

THE IDEAL PROPERTY IN CROSSED PRODUCTS

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ABSTRACT. We describe the lattice of the ideals generated by projections and prove a characterization of the ideal property for “large” classes of crossed products of commutative C^* -algebras by discrete, amenable groups; some applications are also given. We prove that the crossed product of a C^* -algebra with the ideal property by a group with the ideal property may fail to have the ideal property; this answers a question of Shuzhou Wang.

1. INTRODUCTION

A C^* -algebra is said to have the ideal property if its ideals are generated (as ideals) by their projections (in this paper, by an ideal we shall mean a closed, two-sided ideal).

The class of the C^* -algebras with the ideal property is important because it includes the C^* -algebras of real rank zero ([BPe]) and also the simple, unital C^* -algebras. The ideal property has been studied in [Pa1]–[Pa9], [PaR], [S]. On the other hand, the crossed products ([Pe]) represent a valuable source of interesting C^* -algebras, related to the domain of the dynamical systems. Many important nuclear, simple, unital, separable C^* -algebras generated by a subset of elements or many AH algebras are in fact crossed products of commutative, separable C^* -algebras by discrete, countable, amenable groups (e.g. the irrational rotation algebras, the Bunce-Deddens algebras and others (see [Bl])). It became clear that the nuclear crossed products with the ideal property will play a central role in Elliott’s project of the classification of the nuclear, separable C^* -algebras by invariants including K -theory ([Ell]). Therefore, it is clearly important to know when a crossed product has the ideal property. Our main interest is in the case when the group is discrete and it acts on a commutative C^* -algebra. For a discrete, countable, amenable group G and a commutative, separable C^* -algebra A let α be an action of G on A which induces a free action of G on the spectrum of A . Based mainly on results in [EH], [GL] and [Z], we identify the lattice of the ideals generated by projections of $A \rtimes_{\alpha} G$ with a lattice of α -invariant ideals of A (see Theorem 2.1 below), which allows us to characterize the ideal property for $A \rtimes_{\alpha} G$ (see Corollary 2.2 below). Using this result, we deduce that if in addition the spectrum of the C^* -algebra A is zero dimensional, or if $A = C(X)$, where X is a compact, Hausdorff topological space and every

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closed, α -invariant subset of X is also open, it follows that $A \rtimes_{\alpha} G$ has the ideal property (see Corollary 2.3 and Corollary 2.4 below). In [PaR] we proved, jointly with M. Rørdam, that the minimal tensor product of two C^* -algebras each having the ideal property does not necessarily have the ideal property. On the other hand, if (A, G, α) is a C^* -dynamical system, then one may visualize the crossed product $A \rtimes_{\alpha} G$ as a skew tensor product between A and $C^*(G)$, since if α is trivial, then $A \rtimes_{\alpha} G \cong A \otimes_{\max} C^*(G)$. We answer a question on Shuzhou Wang, proving that there are crossed products of C^* -algebras with the ideal property by groups G with the ideal property (i.e., $C^*(G)$ has the ideal property; see Definition 2.5 below) that do not have the ideal property (see Theorem 2.6 below).

If A is a C^* -algebra, the fact that I is an ideal (closed, two-sided) of A will be denoted by $I \triangleleft A$. The projections of a C^* -algebra A will be denoted by $\mathcal{P}(A)$. Let G be a locally compact group. G is amenable if there is a left invariant mean on $L^{\infty}(G)$ (see [Pe]). Note that if G is abelian or if it is compact, then G is amenable ([Pe]). For the definition of $C^*(G)$, the group C^* -algebra of a locally compact group G , see [Pe, 7.1.5]. Let (G, X) be a topological transformation group. Recall that G is said to act freely on X (or we say that the action of G on X is a free action) if for every $x \in X$ and for every $g \in G$, g different from the neutral element of G , we have that $gx \neq x$. Recall also that a C^* -dynamical system is a triple (A, G, α) consisting of a C^* -algebra A , a locally compact group G and a continuous homomorphism α of G into the group $\text{Aut}(A)$ of automorphisms (i.e., $*$ -automorphisms) of A equipped with the topology of pointwise convergence.

