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THE IDEAL PROPERTY IN CROSSED PRODUCTS

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ABSTRACT. We describe the lattice of the ideals generated by projections and prove a characterization of the ideal property for "large" classes of crossed products of commutative C^*-algebras by discrete, amenable groups; some applications are also given. We prove that the crossed product of a C^*-algebra with the ideal property by a group with the ideal property may fail to have the ideal property; this answers a question of Shuzhou Wang.

1. Introduction

A C^*-algebra is said to have the ideal property if its ideals are generated (as ideals) by their projections (in this paper, by an ideal we shall mean a closed, two-sided ideal).

The class of the C^*-algebras with the ideal property is important because it includes the C^*-algebras of real rank zero ([BPe]) and also the simple, unital C^*-algebras. The ideal property has been studied in [Pa1]–[Pa9], [PaR], [S]. On the other hand, the crossed products ([Pe]) represent a valuable source of interesting C^*-algebras, related to the domain of the dynamical systems. Many important nuclear, simple, unital, separable C^*-algebras generated by a subset of elements or many AH algebras are in fact crossed products of commutative, separable C^*-algebras by discrete, countable, amenable groups (e.g. the irrational rotation algebras, the Bunce-Deddens algebras and others (see [Bl])). It became clear that the nuclear crossed products with the ideal property will play a central role in Elliott's project of the classification of the nuclear, separable C^*-algebras by invariants including K-theory ([Ell]). Therefore, it is clearly important to know when a crossed product has the ideal property. Our main interest is in the case when the group is discrete and it acts on a commutative C^*-algebra. For a discrete, countable, amenable group G and a commutative, separable C^*-algebra A let α be an action of G on A which induces a free action of G on the spectrum of A. Based mainly on results in [EH], [GL] and [Z], we identify the lattice of the ideals generated by projections of $A \rtimes_{\alpha} G$ with a lattice of α-invariant ideals of A (see Theorem 2.1 below), which allows us to characterize the ideal property for $A \rtimes_{\alpha} G$ (see Corollary 2.2 below). Using this result, we deduce that if in addition the spectrum of the C^*-algebra A is zero dimensional, or if $A = C(X)$, where X is a compact, Hausdorff topological space and every

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closed, α-invariant subset of X is also open, it follows that $A \underset{\alpha}{\rtimes} G$ has the ideal property (see Corollary 2.3 and Corollary 2.4 below). In [PaR] we proved, jointly with M. Rørdam, that the minimal tensor product of two C^*-algebras each having the ideal property does not necessarily have the ideal property. On the other hand, if (A, G, α) is a C^*-dynamical system, then one may visualize the crossed product $A \underset{\alpha}{\rtimes} G$ as a skew tensor product between A and $C^*(G)$, since if α is trivial, then $A \underset{\alpha}{\rtimes} G \cong A \otimes_{\max} C^*(G)$. We answer a question on Shuzhou Wang, proving that there are crossed products of C^*-algebras with the ideal property by groups G with the ideal property (i.e., $C^*(G)$ has the ideal property; see Definition 2.5 below) that do not have the ideal property (see Theorem 2.6 below).

If A is a C^*-algebra, the fact that I is an ideal (closed, two-sided) of A will be denoted by $I \lhd A$. The projections of a C^*-algebra A will be denoted by $\mathcal{P}(A)$. Let G be a locally compact group. G is amenable if there is a left invariant mean on $L^\infty(G)$ (see [Pe]). Note that if G is abelian or if it is compact, then G is amenable ([Pe]). For the definition of $C^*(G)$, the group C^*-algebra of a locally compact group G, see [Pe, 7.1.5]. Let (G, X) be a topological transformation group. Recall that G is said to act freely on X (or we say that the action of G on X is a free action) if for every $x \in X$ and for every $g \in G$, g different from the neutral element of G, we have that $gx \neq x$. Recall also that a C^*-dynamical system is a triple (A, G, α) consisting of a C^*-algebra A, a locally compact group G and a continuous homomorphism α of G into the group $\mathrm{Aut}(A)$ of automorphisms (i.e., $*$-automorphisms) of A equipped with the topology of pointwise convergence.

