BEHAVIOR OF THE BERGMAN KERNEL
AND METRIC NEAR CONVEX BOUNDARY POINTS

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(Communicated by Mei-Chi Shaw)

ABSTRACT. The boundary behavior of the Bergman metric near a convex boundary point $z_0$ of a pseudoconvex domain $D \subset \mathbb{C}^n$ is studied. It turns out that the Bergman metric at points $z \in D$ in the direction of a fixed vector $X_0 \in \mathbb{C}^n$ tends to infinity, when $z$ is approaching $z_0$, if and only if the boundary of $D$ does not contain any analytic disc through $z_0$ in the direction of $X_0$.

For a domain $D \subset \mathbb{C}^n$ we denote by $L^2_h(D)$ the Hilbert space of all holomorphic functions $f$ that are square-integrable and by $\|f\|_D$ the $L^2$-norm of $f$. Let $K_D(z)$ be the restriction on the diagonal of the Bergman kernel function of $D$. It is well known (cf. [5]) that

$$K_D(z) = \sup\{|f(z)|^2 : f \in L^2_h(D), \|f\|_D \leq 1\}.$$ 

If $K_D(z) > 0$ for some point $z \in D$, then the Bergman metric $B_D(z; X), \ X \in \mathbb{C}^n$, is well defined and can be given by the equality

$$B_D(z; X) = \frac{M_D(z; X)}{\sqrt{K_D(z)}},$$

where

$$M_D(z; X) = \sup\{|f'(z)X| : f \in L^2_h(D), \|f\|_D = 1, f(z) = 0\}.$$ 

We say that a boundary point $z_0$ of a domain $D \subset \mathbb{C}^n$ is convex if there is a neighborhood $U$ of this point such that $D \cap U$ is convex.

In [4], Herbort proved the following

**Theorem 1.** Let $z_0$ be a convex boundary point of a bounded pseudoconvex domain $D \subset \mathbb{C}^n$ whose boundary contains no nontrivial germ of an analytic curve near $z_0$. Then

$$\lim_{z \to z_0} B_D(z; X) = \infty$$

for any $X \in \mathbb{C}^n \setminus \{0\}$.

Herbort’s proof is mainly based on Ohsawa’s $\bar{\partial}$-technique. The main purpose of this note is to generalize Theorem 1 using more elementary methods.
For a convex boundary point \(z_0\) of a domain \(D \subset \mathbb{C}^n\) we denote by \(L(z_0)\) the set of all \(X \in \mathbb{C}^n\) for which there exists a number \(\varepsilon_X > 0\) such that \(z_0 + \lambda X \in \partial D\) for all complex numbers \(\lambda, |\lambda| \leq \varepsilon_X\). Note that \(L(z_0)\) is a complex linear space.

Then our result is the following one.

**Theorem 2.** Let \(z_0\) be a convex boundary point of a bounded pseudoconvex domain \(D \subset \mathbb{C}^n\) and let \(X \in \mathbb{C}^n\). Then

(a) \(\lim \inf_{z \to z_0} K_D(z) \operatorname{dist}^2(z, \partial D) \in (0, \infty)\);

(b) \(\lim_{z \to z_0} B_D(z; X) = \infty\) if and only if \(X \notin L(z_0)\). Moreover, this limit is locally uniform in \(X \notin L(z_0)\);

(c) if \(L(z_0) = \{0\}\), then (a) and (b) are still true without the assumption that \(D\) is bounded.

**Proof of Theorem 2.** To prove (a) and (b) we will use the following localization theorem for the Bergman kernel and metric [2].

**Theorem 3.** Let \(D \subset \mathbb{C}^n\) be a bounded pseudoconvex domain and let \(V \subset \subset U\) be open neighborhoods of a point \(z_0 \in \partial D\). Then there exists a constant \(C \geq 1\) such that

\[
\tilde{C}^{-1} K_{D \cap U}(z) \leq K_D(z) \leq K_{D \cap U}(z),
\]
\[
\tilde{C}^{-1} B_{D \cap U}(z; X) \leq B_D(z; X) \leq C B_{D \cap U}(z; X)
\]

for any \(z \in D \cap V\) and any \(X \in \mathbb{C}^n\). (Here \(K_{D \cap U}(z)\) and \(B_{D \cap U}(z; \cdot)\) denote the Bergman kernel and metric of the connected component of \(D \cap U\) that contains \(z\).)

So, we may assume that \(D\) is convex.