Let (A, G, α) be a C^* -dynamical system, where G is a discrete group with neutral element e . Consider an injective $*$ -representation $\pi : A \rightarrow \mathcal{B}(H)$. We shall identify A with its image by π and $A \rtimes_{\alpha, \tau} G$ with the norm closure of $(\tilde{\pi} \times U)(l^1(G, A))$ in $\mathcal{B}(l^2(G, H))$, where

$$\begin{aligned} (\tilde{\pi}(a)x)(g) &= \pi(\alpha_{g^{-1}}(a))x(g), \\ (U_h x)(g) &= x(h^{-1}g) \end{aligned}$$

for each $a \in A$, $g, h \in G$ and $x \in l^2(G, H)$ (see [Pe, Theorem 7.7.5]). We will denote by $E : A \rtimes_{\alpha, \tau} G \rightarrow A$ the canonical conditional expectation given by

$$E(aU_g) = \begin{cases} a, & \text{if } g = e, \\ 0, & \text{if } g \neq e \end{cases}$$

where $a \in A$, $g \in G$. If $M \subseteq A \rtimes_{\alpha} G$, we shall denote by $\langle M \rangle$ the (closed and two-sided) ideal of $A \rtimes_{\alpha} G$ generated by M . Suppose now that $A = C(X)$, where X is a compact, Hausdorff topological space. A subset F of X is called α -invariant if it is invariant for the action of G on X canonically induced by α .

As usual, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and \mathcal{K} denotes the C^* -algebra of compact operators on $l^2(\mathbb{N})$.

An AF algebra is an inductive limit C^* -algebra of the form $\varinjlim A_n$, where for each $n \in \mathbb{N}$, A_n is a finite-dimensional C^* -algebra ([Br]).

2. RESULTS

Theorem 2.1. *Let (A, G, α) be a C^* -dynamical system such that the C^* -algebra A is commutative and separable and the group G is discrete, countable and amenable. Suppose that α induces a free action of G on the spectrum of A . Then, the map*

$$\theta : \{I \triangleleft A \mid I \text{ is } \alpha\text{-invariant}\} \rightarrow \{J \mid J \triangleleft A \rtimes_{\alpha} G\}$$

given by

$$\theta(I) := I \rtimes_{\alpha} G$$

is a lattice isomorphism and it induces canonically a lattice isomorphism

$$\begin{aligned} & \{I \triangleleft A \mid I \text{ is } \alpha\text{-invariant and } I \subseteq \langle \mathcal{P}(I \rtimes_{\alpha} G) \rangle\} \\ & \cong \{J \triangleleft A \rtimes_{\alpha} G \mid J \text{ is generated by projections}\}. \end{aligned}$$

Proof. For every $x \in A \rtimes_{\alpha, r} G$ and every $g \in G$, define $x_g := E(xU_g^*)$. For an arbitrary $M \subseteq A$, denote $\mathcal{I}(M) = \{x \in A \rtimes_{\alpha, r} G \mid x_g \in M \text{ for every } g \in G\}$ (see [Z, 4.15]). Note that since G is amenable, $A \rtimes_{\alpha, r} G = A \rtimes_{\alpha} G$ ([Z, Théorème 5.1]). Since A is separable and G is a countable, amenable group acting freely on the spectrum of A , then it follows by [EH, Corollary 5.16] (as in [Z, Théorème 5.15]) that

$$\Phi : \{I \triangleleft A \mid I \text{ is } \alpha\text{-invariant}\} \rightarrow \{J \mid J \triangleleft A \rtimes_{\alpha} G\},$$

defined by

$$\Phi(I) := \mathcal{I}(I)$$

for each α -invariant ideal I of A , is an order isomorphism. Since G is amenable, it follows by [Z, 5.10] that for each α -invariant ideal I of A , we have that

