\phi > \psi_p^*$ with

$$\psi_p^* = \frac{\pi}{2}|\frac{1}{p} - \frac{1}{2}| + (1 - |\frac{1}{p} - \frac{1}{2}|) \arcsin(\frac{|p-2|}{2p - |p-2|}).$$

This improves a result due to M. Cowling. We apply our result to obtain maximal regularity for parabolic equations and evolutionary integral equations.

1. Introduction and main result

Let (Ω, μ) be a measure space. We call a linear operator A in $L_2(\Omega)$ a *submarkovian generator* if A is selfadjoint, $A \geq 0$ and the semigroup $(e^{-tA})_{t\geq 0}$ generated by $-A$ satisfies

$$e^{-tA} f \geq 0 \quad \text{for all } f \geq 0,\ f \in L_2(\Omega), \tag{1.1}$$

$$\|e^{-tA} f\|_\infty \leq \|f\|_\infty \quad \text{for all } f \in L_2(\Omega) \cap L_\infty(\Omega). \tag{1.2}$$

Under these assumptions the *submarkovian semigroup* $(e^{-tA})_{t\geq 0}$ acts as a contraction semigroup in all $L_p(\Omega)$-spaces, $1 \leq p \leq \infty$. It is strongly continuous for $1 \leq p < \infty$ and weak*-continuous for $p = \infty$. We denote its generator by $-A_p$.

If $\phi \in (0, \pi)$, the open sector $\{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \phi\}$ is denoted by $\Sigma(\phi)$. By Stein's interpolation theorem the semigroup $(e^{-tA_p})_{t\geq 0}$ extends in $L_p(\Omega)$, $1 < p < \infty$, to a bounded analytic semigroup on the sector $\Sigma(\rho_p)$ where $\rho_p := \pi(\frac{1}{2} - |\frac{1}{p} - \frac{1}{2}|)$. In particular, for $p \in (1, \infty) \setminus \{2\}$, the spectrum $\sigma(A_p)$ of A_p is contained in the sector $\overline{\Sigma(\rho_p^*)}$ where $\rho_p^* = \frac{\pi}{2} - \rho_p = \pi|\frac{1}{p} - \frac{1}{2}|$.

By means of transference and Stein interpolation, M. Cowling showed ([2]) that $m(A)$, originally defined on $L_2(\Omega)$ via the spectral theorem, acts as a bounded operator on $L_p(\Omega)$ whenever m has a bounded analytic extension to a sector $\Sigma(\phi)$

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with $\phi > \rho_p^*$. In addition the following estimate holds:

$$\|m(A)f\|_p \le C_{p,\phi}\|m\|_{\infty,\Sigma(\phi)}\|f\|_p, \quad f \in L_2(\Omega) \cap L_p(\Omega).$$

In other words this multiplier theorem means that A_p has a bounded H^∞-calculus on each sector $\Sigma(\phi)$ with $\phi > \rho_p^*$. For these results, hypothesis (1.1) is not needed. It is sufficient that $-A$ generates a symmetric contraction semigroup. Then each operator e^{-tA_p} is dominated by a positive contraction.

On the other hand, Liskevich and Perelmuter ([10]) showed that for submarkovian generators A the corresponding semigroup $(e^{-tA_p})_{t\ge 0}$ extends in $L_p(\Omega)$, $1 < p < \infty$, to a bounded semigroup of contractions on the sector $\overline{\Sigma(\theta_p)}$ and is analytic in $\Sigma(\theta_p)$ with $\theta_p := \arccos|\frac{2}{p} - 1|$. Hence

$$\sigma(A_p) \subset \overline{\Sigma(\theta_p^*)} \tag{1.3}$$

where $\theta_p^* := \frac{\pi}{2} - \theta_p = \arcsin|\frac{2}{p} - 1|$. Observe that $\theta_p^* < \rho_p^*$ for $p \in (1,\infty) \setminus \{2\}$.