Let (A, G, α) be a C^*-dynamical system, where G is a discrete group with neutral element e. Consider an injective $*$-representation $\pi : A \to \mathcal{B}(H)$. We shall identify A with its image by π and $A \underset{\alpha,\ r}{\rtimes} G$ with the norm closure of $(\tilde{\pi} \times U)(l^1(G, A))$ in $\mathcal{B}(l^2(G, H))$, where

$$(\tilde{\pi}(a)x)(g) = \pi(\alpha_{g^{-1}}(a))x(g),$$
$$(U_h x)(g) = x(h^{-1}g)$$

for each $a \in A,\ g,\ h \in G$ and $x \in l^2(G, H)$ (see [Pe, Theorem 7.7.5]). We will denote by $E : A \underset{\alpha,\ r}{\rtimes} G \to A$ the canonical conditional expectation given by

$$E(aU_g) = \begin{cases} a, \text{ if } g = e, \\ 0, \text{ if } g \neq e \end{cases}$$

where $a \in A,\ g \in G$. If $M \subseteq A \underset{\alpha}{\rtimes} G$, we shall denote by $\langle M \rangle$ the (closed and two-sided) ideal of $A \underset{\alpha}{\rtimes} G$ generated by M. Suppose now that $A = C(X)$, where X is a compact, Hausdorff topological space. A subset F of X is called α-invariant if it is invariant for the action of G on X canonically induced by α.

As usual, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $\mathcal{K}$ denotes the C^*-algebra of compact operators on $l^2(\mathbb{N})$.

An AF algebra is an inductive limit C^*-algebra of the form $\varinjlim A_n$, where for each $n \in \mathbb{N}$, A_n is a finite-dimensional C^*-algebra ([Br]).

2. Results

Theorem 2.1. *Let (A, G, α) be a C^*-dynamical system such that the C^*-algebra A is commutative and separable and the group G is discrete, countable and amenable. Suppose that α induces a free action of G on the spectrum of A. Then, the map*

$$\theta : \{I \lhd A \mid I \text{ is } \alpha\text{-invariant}\} \to \{J \mid J \lhd A \underset{\alpha}{\rtimes} G\}$$

given by

$$\theta(I) := I \underset{\alpha}{\rtimes} G$$

is a lattice isomorphism and it induces canonically a lattice isomorphism

$$\{I \lhd A \mid I \text{ is } \alpha\text{-invariant and } I \subseteq \langle \mathcal{P}(I \underset{\alpha}{\rtimes} G)\rangle\}$$
$$\cong \{J \lhd A \underset{\alpha}{\rtimes} G \mid J \text{ is generated by projections}\}.$$

Proof. For every $x \in A \underset{\alpha,\, r}{\rtimes} G$ and every $g \in G$, define $x_g := E(xU_g^*)$. For an arbitrary $M \subseteq A$, denote $\mathcal{I}(M) = \{x \in A \underset{\alpha,\, r}{\rtimes} G \mid x_g \in M \text{ for every } g \in G\}$ (see [Z, 4.15]). Note that since G is amenable, $A \underset{\alpha,\, r}{\rtimes} G = A \underset{\alpha}{\rtimes} G$ ([Z, Théorème 5.1]). Since A is separable and G is a countable, amenable group acting freely on the spectrum of A, then it follows by [EH, Corollary 5.16] (as in [Z, Théorème 5.15]) that

$$\Phi : \{I \lhd A \mid I \text{ is } \alpha\text{-invariant}\} \to \{J \mid J \lhd A \underset{\alpha}{\rtimes} G\},$$

defined by

$$\Phi(I) := \mathcal{I}(I)$$

for each α-invariant ideal I of A, is an order isomorphism. Since G is amenable, it follows by [Z, 5.10] that for each α-invariant ideal I of A, we have that

