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COMPACT-COVERING MAPS AND k-NETWORKS

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ABSTRACT. In this paper, we show a characterization of compact-covering s-images of metric spaces and prove a theorem about them. Also we give a set theoretical assumption and under the assumption construct a counterexample which gives a negative answer to some questions.

1. INTRODUCTION

Recall that an onto map $f : X \to Y$ is an *s-map* if $f^{-1}(y)$ is separable for each $y \in Y$ and a *compact-covering map* if every compact $K \subset Y$ is the image of some compact $C \subset X$. An intrinsic characterization of quotient s-images of metric spaces has been given by T. Hoshina [4]. E. Michael and K. Nagami [11] asked that if a Hausdorff space Y is a quotient s-image of a metric space, must Y also be a compact-covering quotient s-image of a metric space? Also the question was mentioned by M. E. Rudin [13] and studied in [3, 5, 6, 8] and again in [10]. L. Foged in [2] gave a completely regular space X which has a point-countable base but no point-countable closed k-network. H. Chen in [1] constructed a counterexample which gives a negative answer to Michael-Nagami's question. But the space in the example is Hausdorff. So E. Michael in [9] (see also [1]) refined the problem by insisting that Y be regular—we call this Problem 1.1.

Problem 1.1. If a completely regular T_1 (or paracompact) space Y is a quotient s-image of a metric space, must Y be a compact-covering quotient s-image of a metric space?

On the other hand, S. Lin in [7] asked the following question which was arranged as Problem 38 of the Problem Section in [14].

Problem 1.2. Suppose a regular T_1 space Y is a quotient s-image of a metric space. Does Y have a point-countable closed k-network if every first countable closed subspace of Y is locally compact?

In this paper, we introduce a definition of strong k-networks, use it to characterize compact-covering s-images of metric spaces, and then prove a theorem about these spaces. Finally we give a set theoretical assumption and under the assumption construct a counterexample which gives a negative answer to the questions above.

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2. A characterization of compact-covering s-images of metric spaces

We recall that a cover $\mathcal{P}$ is a *k-network* for Y if, whenever $K \subset U$ with K compact and U open in Y, then $K \subset \bigcup \mathcal{F} \subset U$ for some finite $\mathcal{F} \subset \mathcal{P}$.

Definition 2.1. Let A be a subset of a space Y. A collection $\mathcal{F}$ of subsets of Y is called a *full cover* of A if $\mathcal{F}$ is finite, and for each $F \in \mathcal{F}$ there is a closed set $C(F)$ in Y with $C(F) \subset F$ such that $A \subset \bigcup \{C(F) : F \in \mathcal{F}\}$.

Call a cover $\mathcal{P}$ a *strong k-network* for Y if, whenever $K \subset U$ with K compact and U open in Y, there is a full cover $\mathcal{F} \subset \mathcal{P}$ of K with $\bigcup \mathcal{F} \subset U$.

Theorem 2.2. *The following are equivalent for a Hausdorff space Y:*

1. *Y is a compact-covering s-image of a metric space.*
2. *Y has a point-countable strong k-network.*

Proof. ($1 \Rightarrow 2$). Let M be a metric space and $f : M \to Y$ be an onto compact-covering s-map. Let $\mathcal{B} = \bigcup_n \mathcal{B}_n$ be a σ-locally finite base. Then $\mathcal{P} = f(\mathcal{B})$ is a point-countable k-network. Let $K \subset O \subset Y$ satisfy K compact and O open and $C \subset M$ be compact with $f(C) = K$. Then there is a full cover $\mathcal{F} \subset \mathcal{B}$ of C with $\bigcup \mathcal{F} \subset f^{-1}(O)$. Assume that B'_i contains a closed set H_i for each $B'_i \in \mathcal{F}$ such that $C \subset \bigcup_{i \le n} H_i$. Then $K \subset \bigcup_{i \le n} f(H_i \cap C) \subset \bigcup_{i \le n} f(B'_i) \subset O$. So $\mathcal{P} = \{f(B) : B \in \mathcal{B}\}$ is a strong point-countable k-network.