In this note we show that, for submarkovian generators, the H^∞-angle ρ_p^* in Cowling's result can also be improved. We shall establish the following.

1.1. **Theorem.** *Let A be a submarkovian generator. Then, for $1 < p < \infty$, A_p has a bounded H^∞-calculus on every sector $\Sigma(\phi)$ with $\phi > \psi_p^*$ where*

$$\psi_p^* := \frac{\pi}{2}\Big|\frac{1}{p} - \frac{1}{2}\Big| + \Big(1 - \Big|\frac{1}{p} - \frac{1}{2}\Big|\Big) \arcsin\Big(\frac{|p-2|}{2p - |p-2|}\Big),$$

i.e. whenever m has a bounded analytic extension to a sector $\Sigma(\phi)$ with $\phi > \psi_p^*$, then $m(A)$ acts as a bounded operator on $L_p(\Omega)$ with

$$\|m(A)f\|_p \le C_{p,\phi}\|m\|_{\infty,\Sigma(\phi)}\|f\|_p, \quad f \in L_2(\Omega) \cap L_p(\Omega).$$

Besides the result in [10] and a variant of Stein's interpolation theorem our proof uses the concept of $\mathscr{R}$-boundedness and a recent result due to Kalton and Weis ([7]).

1.2. *Remark.* (a) For $p \in (1,\infty) \setminus \{2\}$ one has $\theta_p^* < \psi_p^* < \rho_p^*$. In the class of submarkovian generators the angle θ_p^* is optimal with respect to (1.3), equality occurs even for Neumann-Laplacians on open subsets of $\mathbb{R}^2$ ([8]). Hence the optimal angle for a bounded H^∞-calculus in the class of submarkovian generators is $\ge \theta_p^*$.

(b) In [6] it is shown that the Ornstein-Uhlenbeck operator A_p has a bounded H^∞-calculus in $L_p(\gamma)$ (γ is the Gauss measure on $\mathbb{R}^d$) on every sector $\Sigma(\phi)$ with $\phi > \theta_p^*$ and that the angle θ_p^* is optimal. It is remarkable that $\sigma(A_p) = \mathbb{N}_0$, $1 < p < \infty$. The proof in [6] relies heavily on properties of the special operator in question and gives no hint for the general case of a submarkovian generator.

(c) If $(e^{-tA})_{t\ge 0}$ is a semigroup of integral operators whose kernels satisfy Gaussian bounds, then, for $1 < p < \infty$, A_p has a bounded H^∞-calculus on every sector $\Sigma(\phi)$ with $\phi > 0$ ([4]).

This paper is organized as follows. In Section 2 we collect the tools we need for the proof of our main result, among them a recent result (Theorem 2.4) about the connection between operators with a bounded H^∞-calculus and those with the so-called property of "maximal L_q-regularity". Due to this result our proof will be considerably simplified, since we can restrict ourselves to the verification of certain square function estimates. Section 3 is devoted to the proof of Theorem 1.1. In the last section we present two applications of our result to maximal regularity problems of parabolic evolution equations and evolutionary integral equations.

2. Preliminary material and notations

In this section we collect notations and definitions we shall use and we cite some theorems that will be needed in the next section where we prove Theorem 1.1.

For arbitrary Banach spaces X, Y we denote by $\mathscr{B}(X,Y)$ (resp. $\mathscr{B}(X)$) the space of all bounded linear operators $T : X \to Y$ ($T : X \to X$). A major role in the proof of Theorem 1.1 is played by the concept of $\mathscr{R}$-boundedness.

