RENORMING OF $C(K)$ SPACES

JAN RYCHTÁŘ

(Communicated by Jonathan M. Borwein)

ABSTRACT. If $K$ is a scattered Eberlein compact space, then $C(K)^*$ admits an equivalent dual norm that is uniformly rotund in every direction. The same is shown for the dual to the Johnson-Lindenstrauss space $JL_2$.

1. Introduction

We will find classes of Banach spaces whose duals admit equivalent dual norms that are uniformly rotund in every direction (URED) or pointwise uniformly rotund (p-UR) (definitions are given below). The notion of p-UR covers both the weak and weak$^*$ uniform rotundity (W$^*$UR). It can be shown from the Šmulyan theorem (see, e.g., [5, p. 63]), that the existence of a dual p-UR norm on $X^*$ implies the existence of a “big” set in $X^{**}$ on which the bidual norm is uniformly Gâteaux differentiable.

In Section 2 we will prove a three-space-like theorem for the following properties of a Banach space $X$: $X^*$ admits an equivalent dual URED (p-UR) norm. This result enables us to renorm duals to spaces, such as the Johnson-Lindenstrauss space or $C(K)$ for $K$ scattered with $K^{(ω)} = ∅$, by dual norms that are simultaneously locally uniformly rotund (LUR) and p-UR. On the example of $C(K)$, where $K$ is the so-called “two arrow” compact space, it is shown that properties of the duals to be equivalently renormed by dual URED norm (or p-UR norm) are not three space properties.

In Section 3, we will apply previous results, use a result from [1] and a method from [9], [11]. It will be proved that if $K$ is an Eberlein and scattered compact space, then $C(K)^*$ admits an equivalent dual LUR and p-UR norm.

Recently it was shown in [6] that if $X^*$ admits weak$^*$ uniformly rotund norm, then $X$ is a subspace of weakly compactly generated space. However, in [12] Th. 1] it is shown that if $X$ has an unconditional Schauder basis and $X^*$ admits an equivalent (not necessarily dual) URED norm, then $X^*$ admits an equivalent dual weak$^*$ uniformly rotund norm. Hence the space $JL_2$ from Section 2 shows that Theorem 1 in [12] does not hold without the assumption of unconditional Schauder
basis. From the result in Section 3 we can deduce that if $K$ is scattered Eberlein compact, that is not uniform Eberlein compact, then $C(K)^*$ is a dual to the weakly compactly generated space and admits an equivalent dual $p$-UR norm, but no equivalent dual weak* uniformly rotund norm, i.e., there is weakly compactly generated Banach space $X$, such that its dual $X^*$ admits an equivalent dual URED and LUR norm and no $W^*$-UR norm.

Let $(X,\|\cdot\|)$ be a Banach space. Let $S_X$ and $B_X$ denote the unit sphere and the unit ball respectively, i.e., $S_X = \{x \in X; \|x\| = 1\}$ and $B_X = \{x \in X; \|x\| \leq 1\}$. The norm $\|\cdot\|$ on a Banach space $X$ is said to be uniformly rotund in every direction (URED for short), if \(\lim_{n \to \infty} \|x_n - y_n\| = 0\) whenever $x_n, y_n \in S_X$ are such that $x_n - y_n = \lambda_n z$ for some $z \in X, \lambda_n \in \mathbb{R}$ and $\lim_{n \to \infty} \|x_n + y_n\| = 2$. We will say that the norm $\|\cdot\|$ on $X$ is pointwise uniformly rotund (p-UR), if there exists a $w^*$-dense set $\mathcal{F} \subseteq X^*$ such that $\lim_{n \to \infty} f(x_n - y_n) = 0$ whenever $x_n, y_n \in S(X,\|\cdot\|)$, $\lim_{n \to \infty} \|x_n + y_n\| = 2$, and $f \in \mathcal{F}$. More precisely, we say that the norm is $p$-UR with $\mathcal{F}$. Clearly, if the norm is p-UR, then it is URED. In the case of a dual Banach space $X = Y^*$ we say that the norm is weak* uniformly rotund ($W^*$ UR), if it is p-UR with $\mathcal{F} = Y \subseteq Y^{**}$. The norm $\|\cdot\|$ is said to be locally uniformly rotund (LUR), if $\lim_{n \to \infty} \|x - x_n\| = 0$ whenever $x, x_n \in S_X$ are such that $\lim_{n \to \infty} \|x + x_n\| = 2$.