($2 \Rightarrow 1$). Let $\mathcal{B}$ be a point-countable strong k-network of Y. Let $\mathcal{B}_n = \mathcal{B}$ with the discrete topology. Then the Tychonoff product $\prod_{n \in N} \mathcal{B}_n$ is a metric space. Let $M \subset \prod_{n \in N} \mathcal{B}_n$ be all (B_n) such that there is a $y \in Y$ with $\bigcap_{n \in N} B_n = \{y\}$ and every neighborhood of y contains some B_n. Let $f : M \to Y$ be such that, for each $(B_n) \in M$, $f((B_n)) = y$ if $\bigcap_{n \in N} B_n = \{y\}$. We may show that f is an onto continuous s-map just as in the proof of Theorem 6.1 of [3]. Let $\mathcal{C}_n = \{(\{B_1\} \times \{B_2\} \times \ldots \times \{B_n\} \times \prod_{j > n} \mathcal{B}_j) \cap M : B_i \in \mathcal{B} \text{ for each } i \le n\}$. Let $\mathcal{C} = \bigcup_{n \in N} \mathcal{C}_n$. Then $\mathcal{C}$ is a σ-discrete base of M. In the following proof, we show that $f : M \to Y$ is a compact-covering map.

Let K be a compact subset of Y. Then K is a metric subset of Y by Theorem 3.3 in [3]. If K is a finite subset of Y, then there is a finite subset C of M with $f(C) = K$. So we assume that K is infinite in the following proof. Let $\mathcal{F} \subset \mathcal{B}$ be a full cover of K, and let $\mathcal{F}(y) = \{B \in \mathcal{F} : y \in B\}$.

Claim 2.3. *If $y \in K$ and O is an open neighborhood of y in Y, then there is a full cover $\mathcal{F} \subset \mathcal{B}$ of K such that $\bigcup \mathcal{F}(y) \subset O$.*

Proof. Let $y \in W_1 \subset \overline{W_1} \subset W \subset \overline{W} \subset O \cap K$, where both W and W_1 are open in K. There is a full cover $\mathcal{F}_1 = \{B_{1i} : i \le n\} \subset \mathcal{B}$ of $\overline{W}$ with $\bigcup \mathcal{F}_1 \subset O$. Since the open set $Y \setminus \overline{W_1}$ contains compact $K \setminus W$, then there is a full cover $\mathcal{F}_2 = \{B_{2i} : i \le m\}$ of $K \setminus W$ with $\bigcup \mathcal{F}_2 \subset Y \setminus \overline{W_1}$. Then $\mathcal{F} = \mathcal{F}_1 \bigcup \mathcal{F}_2$ is a full cover of K such that $\bigcup \mathcal{F}(y) \subset O$.

Call a full cover $\mathcal{F} = \{B_i : i \le n\}$ of B an irreducible full cover if each $B_i \in \mathcal{F}$ contains a closed set C_i in Y such that $\{C_i : i \le n\}$ is an irreducible cover of B. Notice that irreducible full covers do not mean irreducible covers.

Claim 2.4. $|\{\mathcal{F} \subset \mathcal{B} : \mathcal{F} \text{ is an irreducible full cover of } K\}| = \aleph_0$.

Proof. Let $\mathcal{B}(K) = \{B \in \mathcal{B} : Int_K(B \cap K) \neq \emptyset\}$. Then $\mathcal{B}(K)$ is point-countable since $\mathcal{B}$ is point-countable. Since K is compact metrizable, in particular, separable, the point-countability of $\mathcal{B}$ implies that $\mathcal{B}(K)$ is countable. Let $(\mathcal{F}'_n)$ enumerate all finite unions of $\mathcal{B}(K)$.

Let $\mathcal{F}' = \{B_i : i \leq n\}$ be an irreducible full cover of K. Then each $B_i \in \mathcal{F}'$ contains a closed subset C_i in Y such that $\mathcal{C} = \{C_i : i \leq n\}$ is an irreducible cover of K. Because an open set $K \setminus \bigcup_{j \neq i} C_j$ is an open set of K included in C_i and $\mathcal{C}$ is irreducible, $K \setminus \bigcup_{j \neq i} C_j \neq \emptyset$. Then $Int_K(C_i \cap K) \neq \emptyset$ for each $C_i \in \mathcal{C}$, and $\mathcal{F}' \in (\mathcal{F}'_n)$. So all irreducible full covers are *at most* countable.

