Behavior of the Bergman Kernel and Metric Near Convex Boundary Points

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Abstract. The boundary behavior of the Bergman metric near a convex boundary point $z_0$ of a pseudoconvex domain $D \subset \mathbb{C}^n$ is studied. It turns out that the Bergman metric at points $z \in D$ in the direction of a fixed vector $X_0 \in \mathbb{C}^n$ tends to infinity, when $z$ is approaching $z_0$, if and only if the boundary of $D$ does not contain any analytic disc through $z_0$ in the direction of $X_0$.

For a domain $D \subset \mathbb{C}^n$ we denote by $L^2_h(D)$ the Hilbert space of all holomorphic functions $f$ that are square-integrable and by $\|f\|_D$ the $L_2$-norm of $f$. Let $K_D(z)$ be the restriction on the diagonal of the Bergman kernel function of $D$. It is well known (cf. [5]) that

$$K_D(z) = \sup \{|f(z)|^2 : f \in L^2_h(D), \|f\|_D \leq 1\}.$$ If $K_D(z) > 0$ for some point $z \in D$, then the Bergman metric $B_D(z; X)$, $X \in \mathbb{C}^n$, is well defined and can be given by the equality

$$B_D(z; X) = \frac{M_D(z; X)}{\sqrt{K_D(z)},}$$

where

$$M_D(z; X) = \sup \{|f'(z)X| : f \in L^2_h(D), \|f\|_D = 1, f(z) = 0\}.$$ We say that a boundary point $z_0$ of a domain $D \subset \mathbb{C}^n$ is convex if there is a neighborhood $U$ of this point such that $D \cap U$ is convex.

In [4], Herbort proved the following

Theorem 1. Let $z_0$ be a convex boundary point of a bounded pseudoconvex domain $D \subset \mathbb{C}^n$ whose boundary contains no nontrivial germ of an analytic curve near $z_0$. Then

$$\lim_{z \to z_0} B_D(z; X) = \infty$$

for any $X \in \mathbb{C}^n \setminus \{0\}$.

Herbort’s proof is mainly based on Ohsawa’s $\bar{\partial}$-technique. The main purpose of this note is to generalize Theorem 1 using more elementary methods.
For a convex boundary point $z_0$ of a domain $D \subset \mathbb{C}^n$ we denote by $L(z_0)$ the set of all $X \in \mathbb{C}^n$ for which there exists a number $\varepsilon_X > 0$ such that $z_0 + \lambda X \in \partial D$ for all complex numbers $\lambda$, $|\lambda| \leq \varepsilon_X$. Note that $L(z_0)$ is a complex linear space.

Then our result is the following one.

**Theorem 2.** Let $z_0$ be a convex boundary point of a bounded pseudoconvex domain $D \subset \mathbb{C}^n$ and let $X \in \mathbb{C}^n$. Then

(a) $\lim\inf_{z \to z_0} K_D(z) \text{dist}^2(z, \partial D) \in (0, \infty]$;

(b) $\lim_{z \to z_0} B_D(z; X) = \infty$ if and only if $X \notin L(z_0)$. Moreover, this limit is locally uniform in $X \notin L(z_0)$;

(c) if $L(z_0) = \{0\}$, then (a) and (b) are still true without the assumption that $D$ is bounded.

**Proof of Theorem 2.** To prove (a) and (b) we will use the following localization theorem for the Bergman kernel and metric [2].

**Theorem 3.** Let $D \subset \mathbb{C}^n$ be a bounded pseudoconvex domain and let $V \subset U$ be open neighborhoods of a point $z_0 \in \partial D$. Then there exists a constant $C \geq 1$ such that

$$
\tilde{C}^{-1} K_{D \cap U}(z) \leq K_D(z) \leq K_{D \cap U}(z),
$$

$$
\tilde{C}^{-1} B_{D \cap U}(z; X) \leq B_D(z; X) \leq \tilde{C} B_{D \cap U}(z; X)
$$

for any $z \in D \cap V$ and any $X \in \mathbb{C}^n$. (Here $K_{D \cap U}(z)$ and $B_{D \cap U}(z; \cdot)$ denote the Bergman kernel and metric of the connected component of $D \cap U$ that contains $z$.)

So, we may assume that $D$ is convex.