To prove part (a) of Theorem 2, for any \(z \in D\) we choose a point \(\tilde{z} \in \partial D\) such that \(||z - \tilde{z}|| = \operatorname{dist}(z, \partial D)\). We denote by \(l\) the complex line through \(z\) and \(\tilde{z}\). By the Oshawa-Takegoshi extension theorem for \(L^2\)-holomorphic functions [2], it follows that there exists a constant \(C_1 > 0\) only depending on the diameter of \(D\) (not on \(l\)) such that

\[
(1) \quad K_D(z) \geq C_1 K_{D \cap l}(z).
\]

Since \(D \cap l\) is convex, it is contained in an open half-plane \(\Pi\) of the \(l\)-plane with \(\tilde{z} \in \partial \Pi\). Then

\[
(2) \quad K_{D \cap l}(z) \geq K_{\Pi}(z) = \frac{1}{4\pi \operatorname{dist}^2(z, \partial \Pi)}.
\]

Now, part (a) of Theorem 2 follows from the inequalities (1), (2) and the fact that \(\operatorname{dist}(z, \partial \Pi) \leq ||z - \tilde{z}|| = \operatorname{dist}(z, \partial D)\).

To prove part (b) of Theorem 2, we denote by \(N(z_0)\) the complex affine space through \(z_0\) that is orthogonal to \(L(z_0)\). Set \(E(z_0) = D \cap N(z_0)\). Note that \(E(z_0)\) is a nonempty convex set. So, part (b) of Theorem 2 will be a consequence of the following:

**Theorem 4.** Let \(z_0\) be a boundary point of a bounded convex domain \(D \subset \mathbb{C}^n\).

Then:

(i) \(\lim_{z \to z_0} B_D(z; X) = \infty\) locally uniformly in \(X \notin L(z_0)\);
(ii) for any compact set $K \subseteq E(z_0)$ there exists a constant $C > 0$ such that

$$B_D(z; X) \leq C||X||, \quad z \in K^0, \ X \in L(z_0),$$

where $K^0 := \{z_0 + tz : \ z \in K, \ 0 < t \leq 1\}$ is the cone generated by $K$.

Proof of Theorem 4. To prove (i) we will use the well-known fact that the Carathéodory metric $C_D(z; X)$ of $D$ does not exceed $B_D(z; X)$. On the other hand, we have the following simple geometric inequality [1]:

$$C_D(z; X) \geq \frac{1}{2d(z; X)},$$

where $d(z; X)$ denotes the distance from $z$ to the boundary of $D$ in the $X$-direction, i.e., $d(z; X) := \sup\{r : z + \lambda X \in D, \ \lambda \in \mathbb{C}, \ |\lambda| < r\}$. So, if we assume that (i) does not hold, then we may find a number $a > 0$ and sequences $D \ni (z_j), \ z_j \to z_0, \ \mathbb{C}^n \ni (X_j), \ X_j \to X \notin L(z_0)$, such that $B_D(z_j; X_j) \leq \frac{1}{2a}$. Hence $d(z_j; X_j) \geq a$ which implies that for $|\lambda| \leq a$ the points $z_0 + \lambda X$ belong to $D$ and, in view of convexity, they belong to $\partial D$. This means that $X \in L(z_0)$, a contradiction.

To prove part (ii) of Theorem 4, we may assume that $z_0 = 0$ and $L := L(0) = \{z \in \mathbb{C}^n : z_1 = \ldots = z_k = 0\}$ for some $k < n$. Then $N := N(0) = \{z \in \mathbb{C}^n : z_{k+1} = \ldots = z_n = 0\}$. From now on we will write any point $z \in \mathbb{C}^n$ in the form $z = (z', z'')$, $z' \in \mathbb{C}^k$, $z'' \in \mathbb{C}^{n-k}$. Note that $L \subseteq \partial D$ near 0, i.e., there exists a $c > 0$ such that

$$\{0'\} \times \Delta''_c \subseteq \partial D,$$

where $\Delta''_c \subseteq \mathbb{C}^{n-k}$ is the polydisc with center at the origin and radius $c$. Since $K \subseteq E := E(0)$ and since $E$ is convex, there exists an $\alpha > 1$ such that $K \subseteq E_{\alpha}$, where $E_{\alpha} := \{z : az \in E\}$. Note that $K^0 \subseteq E_{\alpha}$. Using (3), the equality

$$(z', z'') = \frac{1}{\alpha}(\alpha z', 0'') + (1 - \frac{1}{\alpha})(0', (1 - \frac{1}{\alpha})^{-1}z''),$$

and the convexity of $D$, it follows that

$$F_{\alpha} \times \Delta''_c \subseteq D,$$

where $\varepsilon := c(1 - \frac{1}{\alpha})$ and where $F_{\alpha}$ is the projection of $E_{\alpha}$ in $\mathbb{C}^k$ (we can identify $E_{\alpha}$ with $F_{\alpha}$). For $\delta := c(\alpha - 1)$ we obtain in the same way that

$$\hat{D} := D \cap (\mathbb{C}^k \times \Delta''_\delta) \subseteq F_{\frac{\alpha}{\delta}} \times \Delta''_\delta.$$