On the other hand, since K is infinite, there is a cluster point $y \in K$. Let d be a metric of K. Let $U_n = \{y' \in K : d(y, y') < 1/n\}$ for each $n \in N$. Let O_n be an open set of Y with $K \cap O_n = U_n$ for each $n \in N$. Then there is a full cover $\mathcal{F}_n \subset \mathcal{B}$ of K such that $\bigcup \mathcal{F}_n(y) \subset O_n$ for each $n \in N$ by Claim 2.3. Since each finite cover contains an irreducible finite cover, then each $\mathcal{F}_n$ contains an irreducible full cover $\mathcal{H}_n$. Note that $y \in Int_K(K \cap (\bigcup \mathcal{H}_n(y)) \subset \bigcup \mathcal{H}_n(y) \subset O_n$ for each $n \in N$. Hence there must be *at least* countably infinitely many different $\mathcal{F}_n$'s.

Let $(\mathcal{F}_n)$ enumerate all irreducible full covers of K. Then $\prod_{n \in N} \mathcal{F}_n$ is a compact subset of $\prod_{n \in N} \mathcal{B}_n$. For each $\mathcal{F}_n = \{B_{ni} : i \leq i(n)\} \in (\mathcal{F}_n)$ and each $B_{ni} \in \mathcal{F}_n$, let $C_{ni} \subset K \cap B_{ni}$ be such that $\mathcal{C}_n = \{C_{ni} : i \leq i(n)\}$ is an irreducible closed cover of K by the definition of $\mathcal{F}_n$. We assign this $\mathcal{C}_n$ to the $\mathcal{F}_n$ for each $\mathcal{F}_n \in (\mathcal{F}_n)$. Let

$$D = \{(B_{n,j(n)}) \in (\prod_{n \in N} \mathcal{F}_n) \cap M : j(n) \leq i(n) \quad \text{for each } n \in N, \text{ and } \bigcap_{n \in N} C_{n,j(n)} \neq \emptyset\}.$$

Claim 2.5. *$f(D) = K$.*

Proof. Pick an $x = (B_n) \in D \subset M$. Suppose $f(x)$ is not in K. Then $Y \setminus K$ is an open neighborhood of $f(x)$, so there is a B_n in (B_n) with $f(x) \in B_n \subset Y \setminus K$ by the definition of the subspace M. It is a contradiction to the definition of $\mathcal{F}_n$, so $f(D) \subset K$.

Pick a $y \in K$. Then for each $n \in N$, there is a $B_{n,j(n)} \in \mathcal{F}_n$ with $y \in C_{n,j(n)}$. Let $x = (B_{n,j(n)})$. Then $x \in \prod_{n \in N} \mathcal{F}_n$ and $y \in \bigcap_n B_n$. Pick an open set $O \subset Y$ with $y \in O$. Then, by Claim 2.3, there is a full cover $\mathcal{F}$ of K with $\bigcup \mathcal{F}(y) \subset O$. Then $\mathcal{F}$ contains an irreducible full cover $\mathcal{F}'$ of K and $\mathcal{F}' \in (\mathcal{F}_n)$. So there is an $n \in N$ with $\mathcal{F}' = \mathcal{F}_n$, and $B_{n,j(n)} \in \mathcal{F}_n$, $C_{n,j(n)} \in \mathcal{C}_n$ and $y \in C_{n,j(n)} \subset B_{n,j(n)} \subset \bigcup \mathcal{F}_n(y) \subset \bigcup \mathcal{F}(y) \subset O$. This implies $\bigcap_{n \in N} C_{n,j(n)} = \bigcap_{n \in N} B_{n,j(n)} = \{y\}$ and $x \in M$. Hence $x \in D$ and $f(x) = y \in f(D)$.