$$\mathcal{I}(I) = \langle I \rangle.$$

But, by [GL, Proposition 3.11 (ii)] it follows that $\langle I \rangle = I \rtimes_{\alpha} G$ for every α -invariant ideal I of A . Again using the fact that G is amenable, by a result of E.C. Gootman and A.J. Lazar ([GL, Theorem 3.4]) it follows that an ideal J of $A \rtimes_{\alpha} G$ is $\hat{\alpha}$ -invariant if and only if $J = I \rtimes_{\alpha} G$, for some unique α -invariant ideal I of A , where $\hat{\alpha}$ is the dual action of \hat{G} on $A \rtimes_{\alpha} G$ ([T]). Hence, there is an order isomorphism

$$\Psi : \{I \triangleleft A \mid I \text{ is } \alpha\text{-invariant}\} \rightarrow \{J \triangleleft A \rtimes_{\alpha} G \mid J \text{ is } \hat{\alpha}\text{-invariant}\}$$

given by

$$\Psi(I) := I \rtimes_{\alpha} G$$

for each α -invariant ideal I of A .

Now, observe that by the above facts, it follows that the order isomorphisms Φ and Ψ , which have the same domains of definition, agree on their domain of definition and hence the range of Φ is equal to the range of Ψ . But this together with the above discussion imply that θ is an order isomorphism. The fact that $\{I \triangleleft A \mid I \text{ is } \alpha\text{-invariant}\}$ and $\{J \mid J \triangleleft A \rtimes_{\alpha} G\}$ are lattices and θ is a lattice isomorphism follows from [Pa4, Remark 4.3].

Now, suppose that $I \triangleleft A$, I is α -invariant and $I \subseteq \langle \mathcal{P}(I \rtimes_{\alpha} G) \rangle$. This implies that $\theta(I) = I \rtimes_{\alpha} G \subseteq \langle \mathcal{P}(I \rtimes_{\alpha} G) \rangle = \langle \mathcal{P}(\theta(I)) \rangle$, from which we conclude that $\theta(I)$ is generated by its projections. On the other hand, if $I \triangleleft A$, I is α -invariant and $\theta(I) = I \rtimes_{\alpha} G$ is generated by projections, then, clearly $I \subseteq I \rtimes_{\alpha} G = \langle \mathcal{P}(I \rtimes_{\alpha} G) \rangle$. These facts, together with [Pa4, Remark 4.3], imply that θ induces canonically a lattice isomorphism

$$\begin{aligned} & \{I \triangleleft A \mid I \text{ is } \alpha\text{-invariant and } I \subseteq \langle \mathcal{P}(I \rtimes_{\alpha} G) \rangle\} \\ & \cong \{J \triangleleft A \rtimes_{\alpha} G \mid J \text{ is generated by projections}\}. \end{aligned}$$

Corollary 2.2. *Let A, G and α be as in the hypothesis of Theorem 2.1. Then, the following are equivalent:*

- (a) $A \rtimes_{\alpha} G$ has the ideal property.
- (b) For every α -invariant ideal I of A , we have that $I \subseteq \langle \mathcal{P}(I \rtimes_{\alpha} G) \rangle$.

Corollary 2.3. *Let A, G and α be as in the hypothesis of Theorem 2.1. Moreover, suppose that the spectrum of A has dimension zero. Then, $A \rtimes_{\alpha} G$ has the ideal property.*

Proof. Since A is a commutative, separable C^* -algebra with zero-dimensional spectrum, then A is an AF algebra. Hence, A has the ideal property. Now, the result follows by also using Corollary 2.2.

Corollary 2.4. *Let A, G and α be as in the hypothesis of Theorem 2.1. Suppose moreover that $A = C(X)$, where X is a compact, Hausdorff topological space, and that every closed α -invariant subset of X is also open. Then, $A \rtimes_{\alpha} G$ has the ideal property.*

Proof. Let I be an α -invariant ideal of A . Then, there is a closed, α -invariant subset F of X such that $I = \{f \in C(X) \mid f|_F = 0\}$. Since $X \setminus F$ is closed (because, by hypothesis, F is open) it follows that the map $A \ni f \mapsto f|_F \oplus f|_{X \setminus F} \in C(F) \oplus C(X \setminus F)$ is an isomorphism, which implies that $I \cong C(X \setminus F)$ and hence, I being unital, is generated by projections. Now, the result follows by also using Corollary 2.2.