A compact space $K$ is an Eberlein compact if $K$ is homeomorphic to a weakly compact subset of a Banach space in its weak topology. A compact space $K$ is a uniform Eberlein compact if $K$ is homeomorphic to a weakly compact subset of a Hilbert space. A compact space is called scattered if every closed subset $L \subseteq K$ has an isolated point in $L$. For scattered compact spaces the Cantor derivative sets are defined as follows: $K^{(0)} = K, K^{(1)} = K'$ is the set of all limit points of $K$. If $\alpha$ is an ordinal and $K^{(\beta)}$ are defined for all $\beta < \alpha$, then we put $K^{(\alpha)} = (K^{(\beta)})'$ for $\alpha = \beta + 1$ and $K^{(\alpha)} = \bigcap_{\beta < \alpha} K^{(\beta)}$ for $\alpha$ a limit ordinal.

If we consider spaces such as $c_0(\Gamma), \ell_1(\Gamma), l_\infty(\Gamma)$, by the symbol $e_\gamma$ we mean the standard unit vector.

For more information in this area we refer to [3], [5], [7, Ch. 12], [10] and [14].

2. The Three Space Problem

**Theorem 1.** Let $X$ be a Banach space such that $c_0(\Gamma) \subseteq X$. Let the dual to $Y = X/c_0(\Gamma)$ admit an equivalent dual $p$-UR (URED) norm. Then $X^*$ admits an equivalent dual $p$-UR (URED) norm.

**Proof.** Let $i : c_0(\Gamma) \to X$ be the inclusion map and $q : X \to Y$ be the quotient map. Then the dual mappings are $i^* : X^* \to c_0(\Gamma)^*$, which is a quotient map and a restriction, and $q^* : Y^* \to X^*$, which is an inclusion. Because of the lifting property of the space $\ell_1(\Gamma) \equiv c_0(\Gamma)^*$ there is a bounded linear map $l : \ell_1(\Gamma) \to X^*$ (the so-called lifting; see, e.g., [11]) such that $i^*(l(e)) = e$ for all $e \in \ell_1(\Gamma)$. Hence we have an isomorphism $X^* \equiv \ell_1(\Gamma) \oplus Y^*$, where the duality between $(f,g) \in \ell_1(\Gamma) \oplus Y^*$ and $x \in X$ is given by the formula

$$\langle (f,g), x \rangle = \langle l(f), x \rangle + \langle q^*(g), x \rangle.$$

Let $\|\cdot\|_{Y^*}$ be a dual norm on $Y^*$ which is p-UR with $\mathcal{F}$. We will prove that there is an equivalent dual norm $\|\cdot\|_u$ on $X^*$, that is, p-UR with $\mathcal{G} = \{(e_\gamma, 0); \gamma \in \Gamma\} \cup \{(0, f); f \in \mathcal{F}\}$, where we identify $X^{**}$ with $l_\infty(\Gamma) \oplus Y^{**}$ and where $\{e_\gamma; \gamma \in \Gamma\}$ denote the standard unit vectors in $c_0(\Gamma) \subseteq l_\infty(\Gamma)$. The proof that the norm $\|\cdot\|_u$ is URED if $\|\cdot\|_{Y^*}$ is URED proceeds in the same way.
Let $\|\cdot\|_{X^*}$ be a dual norm on $X^*$ which extends the norm $\|\cdot\|_{Y^*}$. Let $\|\cdot\|_{\ell_1(\Gamma)}$ be the standard norm on $\ell_1(\Gamma)$. We choose $\alpha > 1$ such that

$$a^{-1}||(f,g)||_{X^*} \leq \|f\|_{\ell_1(\Gamma)} + \|g\|_{Y^*} \leq a||(f,g)||_{X^*}.$$  

Put

$$||(f,g)||_w = (\|f\|_{\ell_1(\Gamma)} + \|f\|_{\ell_1(\Gamma)} + \|g\|_{Y^*})^{\frac{1}{2}}.$$ 

