

ON THE EXISTENCE OF CHAOTIC AND HYPERCYCLIC SEMIGROUPS ON BANACH SPACES

TERESA BERMÚDEZ, ANTONIO BONILLA, AND ANTONIO MARTINÓN

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ABSTRACT. We prove that every separable infinite dimensional complex Banach space admits a hypercyclic uniformly continuous semigroup. We also prove that there exist Banach spaces admitting no chaotic strongly continuous semigroups.

1. INTRODUCTION

Throughout the paper, we work with complex Banach spaces, although some results are also valid in real Banach space.

A bounded linear operator T on a Banach space X is *hypercyclic* provided that there exists $x \in X$ such that $\{T^n x\}_{n \in \mathbb{N}}$ is dense in X . T is *chaotic* provided that it is hypercyclic and the set of periodic points is dense in X . A point x is called *periodic* for T if there exists some $n > 1$ such that $T^n x = x$. This definition of chaos is consistent with the definition given by Devaney [9, page 50] for an arbitrary continuous mapping f on a metric space, which requires that f be transitive, the set of periodic points of f be dense, and f have sensitive dependence on initial conditions. The two first conditions imply the third [3]. On the other hand, Godefroy and Shapiro [12] showed that transitive operators on Fréchet spaces have sensitive dependence on initial conditions.

Interest in hypercyclic operators arises from the invariant subset problem. In fact, it is easy to check that an operator T has no non-trivial invariant closed subset if and only if each non-zero vector is hypercyclic for T .

For more details about hypercyclicity see the excellent survey by Grosse-Erdmann [14].

In 1969, Rolewicz [23] gave the first example of a hypercyclic operator on a Banach space. He showed that if B is the backward shift on $l^2(\mathbb{N})$, then λB is hypercyclic if and only if $|\lambda| > 1$. Also, he wondered if for every separable infinite

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dimensional Banach space there exists a hypercyclic operator. This question was independently answered in the affirmative by Ansari [1] and Bernal [4], and by Bonet and Peris [6] for Fréchet spaces. On the other hand, Bonet, Martínez and Peris have recently proved that the dual of a reflexive separable hereditarily indecomposable complex Banach space admits no chaotic continuous linear operator [5].

A strongly continuous semigroup of bounded linear operator $\{T(t)\}_{t \geq 0}$ is *hypercyclic* provided that there exists $x \in X$ such that $\{T(t)x\}_{t \geq 0}$ is dense in X . Furthermore, it is chaotic if it is hypercyclic and the set $\{x \in X : \exists t > 0, T(t)x = x\}$ is dense in X .

Hypercyclic and chaotic behaviour in strongly continuous semigroups can be viewed as the continuous time analog of the discrete time case mentioned above, and can occur when the infinitesimal generator is bounded or unbounded, but only in separable infinite dimensional spaces [8, pp. 794-795].

The following example gives a strongly continuous semigroup of bounded linear operators in $L^2([0, \infty), \mathbb{C})$ that is chaotic. The linear evolution problem is a convection-diffusion type equation of the form

$$\begin{cases} u_t(x, t) = au_{xx}(x, t) + bu_x(x, t) + cu(x, t), \\ u(0, t) = 0 \text{ for } t \geq 0, \\ u(x, 0) = f(x) \text{ for } x \geq 0 \text{ with some } f \in L^2([0, \infty), \mathbb{C}). \end{cases}$$

If $a, b, c > 0$ with $c < \frac{b^2}{2a} < 1$, then the solution semigroup to this partial differential equation is chaotic. For details see [8, Example 4.12]. So, a natural problem is: *Does every Banach space admit a chaotic strongly continuous semigroup?*

Our aim in this paper is to prove that every separable infinite dimensional complex Banach space admits a hypercyclic uniformly continuous semigroup, and that there exist Banach spaces that admit no chaotic strongly continuous semigroups.