Now, let $(z, X) \in K^0 \times L$. Note that $z = (z', 0'')$ and $X = (0', X'')$. Then, using (4) and the product properties of the Bergman kernel and metric, we have

$$M_D(z; X) \leq M_{F_{\alpha} \times \Delta''_\delta}(z; X) = M_\Delta''(0''; X'') \sqrt{K_{F_{\alpha}}(z')} \leq C_1||X||\sqrt{K_{F_{\alpha}}(z')}$$

for some constant $C_1 > 0$. On the other hand, since $K^0 \subseteq \mathbb{C}^k \times \Delta''_\delta$, by virtue of Theorem 3 there exists a constant $\tilde{C} \geq 1$ such that

$$K_D(z) \geq \tilde{C}^{-1}K_{\hat{D}}(z).$$

Moreover, in view of (5), we have

$$K_D(z) \geq K_{F_{\frac{\alpha}{\delta}}} (z') K_\Delta''(0'').$$
and hence
\begin{equation}
K_D(z) \geq (C_2)^2 K_{F_{\mathbf{1}}}(z')
\end{equation}
for some constant $C_2 > 0$. Now, by (6) and (7), it follows that
\begin{equation}
B_D(z;X) = \frac{M_D(z;X)}{\sqrt{K_D(z)}} \leq \frac{C_1}{C_2} \|X\| \sqrt{\frac{K_{F_{\mathbf{1}}}(z')}}{K_{F_{\mathbf{1}}}(z')}.
\end{equation}
Note that $z' \rightarrow \alpha^{-2}z'$ is a biholomorphic mapping from $F_{\mathbf{1}}$ onto $F_\alpha$ and, therefore,
\begin{equation}
K_{F_{\mathbf{1}}}(z') = \alpha^{-4k} K_{F_\alpha}(\alpha^{-2}z').
\end{equation}
In view of (8) and (9), in order to finish (ii) we have to find a constant $C_3 > 0$ such that
\begin{equation}
K_{F_{\alpha}}(z') \leq C_3 K_{F_{\alpha}}(\alpha^{-2}z')
\end{equation}
for any $z' \in H^0$ with $H^0 := \{tz' : z' \in H, \ 0 < t \leq 1\}$, where $H$ is the projection of $K$ into $\mathbb{C}^k$ (we can identify $K$ with $H$).

To do this, note first that $\gamma := \text{dist}(H, \partial F_{\alpha}) > 0$ since $K \subset E_{\alpha}$. Fix $\tau \in (0,1]$ and $z' \in H^0$, and denote by $T_{\tau,z'}$ the translation that maps the origin in the point $\tau z'$. It is easy to check that
\begin{equation}
T_{\tau,z'}(\bar{F}_{\alpha} \cap B_{\gamma}) \subset F_{\alpha},
\end{equation}
where $B_{\gamma}$ is the ball in $\mathbb{C}^k$ with center at the origin and radius $\gamma$. To prove (10), we will consider the following two cases:

Case I. $z' \in H^0 \setminus B_{2\gamma} \subset F_{\alpha}$: Then
\begin{equation}
K_{F_{\alpha}}(z') \leq \frac{m_1}{m_2} K_{F_{\alpha}}(\alpha^{-2}z'),
\end{equation}
where $m_1 := \sup_{H^0 \setminus B_{2\gamma}} K_{F_{\alpha}}$ and $m_2 := \inf K_{F_{\alpha}}$.