This is an equivalent norm on $X^* \cong \ell_1(\Gamma) \oplus Y^*$. The norm $\|\cdot\|_w$ need not be a dual norm, but it is p-UR with $G$. This convexity property will be used at the end of this proof. To have a dual norm, let us define

$$||(f,g)|| = ||(f,g)||_w + a\|f\|_{\ell_1(\Gamma)}.$$  

**Observation.** The norm $\|\cdot\|$ is a dual norm on $X^*$.

**Proof of the Observation.** We will follow the proof published in \[8\] and show that the unit ball is closed in the weak* topology. To prove this, let $\{(f_\alpha, g_\alpha)\}_{\alpha \in A}$ be a net in the unit ball in $(X^*, \|\cdot\|)$, which weak* converges to $(f,g)$. Because $c_0(\Gamma) \subset X$ and $\ell_1(\Gamma) \cong c_0(\Gamma)^*$, $\{f_\alpha\}_{\alpha \in A}$ converges coordinatewise to $f$. To see this, choose $x \in c_0(\Gamma)$. We have

$$\langle (f_\alpha, g_\alpha), i(x) \rangle = \langle (f_\alpha), i(x) \rangle + \langle g_\alpha, i(x) \rangle = \langle f_\alpha, x \rangle.$$ 

To estimate the norm of $(f,g)$ we will decompose $f_\alpha$ in a special way. For each $\alpha \in A$, we can find elements $f_\alpha^1, f_\alpha^2 \in \ell_1(\Gamma)$ such that $f_\alpha = f_\alpha^1 + f_\alpha^2$, the supports of $f_\alpha^1, f_\alpha^2$ are disjoint and $\inf_{\alpha \in A}\|f_\alpha - f_\alpha^1\|_{\ell_1(\Gamma)} = 0$. By passing to a subnet, we can assume that $\{(f_\alpha^2, 0)\}_{\alpha \in A}$ weak* converges to some $(0, g^1)$ and $\{0, g_\alpha\}_{\alpha \in A}$ weak* converges to $(0, g_2) = (0, g - g_1)$. Thus

$$\|f\|_{\ell_1(\Gamma)} \leq \inf_{\alpha \in A}\|f_\alpha^1\|_{\ell_1(\Gamma)},$$

$$\|g_1\|_{Y^*} = ||0, g_1||_{X^*} \leq \inf_{\alpha \in A}\|f_\alpha^2\|_{\ell_1(\Gamma)},$$

$$\|g_2\|_{Y^*} = ||0, g_2||_{X^*} = \inf_{\alpha \in A}\|g_\alpha\|_{Y^*},$$

where we used that $\|\cdot\|_{X^*}$ is the dual norm. It follows from previous estimates that

$$||(f,g)|| = a\|f\|_{\ell_1(\Gamma)} + (||f||_{\ell_1(\Gamma)}^2 + \|g_1 + g_2\|_{Y^*}^2)^{\frac{1}{2}} 
\leq a\|f\|_{\ell_1(\Gamma)} + \|g_1\|_{Y^*} + (||f||_{\ell_1(\Gamma)}^2 + \|g_2\|_{Y^*}^2)^{\frac{1}{2}} 
\leq \inf_{\alpha \in A}\|f_\alpha\|_{\ell_1(\Gamma)} + \|g_\alpha\|_{Y^*} + (\|f_\alpha^1\|_{\ell_1(\Gamma)}^2 + \|f_\alpha^2\|_{\ell_1(\Gamma)}^2 + \|g_\alpha\|_{Y^*}^2)^{\frac{1}{2}} 
\leq \sup_{\alpha \in A}\|f_\alpha, g_\alpha\| \leq 1.$$ 

Thus dual unit ball is $w^*$-closed and the Observation is proved. \hfill \Box

We will continue with the proof of Theorem 1. Let us define the norm $\|\cdot\|_w$ on $X^*$ by the formula

$$||(f,g)||_w^2 = ||(f,g)||^2 + \|f\|_{\ell_1(\Gamma)}^2.$$ 

It is a dual norm, because it is $w^*$-lower semicontinuous as is the seminorm $\|\cdot\|_{\ell_1(\Gamma)}$ on $X^*$. We prove that it is p-UR with $G$. To do this, we use the following fact, which can be found with the proof in [8], Ch. II.
Fact. Let \( a_n, b_n \) be bounded elements of a Banach space \((E, \|\cdot\|)\) such that
\[
\lim_{n \to \infty} \left( 2\|a_n\|^2 + 2\|b_n\|^2 - \|a_n + b_n\|^2 \right) = 0.
\]
Then \( \lim_{n \to \infty} (\|a_n\| - \|b_n\|) = 0 \) and \( \lim_{n \to \infty} (\|a_n\| + \|b_n\| - \|a_n + b_n\|) = 0. \)