Definition 2.5. Let G be a locally compact group. We say that G has the ideal property if its group C^* -algebra $C^*(G)$ has the ideal property.

Question (Shuzhou Wang). Let (A, G, α) be a C^* -dynamical system such that A and G have the ideal property. Does the crossed product $A \rtimes_{\alpha} G$ (always) have the ideal property?

The answer to the above question is “no”, even in the “separable” case, as it follows from the following:

Theorem 2.6. *There are C^* -dynamical systems (A, G, α) with A a separable, simple, unital C^* -algebra (and hence, with the ideal property) and with G a separable, commutative topological group with the ideal property such that $A \rtimes_{\alpha} G$ does not have the ideal property.*

The proof of the above theorem will use the following proposition:

Proposition 2.7. *Let G be a compact, abelian group. Then, G has the ideal property.*

Proof. The hypothesis implies that \hat{G} (the dual group of G) is a discrete, abelian group. We have that $C^*(G) \cong C_0(\hat{G})$ (see [Pe, Proposition 7.1.6]). On the other hand, since \hat{G} is discrete, then $\dim(\hat{G}) = 0$ and hence $C_0(\hat{G})$ is an approximately finite-dimensional C^* -algebra, i.e., it is the inductive limit of a net of finite-dimensional C^* -algebras. Since a finite-dimensional C^* -algebra has the ideal property, this implies that $C_0(\hat{G})$ has the ideal property (as an inductive limit of C^* -algebras with the ideal property ([Pa4, Proposition 2.3])) and hence, by the above discussion, it follows that $C^*(G)$ has the ideal property, which ends the proof.

Proof of Theorem 2.6. Let (B, \mathbb{Z}, α) be a C^* -dynamical system with $B = C(X)$, where X is an infinite, compact, connected, metrizable topological space and such that $A := B \rtimes_{\alpha} \mathbb{Z}$ is a simple C^* -algebra. (Clearly, A is unital and separable and hence with the ideal property, since, obviously, every simple, unital C^* -algebra has the ideal property.) Note that A is simple if (and only if) the homeomorphism of X induced by $\alpha(1)$ is minimal (see e.g. [D, Theorem VIII. 3.9]). (Remark that the irrational rotation algebras are examples of such kind of C^* -algebras A ([Bl]).)

Now, by Takai duality ([T]), we have that

$$A \rtimes_{\hat{\alpha}} \mathbb{T} \cong B \otimes \mathcal{K}$$

where $\hat{\alpha} : \mathbb{T} = \hat{\mathbb{Z}} \rightarrow \text{Aut}(A)$ is the dual action ([T]). Our claim is that $B \otimes \mathcal{K}$ does not have the ideal property and this fact together with the above isomorphism will imply that $A \rtimes_{\hat{\alpha}} \mathbb{T}$ does not have the ideal property either. Indeed, let $\phi \neq F = \overline{F} \subsetneq X$ and let $I := \{f \in B \otimes \mathcal{K} = C(X, \mathcal{K}) \mid f|_F = 0\}$. Then, clearly $I \triangleleft B \otimes \mathcal{K}$ and $\mathcal{P}(I) = \{0\}$, since X is connected, but I is not generated by its projections (because, obviously, $I \neq \{0\}$). Hence, $B \otimes \mathcal{K}$ does not have the ideal property. The proof of the theorem is over if we observe that since \mathbb{T} is a compact, abelian group, the separable group $G = \mathbb{T}$ has the ideal property by Proposition 2.7.