$$\mathcal{I}(I) = \langle I \rangle.$$

But, by [GL, Proposition 3.11 (ii)] it follows that $\langle I \rangle = I \underset{\alpha}{\rtimes} G$ for every α-invariant ideal I of A. Again using the fact that G is amenable, by a result of E.C. Gootman and A.J. Lazar ([GL, Theorem 3.4]) it follows that an ideal J of $A \underset{\alpha}{\rtimes} G$ is $\hat{\alpha}$-invariant if and only if $J = I \underset{\alpha}{\rtimes} G$, for some unique α-invariant ideal I of A, where $\hat{\alpha}$ is the dual action of $\hat{G}$ on $A \underset{\alpha}{\rtimes} G$ ([T]). Hence, there is an order isomorphism

$$\Psi : \{I \lhd A \mid I \text{ is } \alpha\text{-invariant}\} \to \{J \lhd A \underset{\alpha}{\rtimes} G \mid J \text{ is } \hat{\alpha}\text{-invariant}\}$$

given by

$$\Psi(I) := I \underset{\alpha}{\rtimes} G$$

for each α-invariant ideal I of A.

Now, observe that by the above facts, it follows that the order isomorphisms Φ and Ψ, which have the same domains of definition, agree on their domain of definition and hence the range of Φ is equal to the range of Ψ. But this together with the above discussion imply that θ is an order isomorphism. The fact that $\{I \lhd A \mid I \text{ is } \alpha\text{-invariant}\}$ and $\{J \mid J \lhd A \underset{\alpha}{\rtimes} G\}$ are lattices and θ is a lattice isomorphism follows from [Pa4, Remark 4.3].

Now, suppose that $I \lhd A$, I is α-invariant and $I \subseteq \langle \mathcal{P}(I \rtimes_\alpha G)\rangle$. This implies that $\theta(I) = I \rtimes_\alpha G \subseteq \langle \mathcal{P}(I \rtimes_\alpha G)\rangle = \langle \mathcal{P}(\theta(I))\rangle$, from which we conclude that $\theta(I)$ is generated by its projections. On the other hand, if $I \lhd A$, I is α-invariant and $\theta(I) = I \rtimes_\alpha G$ is generated by projections, then, clearly $I \subseteq I \rtimes_\alpha G = \langle \mathcal{P}(I \rtimes_\alpha G)\rangle$. These facts, together with [Pa4, Remark 4.3], imply that θ induces canonically a lattice isomorphism

$$\{I \lhd A \mid I \text{ is } \alpha\text{-invariant and } I \subseteq \langle \mathcal{P}(I \rtimes_\alpha G)\rangle\}$$
$$\cong \{J \lhd A \rtimes_\alpha G \mid J \text{ is generated by projections}\}.$$

Corollary 2.2. *Let A, G and α be as in the hypothesis of Theorem* 2.1*. Then, the following are equivalent:*

(a) *$A \rtimes_\alpha G$ has the ideal property.*
(b) *For every α-invariant ideal I of A, we have that $I \subseteq \langle \mathcal{P}(I \rtimes_\alpha G)\rangle$.*

Corollary 2.3. *Let A, G and α be as in the hypothesis of Theorem* 2.1*. Moreover, suppose that the spectrum of A has dimension zero. Then, $A \rtimes_\alpha G$ has the ideal property.*

Proof. Since A is a commutative, separable C^*-algebra with zero-dimensional spectrum, then A is an AF algebra. Hence, A has the ideal property. Now, the result follows by also using Corollary 2.2.