Now assume that \( x_n = (x_n^1, x_n^2), y_n = (y_n^1, y_n^2) \in X^* \) satisfy
\[
\lim_{n \to \infty} \left( 2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2 \right) = 0.
\]
By the previous Fact we have
\[
\lim_{n \to \infty} \left( 2\|x_n\|^2 \ell_{1}(\Gamma) + 2\|y_n\|^2 \ell_{1}(\Gamma) - \|x_n + y_n\|^2 \ell_{1}(\Gamma) \right) = 0.
\]
And again by the Fact
\[
\lim_{n \to \infty} \left( \|x_n\| + \|y_n\| \ell_{1}(\Gamma) - \|x_n + y_n\| \ell_{1}(\Gamma) \right) = 0.
\]
By (1) and by the Fact it follows that
\[
\lim_{n \to \infty} \left( 2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2 \right) = 0.
\]
Hence by the Fact
\[
\lim_{n \to \infty} \left( \|x_n\| + \|y_n\| \right) - \lim_{n \to \infty} \|x_n + y_n\| = 0.
\]
By (2) and (3) we have
\[
\lim_{n \to \infty} \left( \|x_n\|_w + \|y_n\|_w \right) = \lim_{n \to \infty} \|x_n + y_n\|_w,
\]
\[
\lim_{n \to \infty} \|x_n\|_w = \lim_{n \to \infty} \|y_n\|_w.
\]
Hence
\[
\lim_{n \to \infty} \left\| \frac{x_n}{\|x_n\|_w} + \frac{y_n}{\|y_n\|_w} \right\|_w = \lim_{n \to \infty} \left\| \frac{x_n}{\|x_n\|_w} + \frac{y_n}{\|y_n\|_w} \right\|_w = 2.
\]
The norm \( \|\cdot\|_w \) on \( \ell_{1}(\Gamma) \oplus Y^* \) is p-UR with \( G \); thus, for all \( G \in G \) we have
\[
\lim_{n \to \infty} G \left( \frac{x_n}{\|x_n\|_w} - \frac{y_n}{\|y_n\|_w} \right) = 0,
\]
which finishes the proof of Theorem 1. \(\square\)

Remark. Moreover, if the norm \( \|\cdot\|_{Y^*} \) on \( Y^* \) is LUR, the norm \( \|\cdot\|_{u} \) on \( X^* \) is LUR as well because the norm \( \|\cdot\|_w \) is.

Corollary 2. Let \( K \) be a scattered compact space with \( K^{(\omega)} = \emptyset \). Then \( C(K)^* \) admits an equivalent dual norm that is simultaneously LUR and p-UR with \( F = \{ e_k; k \in K \} \subset c_0(K) \subset C(K)^{**} \).

Proof. By a compactness argument, there is some \( n \in \mathbb{N} \) such that \( K^{(n)} = \emptyset \). We shall prove the corollary by an induction. If \( n = 0, 1 \), the claim is trivial. Let \( n > 1 \). It is easy to see that the space \( Y = \{ f \in C(K); f|K' = 0 \} \) is isometric to the space \( c_0(K \setminus K') \). Moreover, \( C(K)/Y = C(K') \), so we can use Theorem 1. \(\square\)

In fact, the following theorem holds.

Theorem (Deville). Let \( K \) be a scattered compact space, such that \( K^{(\omega)} = \emptyset \). Then \( C(K)^* \) admits an equivalent dual norm, that is LUR and p-UR.
Proof. See [5, Theorem 7.4.7]. There is an equivalent dual LUR norm constructed on \( C(K)^* \). One can compute that this norm is, moreover, p-UR with \( \mathbb{F} = \{ e_k; k \in K \} \).