Case II. $z' \in H^0 \cap B_{2\gamma}$: By Theorem 3 there exists a constant $\tilde{C}_3 \geq 1$ such that $\tilde{C}_3 K_{F_{\alpha}} \geq K_{F_{\alpha} \cap B_{\gamma}}$ on $F_{\alpha} \cap B_{2\gamma}$. In particular,
\begin{equation}
\tilde{C}_3 K_{F_{\alpha}}(\alpha^{-2}z') \geq K_{F_{\alpha} \cap B_{\gamma}}(\alpha^{-2}z').
\end{equation}
On the other hand, by (11) with data $T := T_{1-\alpha,-z'}$ it follows that
\begin{equation}
K_{F_{\alpha} \cap B_{\gamma}}(\alpha^{-2}z') = K_{T(F_{\alpha} \cap B_{\gamma})}(z') \geq K_{F_{\alpha}}(z').
\end{equation}
Now, (12), (13), and (14) imply that (10) holds for $C_3 := \max\{\frac{m_1}{m_2}, \tilde{C}_3\}$. This completes the proofs of Theorem 4 and part (b) of Theorem 2. \qed

Remark. The approximation (5) of the domain $D \cap (\mathbb{C}^k \times \Delta^0_N)$ by the domain $E_{\mathbf{1}} \times \Delta^0_N$ can be replaced by using the Oshawa-Takegoshi theorem \[\] with the data $D$ and $N$. 

Finally, part (c) of Theorem 2 will be a consequence of the following two theorems.

**Theorem 5.** Let \( D \subset \mathbb{C}^n \) be a pseudoconvex domain and let \( U \) be an open neighborhood of a local (holomorphic) peak point \( z_0 \in \partial D \). Then

\[
\lim_{z \to z_0} \frac{K_D(z)}{K_{D\cap U}(z)} = 1
\]

and

\[
\lim_{z \to z_0} \frac{B_D(z; X)}{B_{D\cup U}(z; X)} = 1
\]

locally uniformly in \( X \in \mathbb{C}^n \setminus \{0\} \).

**Theorem 6.** Let \( z_0 \) be a boundary point of a bounded convex domain \( D \subset \mathbb{C}^n \). Then the following conditions are equivalent:

1. \( z_0 \) is a (holomorphic) peak point;
2. \( z_0 \) is the unique analytic curve in \( \bar{D} \) containing \( z_0 \);
3. \( L(z_0) = \{0\} \).

Note that the only nontrivial implication is (3) \( \implies \) (1). It is contained in [8].

Now, part (c) of Theorem 2 is a consequence of this implication, Theorem 5, and part (b) of Theorem 2.

**Proof of Theorem 6.** The implication (2) \( \implies \) (3) is trivial.

The implication (1) \( \implies \) (2) easily follows by the maximum principle and the fact that there are a neighborhood \( U \) of \( z_0 \) and a vector \( X \in \mathbb{C}^n \) such that \( (\bar{D} \cap U) \cup (0,1]X \subset D \) (cf. (11)).

Denote by \( A^0(D) \) the algebra of holomorphic functions on \( D \) which are continuous on \( \bar{D} \). Now, following [8] we shall prove the implication (3) \( \implies \) (1); namely, (3) implies that \( z_0 \) is a peak point with respect to \( A^0(D) \). This is equivalent to the fact (cf. [3]) that the point mass at \( z_0 \) is the unique element of the set \( A(z_0) \) of all representing measures for \( z_0 \) with respect to \( A^0(D) \), i.e. \( \text{supp} \ \mu = \{z_0\} \) for any \( \mu \in A(z_0) \).

Let \( \mu \in A(z_0) \). Since \( D \) is convex, we may assume that \( z_0 = 0 \) and \( D \subset \{z \in \mathbb{C}^n : \text{Re}(z_1) < 0\} \). Note that if \( a \) is a positive number such that \( a \inf_{z \in D} \text{Re}(z_1) > -1 \) (\( D \) is bounded), then the function \( f_1(z) = \exp(z_1 + az_2^2) \) belongs to \( A^0(D) \) and \( |f_1(z)| < 1 \) for \( z \in \bar{D} \setminus \{z_1 = 0\} \). This easily implies (cf. [3]) that \( \text{supp} \ \mu \subset D_1 := \partial D \cap \{z_1 = 0\} \). Since \( L(0) = 0 \), the origin is a boundary point of the compact convex set \( D_1 \). As above, we may assume that \( D_1 \subset \{z \in \mathbb{C}^n : \text{Re}(z_2) \leq 0\} \) (\( z_2 \) is independent of \( z_1 \)) and then construct a function \( f_2 \in A^0(D) \) such that \( |f_2(z)| < 1 \) for \( z \in D_1 \setminus \{z_2 = 0\} \). This implies that \( \text{supp} \ \mu \subset D_1 \cap \{z_2 = 0\} \). Repeating this argument we conclude that \( \text{supp} \ \mu = \{0\} \), which completes the proofs of Theorems 6 and 2.

**Acknowledgments**

This paper was written during the stay of the first author at the University of Oldenburg, supported by a grant from the DAAD. He thanks both institutions, the DAAD and the Mathematical Department of the University of Oldenburg.
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