A set $\mathcal{T} \subset \mathscr{B}(X)$ is called *$\mathscr{R}$-bounded* if there exists a constant $C > 0$, such that for all $T_1, T_2, \ldots, T_n \in \mathcal{T}$, $x_1, x_2, \ldots, x_n \in X$ and all $n \in \mathbb{N}$,

$$\int_0^1 \| \sum_{k=1}^n \varepsilon_k(t) T_k x_k \| \, dt \ \leq \ C \int_0^1 \| \sum_{k=1}^n \varepsilon_k(t) x_k \| \, dt, \tag{2.1}$$

where $(\varepsilon_k)_{k=1}^\infty$ is a fixed sequence of independent symmetric $\{-1,1\}$-valued random variables on $[0,1]$, e.g. the Rademacher functions $\varepsilon_k(t) := \operatorname{sign}(\sin(2^k \pi t))$. The smallest constant C such that (2.1) holds is denoted by $\mathscr{R}(\mathcal{T})$. Obviously, every $\mathscr{R}$-bounded subset of $\mathscr{B}(X)$ is bounded, but the converse is only true in Hilbert spaces (cf. (2.2) below).

If $\phi \in (0, \frac{\pi}{2})$ and $(e^{-zA})_{z \in \Sigma(\phi)}$ is a bounded analytic semigroup on X with generator $-A$, then this semigroup is called $\mathscr{R}$-analytic if there exists a $\delta \in (0, \phi)$ such that

$$\mathscr{R}(\{e^{-zA} : z \in \Sigma(\delta)\}) < \infty\,.$$

This condition is equivalent to

$$\mathscr{R}(\{z(z+A)^{-1} \ : \ z \in \Sigma(\frac{\pi}{2} + \delta)\}) \ < \ \infty$$

(see Theorem 4.2 in [13]). For such semigroups the *$\mathscr{R}$-analyticity angle* $\omega_{\mathscr{R}}(A)$ is defined by

$$\omega_{\mathscr{R}}(A) := \inf\{\varphi \in (0, \frac{\pi}{2}) : \sigma(A) \subset \overline{\Sigma(\varphi)}, \, \mathscr{R}(\{z(z+A)^{-1} \ : \ z \in \Sigma_{\pi - \varphi}\}) < \infty\}.$$

This should be seen in analogy to the *analyticity angle* $\omega_\sigma(A)$, which is given by

$$\omega_\sigma(A) := \inf\{\varphi \in (0, \frac{\pi}{2}) : \sigma(A) \subset \overline{\Sigma(\varphi)}, \, \sup\{\|z(z+A)^{-1}\| : z \in \Sigma_{\pi - \varphi}\} < \infty\}.$$

For negative generators A of $\mathscr{R}$-analytic semigroups one has the relation $\omega_\sigma(A) \leq \omega_{\mathscr{R}}(A)$. The $\mathscr{R}$-analyticity of submarkovian semigroups in $L_p(\Omega)$, $1 < p < \infty$, follows e.g. from the property of maximal L_q-regularity of its submarkovian generator ([9]). For details we refer to [13].

In $L_p(\Omega)$, $1 < p < \infty$, the Khintchine-Kahane inequalities imply that a set $\mathcal{T} \subset \mathscr{B}(L_p(\Omega))$ with $1 < p < \infty$ is *$\mathscr{R}$-bounded* if and only if there exists a constant $C > 0$, such that for all $T_1, T_2, \ldots, T_n \in \mathcal{T}$, $x_1, x_2, \ldots, x_n \in L_p(\Omega)$ and all $n \in \mathbb{N}$,

$$\Big\| \Big(\sum_{k=1}^n |T_k x_k|^2 \Big)^{1/2} \Big\|_{L_p} \ \leq \ C \Big\| \Big(\sum_{k=1}^n |x_k|^2 \Big)^{1/2} \Big\|_{L_p}. \tag{2.2}$$

This motivated the following definition ([14]): For $p, q \in [1, \infty]$ a set $\mathcal{T} \subset \mathscr{B}(L_p(\Omega))$ is called *$\mathscr{R}_q$-bounded* if there exists a constant $C > 0$, such that for all $T_1, T_2, \ldots, T_n$