To prove part (a) of Theorem 2, for any $z \in D$ we choose a point $\tilde{z} \in \partial D$ such that $||z - \tilde{z}|| = \text{dist}(z, \partial D)$. We denote by $l$ the complex line through $z$ and $\tilde{z}$. By the Oshawa-Takegoshi extension theorem for $L^2$-holomorphic functions [2], it follows that there exists a constant $C_1 > 0$ only depending on the diameter of $D$ (not on $l$) such that

$$
K_D(z) \geq C_1 K_{D \cap l}(z).
$$

Since $D \cap l$ is convex, it is contained in an open half-plane $\Pi$ of the $l$-plane with $\tilde{z} \in \partial \Pi$. Then

$$
K_{D \cap l}(z) \geq K_{\Pi}(z) = \frac{1}{4\pi \text{dist}^2(z, \partial \Pi)}.
$$

Now, part (a) of Theorem 2 follows from the inequalities (1), (2) and the fact that $\text{dist}(z, \partial \Pi) \leq ||z - \tilde{z}|| = \text{dist}(z, \partial D)$.

To prove part (b) of Theorem 2, we denote by $N(z_0)$ the complex affine space through $z_0$ that is orthogonal to $L(z_0)$. Set $E(z_0) = D \cap N(z_0)$. Note that $E(z_0)$ is a nonempty convex set. So, part (b) of Theorem 2 will be a consequence of the following:

**Theorem 4.** Let $z_0$ be a boundary point of a bounded convex domain $D \subset \mathbb{C}^n$. Then:

(i) $\lim_{z \to z_0} B_D(z; X) = \infty$ locally uniformly in $X \notin L(z_0)$;
(ii) for any compact set $K \subset \subset E(z_0)$ there exists a constant $C > 0$ such that

$$B_D(z; X) \leq C||X||, \quad z \in K^0, \quad X \in L(z_0),$$

where $K^0 := \{z_0 + t z : z \in K, \ 0 < t \leq 1\}$ is the cone generated by $K$.

**Proof of Theorem 4.** To prove (i) we will use the well-known fact that the Carathéodory metric $C_D(z; X)$ of $D$ does not exceed $B_D(z; X)$. On the other hand, we have the following simple geometric inequality [1]:

$$C_D(z; X) \geq \frac{1}{2d(z; X)},$$

where $d(z; X)$ denotes the distance from $z$ to the boundary of $D$ in the $X$-direction, i.e., $d(z; X) := \sup\{r : z + \lambda X \in D, \ \lambda \in \mathbb{C}, \ |\lambda| < r\}$. So, if we assume that (i) does not hold, then we may find a number $a > 0$ and sequences $D \supset (z_j), \ z_j \to z_0, \ \mathbb{C}^n \supset (X_j), \ X_j \to X \notin L(z_0)$, such that $B_D(z_j; X_j) \leq \frac{1}{2a}$. Hence $d(z_j; X_j) \geq a$ which implies that for $|\lambda| \leq a$ the points $z_0 + \lambda X$ belong to $D$ and, in view of convexity, they belong to $\partial D$. This means that $X \in L(z_0)$, a contradiction.

To prove part (ii) of Theorem 4, we may assume that $z_0 = 0$ and $L := L(0) = \{z \in \mathbb{C}^n : z_1 = \ldots = z_k = 0\}$ for some $k < n$. Then $N := N(0) = \{z \in \mathbb{C}^n : z_k+1 = \ldots = z_n = 0\}$ and $D := D(z, X)$ from now on we will write any point $z \in \mathbb{C}^n$ in the form $z = (z', z'')$, $z' \in \mathbb{C}^k$, $z'' \in \mathbb{C}^{n-k}$. Note that $L \in \partial D$ near $0$, i.e., there exists a $c > 0$ such that

$$\{0'\} \times \Delta''_c \subset \partial D,$$

where $\Delta'' \subset \mathbb{C}^{n-k}$ is the polydisc with center at the origin and radius $c$. Since $K \subset \subset E := E(0)$ and since $E$ is convex, there exists an $\alpha > 1$ such that $K \subset \subset E_\alpha$, where $E_\alpha := \{z : \alpha z \in E\}$. Note that $K^0 \subset E_\alpha$. Using (3), the equality

$$(z', z'') = \frac{1}{\alpha}(\alpha z', 0') + (1 - \frac{1}{\alpha})(0', (1 - \frac{1}{\alpha})^{-1}z''),$$

and the convexity of $D$, it follows that

$$F_\alpha \times \Delta''_c \subset D,$$

where $\varepsilon := c(1 - \frac{1}{\alpha})$ and where $F_\alpha$ is the projection of $E_\alpha$ in $\mathbb{C}^k$ (we can identify $E_\alpha$ with $F_\alpha$). For $\delta := c(\alpha - 1)$ we obtain in the same way that

$$\tilde{D} := D \cap (\mathbb{C}^k \times \Delta''_\delta) \subset F_\alpha \times \Delta''_\delta.$$