Claim 2.6. *D is a compact subset of $\prod_{n \in N} \mathcal{F}_n$.*

Proof. We may pick an $x = (B_n) \in \overline{D} \subset \prod_{n \in N} \mathcal{F}_n$ since $\prod_{n \in N} \mathcal{F}_n$ is compact. Then $(\{B_1\} \times \ldots \times \{B_n\} \times \prod_{j > n} \mathcal{B}_j) \cap D \neq \emptyset$ for each $n \in N$. Pick an $x' \in (\{B_1\} \times \ldots \times \{B_n\} \times \prod_{j > n} \mathcal{B}_j) \cap D$. Then $x' = (B_1, B_2, \ldots, B_n, *, *, \ldots) \in D$. So $C_i \in \mathcal{C}_i$ and $C_i \subset B_i \in \mathcal{F}_i$ for each $i \leq n$ such that $\bigcap_{i \leq n} C_i \neq \emptyset$ by the definition of D. Then $\bigcap_{n \in N} B_n \supset \bigcap_{n \in N} C_n \neq \emptyset$ since each C_n is compact for $x = (B_n)$. Notice $C_n \in \mathcal{C}_n$ and $C_n \subset B_n \cap K$. Pick a $y \in \bigcap_{n \in N} C_n$; then $y \in K$. Pick an open set $O \subset Y$ with $y \in O$. Then, just as the proof of Claim 2.5, there is a $B_n \in (B_n)$ with $y \in B_n \subset O$. This implies $\bigcap_{n \in N} C_n = \bigcap_{n \in N} B_n = \{y\}$ and $x \in D$.

Proof of Theorem 2.2 (*continued*). If K is an infinite compact subset of Y, then there must be countably infinitely many finite subcollections of $\mathcal{B}$ which are irreducible full covers of K by Claim 2.4. If $(\mathcal{F}_n)$ enumerates all the finite subcollections of $\mathcal{B}$ which are irreducible full covers of K, then D is a compact subset of M by Claim 2.6. Then $f(D) = K$ by Claim 2.5. So $f : M \to Y$ is a compact-covering map.

Corollary 2.7. *The following are equivalent for a Hausdorff space Y:*

1. *Y is a compact-covering quotient s-image of a metric space.*
2. *Y is a sequential space with a point-countable strong k-network.*

Proof. $(1 \Rightarrow 2)$. Y has a strong point-countable k-network by Theorem 2.2. Then Y is a sequential space since Y is a quotient image of a first countable space.

$(2 \Rightarrow 1)$. Let $\mathcal{B}$ be a point-countable strong k-network of Y. Then, by Theorem 2.2, there is a metric space $M \subset \prod_{n\in N} \mathcal{B}_n$ and an onto continuous s-map $f : M \to Y$ such that f is a compact-covering map. Since Y is a sequential space, f is a quotient map.

3. Closed k-networks and strong k-networks

It is easy to see that each closed k-network is a strong k-network.

Theorem 3.1. *Let a Hausdorff space Y have point-countable strong k-network $\mathcal{P}$ with $\overline{P}$ compact for each $P \in \mathcal{P}$. Then Y has a compact point-countable k-network.*

Proof. Let $\mathcal{P}$ be a point-countable strong k-network of Y. Let $\mathcal{P}_n = \mathcal{P}$ with the discrete topology. Then there is a metric space $M \subset \prod_{n\in N} \mathcal{P}_n$ and $f : M \to Y$ is a compact-covering s-map by Theorem 2.2. Let $\mathcal{B}$ be the σ-discrete base of M, which was denoted by $\mathcal{C}$ in Theorem 2.2, consisting of basic open sets. Then since each $f(B)$, $B \in \mathcal{B}$, is contained in some element of $\mathcal{P}$, we have that $\overline{f(B)}$ is compact for each $B \in \mathcal{B}$. We use $\mathcal{B}$ to construct a point-countable collection $\mathcal{C} = \{C_\alpha : \alpha \in \Omega\}$ of compact subsets of M such that each $\overline{f(B)}$, $B \in \mathcal{B}$, can be covered by a finite subcollection of $\{f(C) : C \in \mathcal{C}\}$ by a transfinite induction on $\alpha \in \Omega$ for some ordinal Ω. Let $\prec$ be a well ordering of $\mathcal{B}$.

1. Take the first element B_0 of $(\mathcal{B}, \prec)$. Then $\overline{f(B_0)} = K_{00}$ is compact. So there is a compact subset C_{00} of M with $f(C_{00}) = \overline{f(B_0)}$. Pick a finite subcollection $\mathcal{F}_{00}$ of $\mathcal{B}$ which is an irreducible cover of C_{00}. Then $\bigcup\{\overline{f(B)} : B \in \mathcal{F}_{00}\} = K_{01}$ is compact. So there is a compact subset C_{01} of M with $f(C_{01}) = K_{01}$. Pick a finite subcollection $\mathcal{F}_{01}$ of $\mathcal{B}$ which is an irreducible cover of C_{01}. Then $\bigcup\{\overline{f(B)} : B \in \mathcal{F}_{01}\} = K_{02}$ is compact. Then, by induction, there is a countable collection $\mathcal{C}_0 = \{C_{0n} : n \in \omega\}$ of compact subsets of M and a countable collection $\mathcal{O}_0 = \bigcup\{\mathcal{F}_{0n} : n \in \omega\}$ of open subsets of M such that $C_{0n} \subset \bigcup \mathcal{F}_{0n}$ for each $C_{0n} \in \mathcal{C}_0$.