Corollary 2.4. *Let A, G and α be as in the hypothesis of Theorem* 2.1*. Suppose moreover that $A = C(X)$, where X is a compact, Hausdorff topological space, and that every closed α-invariant subset of X is also open. Then, $A \rtimes_\alpha G$ has the ideal property.*

Proof. Let I be an α-invariant ideal of A. Then, there is a closed, α-invariant subset F of X such that $I = \{f \in C(X) \mid f_{|F} = 0\}$. Since $X \setminus F$ is closed (because, by hypothesis, F is open) it follows that the map $A \ni f \longmapsto f_{|F} \oplus f_{|X\setminus F} \in C(F) \oplus C(X \setminus F)$ is an isomorphism, which implies that $I \cong C(X \setminus F)$ and hence, I being unital, is generated by projections. Now, the result follows by also using Corollary 2.2.

Definition 2.5. Let G be a locally compact group. We say that G has the ideal property if its group C^*-algebra $C^*(G)$ has the ideal property.

Question (Shuzhou Wang). Let (A, G, α) be a C^*-dynamical system such that A and G have the ideal property. Does the crossed product $A \rtimes_\alpha G$ (always) have the ideal property?

The answer to the above question is "no", even in the "separable" case, as it follows from the following:

Theorem 2.6. *There are C^*-dynamical systems (A, G, α) with A a separable, simple, unital C^*-algebra (and hence, with the ideal property) and with G a separable, commutative topological group with the ideal property such that $A \rtimes_\alpha G$ does not have the ideal property.*

The proof of the above theorem will use the following proposition:

Proposition 2.7. *Let G be a compact, abelian group. Then, G has the ideal property.*

Proof. The hypothesis implies that $\hat{G}$ (the dual group of G) is a discrete, abelian group. We have that $C^*(G) \cong C_0(\hat{G})$ (see [Pe, Proposition 7.1.6]). On the other hand, since $\hat{G}$ is discrete, then dim $(\hat{G}) = 0$ and hence $C_0(\hat{G})$ is an approximately finite-dimensional C^*-algebra, i.e., it is the inductive limit of a net of finite-dimensional C^*-algebras. Since a finite-dimensional C^*-algebra has the ideal property, this implies that $C_0(\hat{G})$ has the ideal property (as an inductive limit of C^*-algebras with the ideal property ([Pa4, Proposition 2.3])) and hence, by the above discussion, it follows that $C^*(G)$ has the ideal property, which ends the proof.

Proof of Theorem 2.6. Let $(B, \mathbb{Z}, \alpha)$ be a C^*-dynamical system with $B = C(X)$, where X is an infinite, compact, connected, metrizable topological space and such that $A := B \rtimes_{\alpha} \mathbb{Z}$ is a simple C^*-algebra. (Clearly, A is unital and separable and hence with the ideal property, since, obviously, every simple, unital C^*-algebra has the ideal property.) Note that A is simple if (and only if) the homeomorphism of X induced by $\alpha(1)$ is minimal (see e.g. [D, Theorem VIII. 3.9]). (Remark that the irrational rotation algebras are examples of such kind of C^*-algebras A ([Bl]).)

Now, by Takai duality ([T]), we have that

$$A \rtimes_{\hat{\alpha}} \mathbb{T} \cong B \otimes \mathcal{K}$$

where $\hat{\alpha} : \mathbb{T} = \hat{\mathbb{Z}} \to \mathrm{Aut}(A)$ is the dual action ([T]). Our claim is that $B \otimes \mathcal{K}$ does not have the ideal property and this fact together with the above isomorphism will imply that $A \rtimes_{\hat{\alpha}} \mathbb{T}$ does not have the ideal property either. Indeed, let $\phi \neq F = \overline{F} \subsetneqq X$ and let $I := \{f \in B \otimes \mathcal{K} = C(X, \mathcal{K}) \mid f_{|F} = 0\}$. Then, clearly $I \lhd B \otimes \mathcal{K}$ and $\mathcal{P}(I) = \{0\}$, since X is connected, but I is not generated by its projections (because, obviously, $I \neq \{0\}$). Hence, $B \otimes \mathcal{K}$ does not have the ideal property. The proof of the theorem is over if we observe that since $\mathbb{T}$ is a compact, abelian group, the separable group $G = \mathbb{T}$ has the ideal property by Proposition 2.7.

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