2. ON HYPERCYCLIC SEMIGROUPS

We consider the weighted ℓ^1 -space with weight sequence $\beta = (\beta_i)$ of positive numbers defined by

$$\ell^1(\beta) := \{(x_i)_{i=1}^\infty : x_i \in \mathbb{C}, \sum_{i=1}^\infty \beta_i |x_i| < \infty\}$$

equipped with the norm

$$\|(x_i)\| := \sum_{i=1}^\infty \beta_i |x_i|.$$

If $\sup_{i \in \mathbb{N}} \frac{\beta_i}{\beta_{i+1}} \leq M$ for some constant M , then the backward shift B defined by

$$B(x_1, x_2, \dots) := (x_2, x_3, \dots)$$

is a continuous linear operator on $\ell^1(\beta)$.

Lemma 2.1. *Let $\beta = (\beta_i)$ be a sequence of positive numbers. If $\sup_{i \in \mathbb{N}} \frac{\beta_i}{\beta_{i+1}} \leq M$ for some constant M , then the operator e^B is hypercyclic in $\ell^1(\beta)$.*

Proof. In [8, Theorem 5.2] it is proved that e^{tB} is a hypercyclic semigroup. Moreover, it is derived from the proof that e^B is a hypercyclic operator on $\ell^1(\beta)$, since in [8] it is proved that $T(t)$ is a hypercyclic operator for all t , in particular for $t = 1$. □

Similar results may be found in [7, Theorem 2.13].

The following two results, due to Martínez and Peris [17] and Ovsepian and Pelczynski [18], will allow us to show the existence of a hypercyclic uniformly continuous semigroup on every separable Banach space. A similar version of Lemma 2.2 for semigroups may be found in [10, Theorem 3.2].

Lemma 2.2 ([17, Lemma 2.1]). *Let $T_i : X_i \rightarrow X_i$ be operators on separable Banach spaces X_i , $i = 1, 2$, and let $\Phi : X_1 \rightarrow X_2$ be a continuous mapping with dense range such that $T_2\Phi = \Phi T_1$. If T_1 is hypercyclic (chaotic), then T_2 is also hypercyclic (chaotic).*

Lemma 2.3 ([18]). *Let X be a separable infinite dimensional Banach space. Then there exist $(x_n)_{n \in \mathbb{N}} \subset X$ and $(f_m)_{m \in \mathbb{N}} \subset X^*$ satisfying the following conditions:*

- (1) $f_m(x_n) = \delta_{m,n}$, $m, n \in \mathbb{N}$.
- (2) $\overline{\text{span}\{x_n : n \in \mathbb{N}\}} = X$.
- (3) $f_m(x) = 0$, for all $m \in \mathbb{N} \Rightarrow x = 0$.
- (4) $\|x_n\| = 1$, for all $n \in \mathbb{N}$ and $\sup_{m \in \mathbb{N}} \|f_m\| = c < \infty$.

Now, we are able to prove the main result of this section.

Theorem 2.4. *Every separable infinite dimensional Banach space X admits a hypercyclic uniformly continuous semigroup.*

Proof. Let $(x_n)_{n \in \mathbb{N}} \subset X$ and $(f_m)_{m \in \mathbb{N}} \subset X^*$ as in Lemma 2.3.

Consider the linear continuous operator $S : X \rightarrow X$ defined by

$$(2.1) \quad Sx := \sum_{n=1}^{\infty} \frac{1}{2^n} f_{n+1}(x)x_n,$$

given in [16, page 297].

We will prove that the operator e^S is hypercyclic. Therefore the semigroup e^{St} is a hypercyclic uniformly continuous semigroup.

Define $\Phi : l^1 \rightarrow X$, by $\Phi((\alpha_j)) := \sum_{j=1}^{\infty} \alpha_j x_j$, which is a linear continuous operator with dense range.