2. For some ordinal α, assume that for each $\beta < \alpha$, we have $\mathcal{O}_\beta = \bigcup\{\mathcal{F}_{\beta n} : n \in \omega\}$ and $\mathcal{C}_\beta = \{C_{\beta n} : n \in \omega\}$ such that for each $B \in \mathcal{O}_\beta$, there is a finite subcollection $\mathcal{C}' \subset \bigcup_{\delta\leq\beta} \mathcal{C}_\delta$ with $\overline{f(B)} \subset \bigcup\{f(C) : C \in \mathcal{C}'\}$.

Let $\mathcal{O}'_\alpha = \bigcup_{\beta<\alpha} \mathcal{O}_\beta$ and $\mathcal{C}'_\alpha = \bigcup_{\beta<\alpha} \mathcal{C}_\beta$. Note that if C is a compact subset of M with $C \subset \bigcup \mathcal{O}'_\alpha$, then C can be covered by a finite subcollection of $\{f(C) : C \in \mathcal{C}'_\alpha\}$.

Case 1. For each $B \in \mathcal{B} \setminus \mathcal{O}'_\alpha$, there is a compact subset C of M with $C \subset \bigcup \mathcal{O}'_\alpha$ and $\overline{f(B)} = f(C)$. In this case, we finish the induction step. Let $\alpha = \Omega$. Then $\mathcal{C} = \mathcal{C}'_\alpha = \{C_\beta : \beta \in \Omega\}$ is the desired collection. Note that it follows from the note above Case 1 that each $\overline{f(B)}$, $B \in \mathcal{B}$, can be covered by a finite subcollection of $\{f(C) : C \in \mathcal{C}\}$.

Case 2. There is a $B \in \mathcal{B} \setminus \mathcal{O}'_\alpha$, such that if compact subset C of M satisfies $f(C) = \overline{f(B)}$, then $C \setminus \bigcup \mathcal{O}'_\alpha \neq \emptyset$. Let B_α be the first one in $\mathcal{B} \setminus \mathcal{O}'_\alpha$ with respect to the order of $(\mathcal{B}, \prec)$.

A. Let $C'_{\alpha 0}$ be a compact subset of M with $f(C'_{\alpha 0}) = \overline{f(B_\alpha)}$ since f is compact-covering. Let $\mathcal{F}'_{\alpha 0} \subset \mathcal{B}$ be an irreducible finite cover of $C'_{\alpha 0}$ and let $\mathcal{F}_{\alpha 0} = \mathcal{F}'_{\alpha 0} \setminus \mathcal{O}'_\alpha$. Then $\mathcal{F}_{\alpha 0} \neq \emptyset$ because of our assumption of Case 2. Take an open set U of $C'_{\alpha 0}$ satisfying $C'_{\alpha 0} \setminus \bigcup \mathcal{O}'_\alpha \subset U \subset \overline{U} \subset \bigcup \mathcal{F}_{\alpha 0}$. Put $C_{\alpha 0} = \overline{U}$ and $C_1 = C'_{\alpha 0} \setminus U$. Then C_1 is a compact set contained in $\bigcup \mathcal{O}'_\alpha$. Hence by the note above Case 1, there is a finite subcollection $\mathcal{C}'_{\alpha 1}$ of $\mathcal{C}'_\alpha$ such that $\{f(C) : C \in \mathcal{C}'_{\alpha 1}\}$ covers C_1. Then $\overline{f(B_\alpha)} = f(C'_{\alpha 0}) \subset f(C_{\alpha 0}) \cup (\bigcup \{f(C) : C \in \mathcal{C}'_{\alpha 1}\})$ and $\mathcal{F}_{\alpha 0} \cap \mathcal{O}'_\alpha = \emptyset$.