Now, let $(z, X) \in K^0 \times L$. Note that $z = (z', 0')$ and $X = (0', X'')$. Then, using (4) and the product properties of the Bergman kernel and metric, we have

$$M_D(z; X) \leq M_{F_\alpha \times \Delta''}(z; X)$$

$$= M_{\Delta''_\delta}(0'; X'')\sqrt{K_{F_\alpha}(z')} \leq C_1||X||\sqrt{K_{F_\alpha}(z')}$$

for some constant $C_1 > 0$. On the other hand, since $K^0 \subset \subset \mathbb{C}^k \times \Delta''_\delta$, by virtue of Theorem 3 there exists a constant $\tilde{C} \geq 1$ such that

$$K_D(z) \geq \tilde{C}^{-1}K_{\tilde{D}}(z).$$

Moreover, in view of (5), we have

$$K_{\tilde{D}}(z) \geq K_{F_\alpha}(z')K_{\Delta''_\delta}(0')$$

and since $K^0 \subset \subset E_\alpha$, we have

$$K_D(z) \geq \tilde{C}^{-1}K_{\tilde{D}}(z).$$
and hence

$$K_D(z) \geq (C_2)^2 K_{F_1}(z')$$

for some constant $C_2 > 0$. Now, by (6) and (7), it follows that

$$B_D(z; X) = \frac{M_D(z; X)}{\sqrt{K_D(z)}} \leq \frac{C_1}{C_2} ||X|| \sqrt{\frac{K_{F_2}(z')}{K_{F_2}(z')}}.$$  

Note that $z' \rightarrow \alpha^{-2}z'$ is a biholomorphic mapping from $F_1$ onto $F_\alpha$ and, therefore,

$$K_{F_1}(z') = \alpha^{-4k} K_{F_\alpha}(\alpha^{-2}z').$$

In view of (8) and (9), in order to finish (ii) we have to find a constant $C_3 > 0$ such that

$$K_{F_\alpha}(z') \leq C_3 K_{F_\alpha}(\alpha^{-2}z')$$

for any $z' \in H^0$ with $H^0 := \{tz' : z' \in H, \ 0 < t \leq 1\}$, where $H$ is the projection of $K$ into $\mathbb{C}^k$ (we can identify $K$ with $H$).

To do this, note first that $\gamma := \text{dist}(H, \partial F_\alpha) > 0$ since $K \subset \subset E_\alpha$. Fix $\tau \in (0, 1]$ and $z' \in H^0$, and denote by $T_{\tau z'}$ the translation that maps the origin in the point $\tau z'$. It is easy to check that

$$T_{\tau z'}(\bar{F}_\alpha \cap B_\alpha) \subset F_\alpha,$$

where $B_\alpha$ is the ball in $\mathbb{C}^k$ with center at the origin and radius $\gamma$. To prove (10), we will consider the following two cases:

Case I. $z' \in H^0 \setminus B_\alpha \subset \subset F_\alpha$: Then

$$K_{F_\alpha}(z') \leq \frac{m_1}{m_2} K_{F_\alpha}(\alpha^{-2}z'),$$

where $m_1 := \sup_{H^0 \setminus B_\alpha} K_{F_\alpha}$ and $m_2 := \inf K_{F_\alpha}$.

Case II. $z' \in H^0 \cap B_\alpha$: By Theorem 3 there exists a constant $\hat{C}_3 \geq 1$ such that $\hat{C}_3 K_{F_\alpha} \geq K_{F_\alpha \cap B_\alpha}$ on $F_\alpha \cap B_\alpha$. In particular,

$$\hat{C}_3 K_{F_\alpha}(\alpha^{-2}z') \geq K_{F_\alpha \cap B_\alpha}(\alpha^{-2}z').$$

On the other hand, by (11) with data $T := T_{1-\alpha^{-2}z'}$ it follows that

$$K_{F_\alpha \cap B_\alpha}(\alpha^{-2}z') = K_{T(F_\alpha \cap B_\alpha)}(z') \geq K_{F_\alpha}(z').$$

Now, (12), (13), and (14) imply that (10) holds for $C_3 := \max\{\frac{m_1}{m_2}, \hat{C}_3\}$. This completes the proofs of Theorem 4 and part (b) of Theorem 2.