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REFERENCES

- [Bl] B. Blackadar, *K-theory for operator algebras*, Second edition, Mathematical Sciences Research Institute Publications, 5, Cambridge University Press, Cambridge, 1998. MR **99g**:46104
- [Br] O. Bratteli, *Inductive limits of finite dimensional C^* -algebras*, Trans. Amer. Math. Soc. **171** (1972), 195-234. MR **47**:844
- [BPe] L. G. Brown and G. K. Pedersen, *C^* -algebras of real rank zero*, J. Funct. Anal. **99** (1991), no. 1, 131-149. MR **92m**:46086
- [D] K. Davidson, *C^* -algebras by example*, Fields Institute Monographs, 6, American Mathematical Society, Providence, RI, 1996. MR **97i**:46095

- [EH] E. G. Effros and F. Hahn, *Locally compact transformation groups and C^* -algebras*, Memoirs of the American Mathematical Society, No. 75, American Mathematical Society, Providence, RI, 1967. MR **37**:2895
- [Ell] G. A. Elliott, *The classification problem for amenable C^* -algebras*, Proceedings of the International Congress of Mathematicians, Vol. **1, 2** (Zürich, 1994), 922-932, Birkhäuser, Basel, 1995. MR **97g**:46072
- [GL] E. C. Gootman and A. J. Lazar, *Applications of non-commutative duality to crossed product C^* -algebras determined by an action or coaction*, Proc. London Math. Soc. (3) **59** (1989), no. 3, 593-624. MR **91b**:46069
- [Pa1] C. Pasnicu, *AH algebras with the ideal property*, Operator algebras and operator theory (Shanghai, 1997), 277-288, Contemp. Math., **228**, American Mathematical Society, Providence, RI, 1998. MR **99m**:46141
- [Pa2] C. Pasnicu, *Extensions of AH algebras with the ideal property*, Proc. Edinburgh Math. Soc. (2) **42** (1999), no. 1, 65-76. MR **2001c**:46112
- [Pa3] C. Pasnicu, *Shape equivalence, nonstable K -theory and AH algebras*, Pacific J. Math. **192** (2000), no. 1, 159-182. MR **2001d**:46102
- [Pa4] C. Pasnicu, *On the AH algebras with the ideal property*, J. Operator Theory **43** (2000), no. 2, 389-407. MR **2001c**:46111
- [Pa5] C. Pasnicu, *On the (strong) GAH algebras*, Rev. Roumaine Math. Pures Appl. **46** (2001), no. 4, 489-498.
- [Pa6] C. Pasnicu, *The ideal property and traces*, Math. Nachr. **227** (2001), 127-132. MR **2002g**:46101
- [Pa7] C. Pasnicu, *Ideals generated by projections and inductive limit C^* -algebras*, Rocky Mountain J. Math. **31** (2001), no. 3, 1083-1095.
- [Pa8] C. Pasnicu, *The projection property*, Glasg. Math. J. **44** (2002), no. 2, 293-300.
- [Pa9] C. Pasnicu, *LB algebras*, J. Operator Theory (to appear).
- [PaR] C. Pasnicu and M. Rørdam, *Tensor products of C^* -algebras with the ideal property*, J. Funct. Anal. **177** (2000), no. 1, 130-137. MR **2001m**:46124
- [Pe] G. K. Pedersen, *C^* -algebras and their automorphism groups*, London Mathematical Society Monographs, **14**, Academic Press, Inc. (Harcourt Brace Jovanovich, Publishers), London-New York, 1979. MR **81e**:46037
- [S] K. H. Stevens, *The classification of certain non-simple approximate interval algebras*, Fields Inst. Commun., **20**, American Mathematical Society, Providence, RI, 1998. MR **2002d**:46052
- [T] H. Takai, *On a duality for crossed products of C^* -algebras*, J. Funct. Anal. **19** (1975), 25-39. MR **51**:1413
- [Z] G. Zeller - Meier, *Products croisés d'une C^* -algèbre par un groupe d'automorphismes*, J. Math. Pures Appl. (9) **47** (1968), 101-239. MR **39**:3329

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