$\in \mathcal{T}$, $x_1, x_2, \ldots, x_n \in L_p(\Omega)$ and all $n \in \mathbb{N}$,

$$\left\| \left(\sum_{k=1}^{n} |T_k x_k|^q \right)^{1/q} \right\|_{L_p} \leq C \left\| \left(\sum_{k=1}^{n} |x_k|^q \right)^{1/q} \right\|_{L_p} \qquad \text{if } q \in [1, \infty),$$

$$\left\| \sup_{k=1}^{n} |T_k x_k| \right\|_{L_p} \leq C \left\| \sup_{k=1}^{n} |x_k| \right\|_{L_p} \qquad \text{if } q = \infty.$$

2.1. *Remark.* (a) By Fubini's theorem bounded sets in $\mathscr{B}(L_p(\Omega))$ are $\mathscr{R}_p$-bounded and vice versa.

(b) For $p, q \in (1, \infty)$ a set $\mathcal{T} \subset \mathscr{B}(L_p(\Omega))$ is $\mathscr{R}_q$-bounded if and only if the set $\mathcal{T}' := \{T' : T \in \mathcal{T}\} \subset \mathscr{B}(L_{p'}(\Omega))$ is $\mathscr{R}_{q'}$-bounded with $1 = \frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'}$. For details see [14].

2.2. *Remark.* If A is a submarkovian generator, then the following holds:

(a) The set $\{e^{-zA_2} : z \in \overline{\Sigma(\pi/2)}\}$ is $\mathscr{R}_2$-bounded in $L_2(\Omega)$.

(b) The set $\{e^{-zA_p} : z \in \Sigma(\theta_p)\}$, $1 < p < \infty$, is $\mathscr{R}_p$-bounded in $L_p(\Omega)$.

(c) For $p \in (1, \infty)$ the set $\{T(t) : t \geq 0\} \subset \mathscr{B}(L_p(\Omega))$ with

$$T(t) := \frac{1}{t} \int_0^t e^{-sA_p}\, ds$$

is $\mathscr{R}_\infty$-bounded.

Property (a) is clear from the definition, (b) is due to Liskevich and Perelmuter ([10]), and (c) is due to Stein. A proof can be found e.g. in [5] Theorem 5.4.3.

We shall use the concept of $\mathscr{R}_q$-boundedness since it admits the application of the following variant of Stein's interpolation theorem which is due to Weis ([14]). Put $\Sigma(\theta_0, \theta_1) := \{z \in \mathbb{C} \setminus \{0\} : \theta_0 < \arg z < \theta_1\}$ and $X_p = L_p(\Omega), p \in [1, \infty]$.

2.3. **Theorem.** *Let $N : \Sigma(\theta_0, \theta_1) \to \mathscr{B}(X_1 \cap X_\infty, X_1 + X_\infty)$ be an operator-valued function such that for all $x \in X_1 \cap X_\infty$ the function $\lambda \mapsto N(\lambda)x \in X_1 + X_\infty$ is continuous and bounded in $\overline{\Sigma(\theta_0, \theta_1)}$ and analytic in $\Sigma(\theta_0, \theta_1)$. Assume moreover that for $p_0, p_1, q_0, q_1 \in [1, \infty]$ and $j = 0, 1$ the set*

$$\{N(te^{i\theta_j}) : t > 0\} \subset \mathscr{B}(X_{p_j})$$

is $\mathscr{R}_{q_j}$-bounded. Then for every $\alpha \in (0, 1)$ and

$$\theta = (1 - \alpha)\theta_0 + \alpha\theta_1, \qquad \frac{1}{p} = (1 - \alpha)\frac{1}{p_0} + \alpha\frac{1}{p_1}, \qquad \frac{1}{q} = (1 - \alpha)\frac{1}{q_0} + \alpha\frac{1}{q_1},$$

the set $\{N(te^{i\theta}) : t > 0\} \subset \mathscr{B}(X_p)$ is $\mathscr{R}_q$-bounded.