Claim. The operator $e^{\tilde{S}} : l^1 \rightarrow l^1$ is hypercyclic, where

$$\tilde{S}((\alpha_j)) := \left(\frac{\alpha_2}{2}, \dots, \frac{\alpha_{n+1}}{2^n}, \dots \right).$$

From the definition it follows that $S\Phi = \Phi\tilde{S}$ on l^1 and therefore $e^S\Phi = \Phi e^{\tilde{S}}$. Applying Lemma 2.2 we have that e^S is hypercyclic in X .

Proof of the Claim. We define $\beta_1 := 1$, and $\beta_i := 2^{1+\dots+(i-1)}$ for $i > 1$ and $\Phi_\beta : l^1(\beta) \rightarrow l^1$, by $\Phi_\beta(\alpha_1, \alpha_2, \dots) := (\beta_1\alpha_1, \beta_2\alpha_2, \dots)$, which is a linear continuous and surjective mapping.

According to Lemma 2.1 we have that e^B is hypercyclic on $l^1(\beta)$, since $\sup \frac{\beta_i}{\beta_{i+1}} \leq M$ for some constant M . Therefore by Lemma 2.2, the operator $e^{\tilde{S}}$ is hypercyclic because $e^{\tilde{S}}\Phi_\beta = \Phi_\beta e^B$. □

Remark 1. The above result implies that e^S is hypercyclic, where S is given by (2.1). In particular, this result provides a new proof of Ansari's and Bernal's result without using Salas' theorem [24, Section 3].

As a consequence of the result of Oxtoby and Ulam [19, Theorem 6] it is obtained that if $\{T(t)\}$ is a hypercyclic strongly continuous semigroup, then there exists a residual set $G \subset \mathbb{R}^+$, i.e. the complement of G is a set of first category, such that $T(t)$ is a hypercyclic operator for all $t \in G$. This was pointed out by A. Peris to the authors in a personal communication [21] where he also proposed the following open problem.

Open Problem 1. If $\{T(t)\}$ is a hypercyclic strongly continuous semigroup, then are all $T(t)$ hypercyclic operators?

3. ON CHAOTIC SEMIGROUPS

Our approach to prove the existence of Banach spaces admitting no chaotic semigroups is similar to that of [5], since it is based on hereditarily indecomposable Banach spaces.

A Banach space X is called *hereditarily indecomposable* if whenever Y and Z are closed infinite dimensional subspaces of X satisfying $Y \cap Z = \{0\}$, then $Y + Z$ is non-closed.

Gowers and Maurey provided the first example of hereditarily indecomposable Banach space [13]. In particular, they proved that every linear bounded operator T on X can be written as $T = \lambda I + S$, where $\lambda \in \mathbb{C}$ and S is *strictly singular* (i.e. it has no bounded inverse on any infinite-dimensional subspace). Indeed, the spectrum of T is finite or consists of λ and a sequence $(\lambda_n)_{n \in \mathbb{N}}$ of eigenvalues with finite multiplicity converging to λ . Therefore, if there are not eigenvalues, the spectrum is a singleton. This fact will be used later on.

Notice that the dual of a hereditarily indecomposable space may be far from being hereditarily indecomposable [2]. However, Ferenczi proved that the space defined by Gowers and Maurey in [13], X_{GM} (which is hereditarily indecomposable), satisfies that X_{GM}^* is also hereditarily indecomposable [11, Corollary 22].

In [22], Råbiger and Ricker studied the strongly continuous semigroups on hereditarily indecomposable spaces. This reference will be an important tool for obtaining the main result of this section. In the next proposition we give some useful results of Råbiger and Ricker.

Proposition 3.1 ([22]). *Let X be a hereditarily indecomposable space.*

- (1) *If T is a linear operator defined in $D(T) \subset X$ with a non-empty resolvent set, then the spectrum set $\sigma(T)$ is either a finite set (possibly empty) or consists of a convergent sequence in the extended complex plane $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ where all except at most one point in $\sigma(T)$ are eigenvalues of T .*
- (2) *If T is the generator of a strongly continuous group on X , then T is a bounded linear operator.*
- (3) *If T is the generator of a strongly continuous semigroup on X , then the generator satisfies the spectral mapping theorem, that is,*

$$e^{t\sigma(A)} = \sigma(T(t)) \setminus \{0\}$$

for all $t > 0$.