Note that it is shown in [8] (see also [13]) that there is a Banach space \( JL_2 \) with the following properties:

1. \( c_0 \subset JL_2, JL_2/c_0 = \ell_2(\Gamma) \), where the cardinality of the set \( \Gamma \) is a continuum,
2. \( JL_2 \) is not a subspace of any WCG space; in particular, \( JL_2 \) is not isomorphic to the \( c_0 \oplus \ell_2(\Gamma) \),
3. there is an equivalent dual LUR norm on \( JL_2^* \).

From Theorem 1 we can obtain a stronger result.

**Theorem 3.** There is an equivalent dual norm on the space \( JL^* \) that is LUR and p-UR with \( \mathbb{F} \), where \( \mathbb{F} \) is the canonical imbedding of \( c_0 \oplus \ell_2(\Gamma) \) into \( \ell_\infty \oplus \ell_2(\Gamma) \equiv JL^{**} \).

It is shown in [5] pp. 299–305, that if \( K \) is a so-called “two arrow” compact space, then \( C([0, 1]) \subset C(K), C(K)/C([0, 1]) = c_0([0, 1]) \) and \( C(K) \) has no equivalent Gateaux smooth norm. It means that there is no dual equivalent strictly convex norm on \( C(K)^* \). It means that \( C(K)^* \) does not admit an equivalent dual URED norm, although both \( C([0, 1])^* \) and \( c_0([0, 1])^* \) do admit an equivalent dual p-UR norms.

3. SCATTERED EBERLEIN COMPACT SPACES

**Theorem 4.** Let \( K \) be a scattered compact space such that \( K = \bigcup_{n=1}^{\infty} K_n \), and for all \( n \in \mathbb{N} \) let \( C(K_n)^* \) admit an equivalent dual p-UR norm with \( \mathbb{F}_n = \{ e_k; k \in K_n \} \). Then \( C(K)^* \) admits an equivalent dual p-UR norm with \( \mathbb{F} = \{ e_k; k \in K \} \).

**Proof.** This proof is similar to the proof of Theorem 2.7.16 in [5], which states, that the space \( L_1(\Omega) \) admits a norm that is LUR and URED.

As in [9], we can define the operator \( T: C(K) \to \sum_2 C(K_n) \) by the formula \( T(f) = (\frac{1}{n} f|_{K_n}) \). For \( k \in K \) put \( N(k) = \{ n \in \mathbb{N}; k \in K_n \} \). For \( k \in K, n \in N(k) \) let \( k_n \) denote a copy of \( k \) in \( K_n \). For \( A \subset K \) put \( A = \{ k_n; k \in A, n \in N(k) \} \). By Rudin’s Theorem (see [7]) \( C(K)^* \) is isometric to the space \( \ell_1(K) \) and the canonical norm \( \| \cdot \|_1 \) is a dual norm on \( C(K)^* \), the same holds for \( K_n \)’s. Without loss of generality, we can assume, that the p-UR norms are uniformly close to the original norms on \( C(K_n)^* \). Hence \( (\sum_2 C(K_n))^* = \sum_2 \ell_1(K_n) \) and \( (\sum_2 C(K_n))^* \) admits an equivalent dual norm \( \| \cdot \|_{\Sigma} \), which is p-UR with \( \mathbb{G} = \{ e_k; k \in K, n \in N(k) \} \).

The dual operator \( T^*: (\sum_2 C(K_n))^* \to C(K)^* \) is given by

\[
T^*(y^*) = \left( \sum_{n \in N(k)} \frac{1}{n^2} y^*(k_n) \right)_{k \in K}.
\]

The range of \( T^* \) is a dense set in \( C(K)^* \). Now, we shall use the standard LUR renorming method. For \( n \in \mathbb{N} \) and \( x \in \ell_1(K) \) we define

\[
|x|_n^2 = \inf \left\{ \| x - T^* y \|_1^2 + \frac{1}{n} \| y \|_2^2; y \in \sum_2 C(K_n)^* \right\},
\]

\[
\| x \|_2^2 = \| x \|_1^2 + \sum_{n=1}^{\infty} 2^{-n} |x|_n^2.
\]
Choose \( \varepsilon > 0 \) and passing to a subsequence again, we can assume, moreover, that 
\[
\lim_{i \to \infty}(2\|x_i\|^2 + 2\|y_i\|^2 - \|x_i + y_i\|^2) = 0.
\]