We now sketch the construction of a bounded H^∞-calculus as given in [3] but we restrict ourselves to negative generators of bounded analytic semigroups. For $0 < \mu \leq \pi$ define the following sets of functions:

$$H^\infty(\Sigma(\mu)) \;:=\; \{m : \Sigma(\mu) \to \mathbb{C} \,|\, m \text{ is bounded and analytic in } \Sigma(\mu)\};$$
$$\Psi(\Sigma(\mu)) \;:=\; \{\psi \in H^\infty(\Sigma(\mu)) \,|\, \exists\, c, s > 0\, \forall\, z \in \Sigma(\mu) : |\psi(z)| \leq c|z|^s/(1 + |z|^{2s})\}.$$

If A is injective and has dense range in X and $-A$ generates a bounded analytic semigroup, then fix $\omega_\sigma(A) < \theta < \mu < \pi$. For arbitrary $\psi \in \Psi(\Sigma(\mu))$ define the operator $\psi(A)$ by

$$\psi(A) \;:=\; \frac{1}{2\pi i} \int_{\Gamma_\theta} \psi(z)(z - A)^{-1}\, dz\,. \tag{2.3}$$

Here the contour $\Gamma_\theta : \mathbb{R} \to \mathbb{C}$ has the representation $t \in \mathbb{R} \to |t|e^{-\operatorname{sign}(t)i\theta}$. By assumption the definition of $\psi(A)$ makes sense and $\psi(A) \in \mathscr{B}(X)$ since the integral converges absolutely. Moreover $\psi(A)$ is independent of θ. The operator A is said to have a (bounded) H^∞-*calculus* if there exist constants $\mu \in (\omega_\sigma(A), \pi), C > 0$ such that

$$\|\psi(A)\|_{\mathscr{B}(X)} \;\le\; C\,\|\psi\|_{\infty,\Sigma(\mu)} \qquad \text{for all } \psi \in \Psi(\Sigma(\mu)).$$

An approximation procedure then allows us to extend the functional calculus to all of $H^\infty(\Sigma(\mu))$. For details we refer to [3]. The H^∞-angle $\omega_{H^\infty}(A)$ of A is defined by

$$\omega_{H^\infty}(A) := \inf\{\varphi \in (\omega_\sigma(A), \pi) \,:\, A \text{ has a } H^\infty\text{-calculus on } \Sigma(\varphi)\}.$$

We shall use the following theorem which is a special case of a recent result due to Kalton and Weis (Theorem 5.3 in [7]).

2.4. **Theorem.** *Let A be an injective operator in $L_p(\Omega), 1 < p < \infty$, with dense range. If A has a bounded H^∞-calculus, then $\omega_{\mathscr{R}}(A) = \omega_{H^\infty}(A)$.*

Now for submarkovian generators A which are injective and have dense range, Cowling's result enables us to apply Theorem 2.4 in $L_p(\Omega)$, $1 < p < \infty$. For the proof of Theorem 1.1 we thus only have to control the angle $\omega_{\mathscr{R}}(A_p)$ which corresponds to the control of

$$\sup\{\delta \in (0, \frac{\pi}{2}) : \{e^{-zA_p} : |\arg z| \le \delta\} \text{ is } \mathscr{R}_2\text{-bounded in } \mathscr{B}(L_p(\Omega))\}$$

by the considerations above. This is the subject of the next section.