Recall that the *spectrum* of a linear operator T on a normed space X , $\sigma(T)$, is defined by the complement in \mathbb{C} of the *resolvent set* $\rho(T)$, and $\lambda \in \rho(T)$ if the operator $\lambda I - T$ has zero kernel, dense range and $(\lambda I - T)^{-1}$ is bounded.

In order to present the main result, we need an additional property.

In a similar way to [15, Proposition 2.2] we obtain the following lemma.

Lemma 3.2. *Let X be a Banach space and $\{T(t)\}_{t \geq 0}$ a strongly continuous semigroup such that*

$$(3.1) \quad X = M_1 \oplus M_2$$

where M_i , $i = 1, 2$, are closed subspaces with

$$(3.2) \quad T(t)M_i \subset M_i, \quad i = 1, 2,$$

for all $t \geq 0$. If $\{T(t)\}_{t \geq 0}$ is a chaotic semigroup on X , then its restriction on M_i , $\{T(t)_{M_i}\}_{t \geq 0}$ is chaotic on M_i , for $i = 1, 2$.

Proof. Notice that condition (3.2) is equivalent to the fact that the projection P_{M_i} , associated to the subspace M_i , commutes with $T(t)$ for all $t \geq 0$, i.e.

$$P_{M_i}T(t) = T(t)P_{M_i}$$

for all $t \geq 0$. Hence the proof can be readily obtained by applying a version of Lemma 2.2 for semigroups. □

Remark 2. Since there is no hypercyclic strongly continuous semigroup in a finite dimensional space, the lemma implies that the decomposition given in (3.1) cannot be obtained with any finite dimensional subspace if the semigroup is hypercyclic.

Theorem 3.3. *Let X be a hereditarily indecomposable Banach space such that X^* is also a hereditarily indecomposable space. Then X admits no chaotic strongly continuous semigroups.*

Proof. Let us suppose there exists a chaotic strongly continuous semigroup $\{T(t)\}_{t \geq 0}$, and we denote by A its infinitesimal generator with domain $D(A)$. By Proposition 3.1 the strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ satisfies the *spectral mapping theorem*, that is,

$$(3.3) \quad e^{t\sigma(A)} = \sigma(T(t)) \setminus \{0\},$$

for every $t > 0$. Moreover, the resolvent set $\rho(A)$ cannot be an empty set, by (3.3) and since $T(t)$ is a bounded operator for all $t \geq 0$. As an application of Proposition 3.1, the spectrum of A is either a finite set (possibly empty) or consists of a convergent sequence in the extended complex plane \mathbb{C}_∞ . All points, except at most one point in $\sigma(A)$, are eigenvalues of A .

We will proceed with the proof in four steps.

Step 1: $\sigma(A) = \{0\}$.

Since the semigroup is chaotic, there exist $x \in X \setminus \{0\}$ and $t_0 > 0$ such that $T(t_0)x = x$. Hence $1 \in \sigma_p(T(t_0))$ and by the spectral mapping theorem we have that $\sigma(A) \neq \emptyset$. By [25, Theorem V.2.4] we obtain that $\sigma(A) = \sigma(A^*) \neq \emptyset$, and the spectrum of A^* is a finite set or a convergent sequence in \mathbb{C}_∞ , which are eigenvalues of A^* . Notice that the limit of the sequence could be out of the point spectrum. By [8, Theorem 3.3] the point spectrum of A^* is empty, hence $\sigma(A) = \sigma(A^*) = \{\lambda\}$.

Let us prove that $\lambda = 0$. We have that $\{T(t)\}_{t \geq 0}$ is chaotic, hence $1 \in \sigma_p(T(t))$ for some $t > 0$. Using the spectral mapping theorem we get that $\lambda = 0$.