Then for all \( n \in \mathbb{N} \),
\[
(1) \quad \lim_{i \to \infty}(2|x_i|^2 + 2|y_i|^2 - |x_i + y_i|^2) = 0.
\]

The infimum in the definition of \( |f_n| \) is attained (see [5, p. 44]); e.g., for all \( i, n \in \mathbb{N} \) there are \( u_i^{(n)}, v_i^{(n)} \in \sum_{l} C(K_n)^* \) such that
\[
|x_i|^2 = \|x_i - T^*u_i^{(n)}\|^2 + \frac{1}{n}\|u_i^{(n)}\|^2,
\]
\[
|y_i|^2 = \|y_i - T^*v_i^{(n)}\|^2 + \frac{1}{n}\|v_i^{(n)}\|^2.
\]

From (2) we get
\[
(3) \quad \|u_i^{(n)}\| \leq n|x_i|, \quad \|v_i^{(n)}\| \leq n|x_i|,
\]
and by the same manner we have \( \|v_i^{(n)}\| \leq n; \) therefore, for all \( k \in K \) and \( l \in N(k) \)
\[
(4) \quad (u_i^{(n)} - v_i^{(n)})(k_i) \leq \|u_i^{(n)} - v_i^{(n)}\|_1 \leq c, \quad \|u_i^{(n)} - v_i^{(n)}\|_{\Sigma} \leq 2cn,
\]
where \( c \) is a constant of the equivalence of norms \( \|\cdot\|_1 \) and \( \|\cdot\|_{\Sigma} \). From (1), (2), (3)
we have
\[
\lim_{i \to \infty}(2\|u_i^{(n)}\|^2 + 2\|v_i^{(n)}\|^2 - \|u_i^{(n)} + v_i^{(n)}\|^2) = 0.
\]

The norm \( \|\cdot\|_{\Sigma} \) is p-UR and hence for all \( k \in K, m \in \mathbb{N} \) and \( n \in N(k) \)
\[
(5) \quad \lim_{i \to \infty}(u_i^{(m)} - v_i^{(m)})(k_n) = 0.
\]

We can assume (by passing to a subsequence), that \( \lim_{i \to \infty}|x_i| = d_n \). For every
\( x \in l_1(K), |x| \) is a nonincreasing sequence, hence there is \( d = \lim_{n \to \infty}d_n \). By
passing to a subsequence again, we can assume, moreover, that \( \lim_{i \to \infty}|y_i| = d_n \). Choose \( \varepsilon > 0 \) and \( k \in K \). Put \( A = K \setminus \{k\} \). Let \( m \in \mathbb{N} \) be such that \( d_m < d + \varepsilon \). Then
\[
|(x_i - y_i)(k)| \leq |(x_i - T^*u_i^{(m)})(k)| + |(T^*u_i^{(m)} - T^*v_i^{(m)})(k)| + |(T^*v_i^{(m)} - y_i)(k)|.
\]