3. Proof of Theorem 1.1

Let A be a submarkovian generator with semigroup $(e^{-tA_p})_{t\ge 0}$ acting on $L_p(\Omega)$, $p \in [1,\infty]$. Let us first assume that A_p is injective and has dense range in $L_p(\Omega)$, $p \in (1,\infty)$. Now define $D \subset \mathbb{R}^3$ as the set of all triples $(\frac{1}{p}, \frac{1}{q}, \theta) \in [0,1]\times[0,1]\times[0,\frac{\pi}{2}]$ such that the semigroup $(e^{-tA_p})_{t\ge 0}$ has a bounded and continuous extension to the sector $\overline{\Sigma(\theta)}$, is analytic in $\Sigma(\theta)$ and the set $\{T(z) : z \in \overline{\Sigma(\theta)}\} \subset \mathscr{B}(L_p(\Omega))$ is $\mathscr{R}_q$-bounded. Here the operator-valued function T is defined by

$$T(te^{i\varphi}) := \frac{1}{t}\int_0^t e^{-(e^{i\varphi}s)A}\,ds, \quad t > 0,\ |\varphi| \le \theta.$$

An application of Theorem 2.3 shows that D is convex. Hence $\overline{D}$ is also convex, and the symmetry of submarkovian generators yields that

$$(\frac{1}{p}, \frac{1}{q}, \theta) \in \overline{D} \implies (\frac{1}{p'}, \frac{1}{q'}, \theta) \in \overline{D},$$

i.e., $\overline{D}$ is invariant under the transformation $S(x,y,\theta) := (1-x, 1-y, \theta)$. The optimal angle of $\mathscr{R}_2$-boundedness for T in $L_p(\Omega)$, $1 < p < \infty$, is then given by

$$\phi_p := \sup\{\theta : (\frac{1}{p}, \frac{1}{2}, \theta) \in D\} = \max\{\theta : (\frac{1}{p}, \frac{1}{2}, \theta) \in \overline{D}\}.$$

It is clear that $\phi_p \le \theta_p$ since, in $L_p(\Omega)$, $\mathscr{R}_2$-boundedness implies $\mathscr{R}_p$-boundedness.

By definition and Remark 2.2 (a) we know that $(\frac{1}{2}, \frac{1}{2}, \frac{\pi}{2}) \in D$ and that $(\frac{1}{p}, \frac{1}{p}, 0) \in D$ for all $1 \le p \le \infty$. In addition Remark 2.2 (b) and (c) state that $(\frac{1}{p}, \frac{1}{\infty}, 0) \in D$

for all $1 < p < \infty$ and $(\frac{1}{p}, \frac{1}{p}, \theta_p) \in D$ for $1 < p < \infty$. Setting

$$M := \{(\frac{1}{2}, \frac{1}{2}, \frac{\pi}{2})\} \cup \{(\frac{1}{p}, \frac{1}{\infty}, 0) : 1 < p < \infty\} \cup \{(\frac{1}{p}, \frac{1}{p}, \theta_p) : 1 < p < \infty\}$$

and letting E denote the closed convex hull of $M \cup S(M)$ it is clear that $E \subset \overline{D}$. By convexity we have

$$(\frac{1}{p}, \frac{1}{p'}, \pi \min(\frac{1}{p}, \frac{1}{p'})) \in E \quad \text{for all } 1 \le p \le \infty.$$

Now if we set

$$\psi_p := \max\{\theta : (\frac{1}{p}, \frac{1}{2}, \theta) \in E\}, \quad 1 < p < \infty,$$

Theorem 2.3 implies that, in $L_p(\Omega)$, $1 < p < \infty$,

$$\mathscr{R}(\{T(z) \,:\, z \in \Sigma(\varphi)\}) < \infty \quad (0 < \varphi < \psi_p).$$

By [13, Corollary 4.4 and its proof] this implies

$$\mathscr{R}(\{e^{-zA_p} \,:\, z \in \Sigma(\varphi)\}) < \infty \quad (0 < \varphi < \psi_p, 1 < p < \infty).$$

An application of Theorem 2.4 now yields the desired relation $\omega_{H^\infty}(A_p) \le \psi_p^*$, $1 < p < \infty$.