Step 2: $\sigma(T(t)) = \{1\}$ for all $t > 0$.

Suppose that there exists $t_0 > 0$ such that $\sigma(T(t_0)) = \{0, 1\}$. We denote by P_0 and P_1 the projections associated to the spectral sets $\sigma_0 = \{0\}$ and $\sigma_1 = \{1\}$, respectively, and $X_i = R(P_i)$ is the range of P_i for $i = 0, 1$. Applying [25, Theorem V.9.1] we have the decomposition $X = X_0 \oplus X_1$ with $\sigma(T(t_0)|_{X_0}) = \{0\}$ and $\sigma(T(t_0)|_{X_1}) = \{1\}$. By properties of the holomorphic functional calculus, the projections P_i commute with every operator that commutes with $T(t_0)$, in particular with $T(t)$ for every $t > 0$. From the definition of hereditarily indecomposable spaces it follows that at least one of X_i must be finite dimensional. On the other hand, by Lemma 3.2 and Remark 2 we have that the dimensions of X_i must be infinite. So we conclude a contradiction. Hence $\sigma(T(t)) = \{1\}$ for all $t > 0$.

Step 3: $\{T(t)\}_{t \geq 0}$ is a uniformly continuous semigroup.

By [20, Theorem I.6.5] the semigroup $\{T(t)\}_{t \geq 0}$ can be extended to a strongly continuous group, since we have that $0 \notin \sigma(T(t)) = \{1\}$ for all $t > 0$. According to Proposition 3.1 the infinitesimal generator is bounded and therefore the semigroup is uniformly continuous (see [20, Theorem I.1.2]).

Step 4: X admits no chaotic uniformly continuous semigroup.

Suppose that there exists a chaotic uniformly continuous semigroup $\{T(t)\}_{t \geq 0}$ on X . By [20, Theorem I.1.2] the infinitesimal generator A of $\{T(t)\}_{t \geq 0}$ is a bounded operator on X , $T(t) = e^{tA}$, and by Step 1 we have that $\sigma(A) = \{0\}$.

Given a periodic point x of $\{T(t)\}_{t \geq 0}$, there exists $t_0 > 0$ such that

$$0 = (T(t_0) - I)x = (e^{t_0 A} - I)x = t_0 A \left(\sum_{n=1}^{\infty} \frac{(t_0 A)^{n-1} x}{n!} \right).$$

Define

$$g(z) := \sum_{n=1}^{\infty} \frac{(t_0 z)^{n-1}}{n!},$$

which is holomorphic on \mathbb{C} and $g(0) = 1$. Hence by using the holomorphic functional calculus, there exists the inverse of $g(A)$ in $L(X)$, since $\sigma(A) = \{0\}$ and $g(0) \neq 0$. Therefore $0 = t_0 A g(A)x = g(A)t_0 Ax$, and consequently $Ax = 0$. Thus $Ax = 0$ for every x in the dense set of periodic points of $\{T(t)\}_{t \geq 0}$. Hence $A = 0$. The contradiction follows because $T(t) = I$ is not chaotic. \square

Remark 3. The same proof, of Theorem 3.3, works for X to be a dual of a reflexive separable hereditarily indecomposable complex Banach space of Gowers and Maurey, using [20, Theorem II.2.4] instead of the spectral mapping theorem.

To finish, we want to pose the following problem.

Open Problem 2. Characterize those Banach spaces X which admit a chaotic strongly continuous semigroup.

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DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, UNIVERSIDAD DE LA LAGUNA, 38271 LA LAGUNA (TENERIFE), SPAIN

E-mail address: tbermude@ull.es

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, UNIVERSIDAD DE LA LAGUNA, 38271 LA LAGUNA (TENERIFE), SPAIN

E-mail address: abonilla@ull.es

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, UNIVERSIDAD DE LA LAGUNA, 38271 LA LAGUNA (TENERIFE), SPAIN

E-mail address: anmarce@ull.es