Considering the second term, we have
\[
\left|T^*(u_i^{(m)} - v_i^{(m)})(k)ight| = \left|\sum_{n \in N(k)} \frac{1}{n^\varepsilon}(u_i^{(m)} - v_i^{(m)})(k_n)\right|
\]
\[
\leq \sum_{n \leq n_0, n \in N(k)} \left|\frac{1}{n^\varepsilon}(u_i^{(m)} - v_i^{(m)})(k_n)\right| + \varepsilon,
\]
where \( n_0 \) depends only on \( \varepsilon \) (because of (4)) and the sum is finite and therefore tends
to 0 for \( i \to \infty \) because of (5). It remains to prove, that \( |(x_i - T^*u_i^{(m)})(k)| < \varepsilon \). We
can assume that \( k \in K_{n_0} \). Put \( y = s_i + (u_i^{(m)}|_{\tilde{A}}) \), where \( s_i(l) = n_0^2x_i(k) \) if \( l = k_{n_0} \),
and \( s_i(l) = 0 \) otherwise. Considering this \( y \) in the definition of \( |x_i|_n \) we get
\[
|x_i|_n^2 \leq \| (x_i - T^*u_i^{(m)})|_A \|^2 + \frac{1}{n} \| s_i + (u_i^{(m)}|_A) \|^2_S
\]
\[
\leq \| (x_i - T^*u_i^{(m)})|_A \|^2 + \frac{1}{n} (\|s_i\|_S + \|u_i^{(m)}|_A\|)^2,
\]
\[
\leq \| (x_i - T^*u_i^{(m)})|_A \|^2 + \frac{1}{n} (n_0^2 c + mc^2)^2
\]
because
\[
\|u_i^{(m)}|_A\|_S \leq c\|u_i^{(m)}|_A\|_1 \leq c\|u_i^{(m)}\|_1 \leq mc^2,
\]
where we used that the canonical norm \( \|\cdot\|_1 \) on \( X^* \) is a lattice norm.
Hence for all \( n \in \mathbb{N} \)
\[
\limsup_{i \to \infty} \| (x_i - T^*u_i^{(m)})|_A \|_1^2 \geq \lim_{i \to \infty} \limsup_{i \to \infty} |x_i|_n^2 - \frac{1}{n} (cn_0^2 + mc^2)^2.
\]
Finally,
\[
\limsup_{i \to \infty} \| (x_i - T^*u_i^{(m)})|_A \|_1^2 \geq d^2.
\]
For all \( i \in \mathbb{N} \) we have
\[
\left| (x_i - T^*u_i^{(m)})(k) \right| = \left| (x_i - T^*u_i^{(m)})|_1 - (x_i - T^*u_i^{(m)})|_A \right|_1, \\
\leq |x_i|_m - \left| (x_i - T^*u_i^{(m)})|_A \right|.
\]
Hence we get
\[
\liminf_{i \to \infty} \left| (x_i - T^*u_i^{(m)})(k) \right| \leq d_m - d \leq \varepsilon.
\]
The same holds for the third term and this concludes the proof.

\[\square\]

**Theorem 5.** Let \( K \) be a scattered Eberlein compact space. Then \( C(K)^* \) admits an equivalent dual norm that is LUR and \( p \)-UR with \( \mathbb{P} = \{e_k; k \in K\} \). In particular, \( C(K)^* \) admits an equivalent dual norm that is LUR and URED.

**Proof.** K. Alster proved in [1] that if \( K \) is a scattered Eberlein compact space, then \( K \) is a strong Eberlein compact space, e.g., \( K \subseteq \{0, 1\}^\Gamma \) for some \( \Gamma \). Hence \( K = \bigcup_{n=1}^\infty K_n \), where \( K_n = \{ x \in K; \text{card}(\gamma \in \Gamma; x(\gamma) = 1) \leq n \} \).

The \( K_n \)'s are uniform Eberlein compact spaces, they are scattered and \( K^{(n+1)} = \emptyset \). Hence by Corollary 2, \( C(K_n)^* \) admits an equivalent dual norm that is both \( p \)-UR with \( \mathbb{P} = \{e_k; k \in K_n\} \) and LUR. Thus we can use the preceding theorem to finish the proof.

\[\square\]

4. **Open question**

It is shown in [12] Th. 1 that if \( X \) has an unconditional Schauder basis and \( X^* \) admits an equivalent URED norm, then \( X^* \) admits an equivalent dual weak* uniformly rotund norm. Because there is a scattered Eberlein compact space \( K \) that is not uniform Eberlein compact (see, e.g., [2] Example 1.10), the space \( C(K)^* \) admits an equivalent dual \( p \)-UR (and hence URED norm) but does not admit any equivalent dual \( W^* \)UR norm. But we do not know the answer to the following questions. Is there any reflexive Banach space \( X \) such that \( X \) admits an equivalent URED norm and does not admit any equivalent \( p \)-UR (and hence \( W^* \)UR) norm?
Is there any Banach space that admits an equivalent URED norm and does not admit any p-UR norm?

References


Department of Mathematical Analysis, Charles University, Faculty of Mathematics and Physics, Sokolovská 83, 186 75 Praha 8, Czech Republic
E-mail address: rychtar@karlin.mff.cuni.cz
Current address: Department of Mathematics and Statistics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1
E-mail address: jrychtar@math.ualberta.ca