Let us calculate ψ_p. By symmetry we have $\psi_p = \psi_{p'}$ and so we can restrict ourselves to the case $1 < p < 2$. For those p, convexity of E yields

$$\psi_p = \max\{\lambda\pi(1 - \frac{1}{q}) + (1-\lambda)\arccos(\frac{2}{r} - 1) : 1 \le r, q \le 2, \lambda \in [0,1], \\ \frac{1}{p} = \lambda\frac{1}{q} + (1-\lambda)\frac{1}{r}, \frac{1}{2} = \lambda(1 - \frac{1}{q}) + (1-\lambda)\frac{1}{r}\}.$$

We fix $x_0 = \frac{1}{p} \in (\frac{1}{2}, 1)$, define $y := \frac{1}{q}, z := \frac{1}{r}, \mu := \frac{1}{p} - \frac{1}{2}$ and introduce the variable u via $2y - 1 = (\frac{1}{\mu} - \frac{1}{u})^{-1}$. Then

$$1 - \lambda = \frac{\mu}{u}, \quad \lambda = \frac{u - \mu}{u}, \quad 2z - 1 = u, \quad \lambda(1 - y) = \frac{1}{2}(1 - \mu) - \frac{\mu}{2u}.$$

The given restriction $\frac{1}{2} \le y \le 1, \frac{1}{2} \le z \le 1$ is equivalent to $\frac{\mu}{1-\mu} \le u \le 1$. Hence

$$\begin{aligned}\psi_p &= \max\{\frac{\pi}{2}(1-\mu) + \frac{\mu}{u}(\arccos u - \frac{\pi}{2}) : \frac{\mu}{1-\mu} \le u \le 1\} \\ &= \frac{\pi}{2}(1-\mu) + \mu \cdot \max\{\frac{\arccos u - \arccos 0}{u} : \frac{\mu}{1-\mu} \le u \le 1\}.\end{aligned}$$

Since the Arcuscosine-function is concave on $[0,1]$, the difference quotient $u \mapsto (\arccos u - \arccos 0)/u$ is decreasing and therefore

$$\psi_p = \frac{\pi}{2}(1-\mu) + \mu\frac{1-\mu}{\mu}(\arccos(\frac{\mu}{1-\mu}) - \frac{\pi}{2}) = (1-\mu)\arccos(\frac{\mu}{1-\mu}).$$

Recalling $\mu = x_0 - 1/2 = 1/p - 1/2$ we have proved that

$$\psi_p = (\frac{3}{2} - \frac{1}{p})\arccos(\frac{2-p}{3p-2}), \quad \text{for } 1 < p < 2.$$

Then $\psi_p = \psi_{p'}$ and $\mu = |1/p - 1/2|$ yield

$$\psi_p = (\frac{1}{2} + \frac{1}{p})\arccos(\frac{p-2}{p+2}), \quad \text{for } p > 2$$

which ends the calculation of ψ_p.

If A is not injective or does not have dense range we argue with the operator $A_{(\varepsilon)} := A+\varepsilon I$, $\varepsilon > 0$, in place of A. The assumptions made on A and the arguments in our proof then imply a uniform estimate ($\psi_p^* < \mu < \pi$)

$$\|m(A_{(\varepsilon)})\|_{\mathscr{B}(L_p)} \le C_{p,\mu}\,\|m\|_{\infty,\Sigma(\mu)} \qquad \text{for all } m \in H^\infty(\Sigma(\mu)) \text{ and all } \varepsilon > 0.$$

An application of Corollary 2.3 in [6] now concludes the proof for general submarkovian generators.

4. Applications to maximal regularity

We indicate in this section how the improvement of the H^∞-angle, i.e. the improvemant of the $\mathscr{R}$-sectoriality angle in Theorem 1.1, may be applied to obtain maximal regularity for parabolic equations or evolutionary integral equations. For reasons of simplicity we assume in this section that A is a submarkovian generator with semigroups $(e^{-tA_p})_{t\ge0}$ in $L_p(\Omega)$ and $0 \notin \sigma(A_p)$, $1 < p < \infty$.