Remark. The approximation (5) of the domain $D \cap (\mathbb{C}^k \times \Delta''_\alpha)$ by the domain $E_\alpha \times \Delta''_\alpha$ can be replaced by using the Oshawa-Takegoshi theorem with the data $D$ and $N$. 

\[\square\]
Finally, part (c) of Theorem 2 will be a consequence of the following two theorems.

**Theorem 5** \((\ref{thm:5})\). \textit{Let} \(D \subset \mathbb{C}^n\) \textit{be a pseudoconvex domain and let} \(U\) \textit{be an open neighborhood of a local (holomorphic) peak point} \(z_0 \in \partial D\). \textit{Then}
\[
\lim_{z \to z_0} \frac{K_D(z)}{K_{D \cap U}(z)} = 1
\]
\[\text{and}\]
\[
\lim_{z \to z_0} \frac{B_D(z;X)}{B_{D \cap U}(z;X)} = 1
\]
locally uniformly in \(X \in \mathbb{C}^n \setminus \{0\}\).

**Theorem 6.** \textit{Let} \(z_0\) \textit{be a boundary point of a bounded convex domain} \(D \subset \mathbb{C}^n\). \textit{Then the following conditions are equivalent:}

1. \(z_0\) is a (holomorphic) peak point;
2. \(z_0\) is the unique analytic curve in \(\bar{D}\) containing \(z_0\);
3. \(L(z_0) = \{0\}\).

Note that the only nontrivial implication is \((3) \Rightarrow (1)\). It is contained in \([8]\).

Now, part (c) of Theorem 2 is a consequence of this implication, Theorem 5, and part (b) of Theorem 2.

**Proof of Theorem 6.** The implication \((2) \Rightarrow (3)\) is trivial.

The implication \((1) \Rightarrow (2)\) easily follows by the maximum principle and the fact that there is a neighborhood \(U\) of \(z_0\) and a vector \(X \in \mathbb{C}^n\) such that \((\bar{D} \cap U) + (0,1]X \subset D\) (cf. \((11)\)).

Denote by \(A^0(D)\) the algebra of holomorphic functions on \(D\) which are continuous on \(\bar{D}\). Now, following \([8]\) we shall prove the implication \((3) \Rightarrow (1)\); namely, \((3)\) implies that \(z_0\) is a peak point with respect to \(A^0(D)\). This is equivalent to the fact (cf. \([3]\)) that the point mass at \(z_0\) is the unique element of the set \(A(z_0)\) of all representing measures for \(z_0\) with respect to \(A^0(D)\), i.e. \(\text{supp} \ \mu = \{z_0\}\) for any \(\mu \in A(z_0)\).

Let \(\mu \in A(z_0)\). Since \(D\) is convex, we may assume that \(z_0 = 0\) and \(D \subset \{z \in \mathbb{C}^n : \text{Re}(z_1) < 0\}\). Note that if \(a\) is a positive number such that \(a \inf_{z \in D} \text{Re}(z_1) > -1\) \((D\) is bounded), then the function \(f(z) = \exp(z_1 + az_1^2)\) belongs to \(A^0(D)\) and \(|f(z)| < 1\) for \(z \in \bar{D} \setminus \{z_1 = 0\}\). This easily implies (cf. \([3]\)) that \(\text{supp} \ \mu \subset D_1 := \partial D \cap \{z_1 = 0\}\). Since \(L(0) = 0\), the origin is a boundary point of the compact convex set \(D_1\). As above, we may assume that \(D_1 \subset \{z \in \mathbb{C}^n : \text{Re}(z_2) \leq 0\}\) \((z_2\) is independent of \(z_1)\) and then construct a function \(f_2 \in A^0(D)\) such that \(|f_2(z)| < 1\) for \(z \in D_1 \setminus \{z_2 = 0\}\). This implies that \(\text{supp} \ \mu \subset D_1 \cap \{z_2 = 0\}\). Repeating this argument we conclude that \(\text{supp} \ \mu = \{0\}\), which completes the proofs of Theorems 6 and 2. \(\square\)

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References


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