(a) Parabolic evolution equations. We fix $p \in (1,\infty)\setminus\{2\}$. For operators satisfying the above assumptions one can define the fractional powers A_p^α, $\alpha > 0$ (see [11]). It is known that $-A_p^\alpha$ is the generator of an analytic semigroup for $\alpha \in (0, \frac{\pi}{2\theta_p^*})$. Observe that, by Remark 1.2(a) and the spectral mapping theorem, this range of α is optimal, even for Neumann Laplacians.

For α in the given range we consider the problem of "maximal L_q-regularity" for the operator A_p^α. By this we mean that for every $f \in L_q(\mathbb{R}_+; L_p(\Omega))$, where $1 < q < \infty$, the Cauchy problem

$$\frac{d}{dt}u(t) + A_p^\alpha u(t) = f(t), \qquad t > 0, \quad u(0) = 0, \tag{4.1}$$

has a (unique) solution $u : \mathbb{R}_+ \to L_p(\Omega)$ which satisfies the estimate

$$\|u'\|_{L_q(\mathbb{R}_+;L_p(\Omega))} + \|A_p^\alpha u\|_{L_q(\mathbb{R}_+;L_p(\Omega))} \le C\|f\|_{L_q(\mathbb{R}_+;L_p(\Omega))}.$$

It is known that this property does not depend on $q \in (1,\infty)$ and that it holds if and only if $\omega_{\mathscr{R}}(A_p^\alpha) < \frac{\pi}{2}$ (see Theorem 6.5 in [7]).

By Cowling's result, A_p^α has maximal regularity for $0 < \alpha < \frac{\pi}{2\rho_p^*}$. By Theorem 1.1 we obtain maximal L_q-regularity for A_p^α also for $\frac{\pi}{2\rho_p^*} \le \alpha < \frac{\pi}{2\psi_p^*}$, i.e. we obtain maximal L_q-regularity for operators for which this property was unknown so far.

Taking another point of view, i.e. fixing $\alpha \in (0,2)$, Theorem 1.1 increases the range of $p \in (1,\infty)$ for which A_p^α has maximal L_q-regularity in $L_p(\Omega)$.

(b) Evolutionary integral equations. Here we consider L_q-regularity conditions for the evolutionary integral equation

$$u(t) + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} A_p\, u(s)\, ds = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s)\, ds, \qquad t > 0. \tag{4.2}$$

Roughly speaking, this is equivalent to considering the problem

$$(\frac{d}{dt})^\beta u(t) + A_p u(t) = f(t), \qquad t > 0, \quad u(0) = 0$$

(see [1], [12] for details).

Again, we fix $p \in (1,\infty)\setminus\{2\}$. We are interested in maximal L_q-regularity in the following sense: for every $f \in L_q(\mathbb{R}_+; L_p(\Omega)), 1 < q < \infty$, the equation (4.2) has a (unique) solution $u \in L_q(\mathbb{R}_+; D(A_p))$. Again, this property does not depend on $q \in (1,\infty)$.

By Theorem 8 in [1], we have maximal L_q-regularity for (4.2) if $\beta\frac{\pi}{2}+\omega_{\mathscr{R}}(A_p)<\pi$ holds. By Theorem 1.1 this means that we have maximal L_q-regularity for (4.2) if $\beta<2(1-\psi_p^*/\pi)$, whereas Cowling's result gives maximal L_q-regularity for the smaller range $\beta<2(1-\rho_p^*/\pi)$. Hence our result increases the range of β for which (4.2) has maximal L_q-regularity in $L_p(\Omega)$.

Taking another point of view, i.e. fixing $\beta\in(0,2)$, our result increases the range of p for which (4.2) has maximal L_q-regularity in $L_p(\Omega)$.

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