B. Assume that we have $\mathcal{F}_{\alpha n}$ and $\mathcal{C}_{\alpha n}$. Let $C'_{\alpha n+1}$ be a compact subset of M with $f(C'_{\alpha n+1}) = \bigcup \{\overline{f(B)} : B \in \mathcal{F}_{\alpha n}\}$. Let $\mathcal{F}'_{\alpha n+1} \subset \mathcal{B}$ be an irreducible finite cover of $C'_{\alpha n+1}$. Then, just as the proof of A of Case 2, let $\mathcal{F}_{\alpha n+1} = \mathcal{F}'_{\alpha n+1} \setminus \mathcal{O}'_\alpha$. Then there is a compact subset $C_{\alpha n+1}$ of $C'_{\alpha n+1}$ and a finite subcollection $\mathcal{C}'_{\alpha n+1}$ of $\mathcal{C}'_\alpha$ such that $f(C'_{\alpha n+1}) \subset f(C_{\alpha n+1}) \cup (\bigcup \{f(C) : C \in \mathcal{C}'_{\alpha n+1}\})$ and $\mathcal{F}_{\alpha n+1} \cap \mathcal{O}'_\alpha = \emptyset$.

Then, by induction, we have $\mathcal{C}_{\alpha n}$ $(n \in \omega)$ and $\mathcal{F}_{\alpha n}$ $(n \in \omega)$ such that for each $B \in \bigcup_{n \in \omega} \mathcal{F}_{\alpha n}$, $\overline{f(B)} \subset f(C_{\alpha n+1}) \cup (\bigcup \{f(C) : C \in \mathcal{C}'\})$ for some finite subcollection $\mathcal{C}' \subset \mathcal{C}'_\alpha$. Let $\mathcal{O}_\alpha = \bigcup_{n \in \omega} \mathcal{F}_{\alpha n}$ and $\mathcal{C}_\alpha = \{C_{\alpha n} : n \in \omega\}$.

Then, by induction, for each $\alpha \in \Omega$ we have $\mathcal{O}_\alpha$ and $\mathcal{C}_\alpha$ such that for each $B \in \mathcal{O}_\alpha$, there is a $C_{\alpha n+1} \in \mathcal{C}_\alpha$ and a finite subcollection $\mathcal{C}' \subset \bigcup_{\beta < \alpha} \mathcal{C}_\beta$ with $\overline{f(B)} \subset f(C_{\alpha n+1}) \cup (\bigcup \{f(C) : C \in \mathcal{C}'\})$ and $\mathcal{O}_\alpha \cap \mathcal{O}'_\alpha = \emptyset$. Let $\mathcal{O} = \bigcup_{\alpha \in \Omega} \mathcal{O}_\alpha$ and $\mathcal{C} = \bigcup_{\alpha \in \Omega} \mathcal{C}_\alpha$.

Claim 3.2. *$\mathcal{K} = \{K = f(C) : C \in \mathcal{C}\}$ is point-countable.*

Proof. Pick a $y \in Y$. Let $\mathcal{B}_y = \{B \in \mathcal{B} : f^{-1}(y) \cap B \neq \emptyset\} = \{B_n : n \in \omega\}$ since f is an s-map. Then $f^{-1}(y) \cap B = \emptyset$ if B is not in $\mathcal{B}_y$. Assume $B_n \in \mathcal{O}_{\alpha(n)}$ for $n \in \omega$. Then $f^{-1}(y) \cap (\bigcup \mathcal{O}_\alpha) = \emptyset$ if $\alpha \notin \{\alpha(n) : n \in \omega\}$ since $\mathcal{O}_\alpha \cap \mathcal{B}_y = \emptyset$. If $C_{\alpha m} \subset \bigcup \mathcal{F}_{\alpha m}$ and $\mathcal{F}_{\alpha m} \subset \mathcal{O}_\alpha$, then $C_{\alpha m} \cap f^{-1}(y) = \emptyset$ for each $m \in \omega$. So $C_{\alpha m} \cap f^{-1}(y) = \emptyset$ for each $C_{\alpha m} \in \bigcup_{\alpha \notin \{\alpha(n) : n \in \omega\}} \mathcal{C}_\alpha$. Notice that $\mathcal{C} \setminus \bigcup_{\alpha \notin \{\alpha(n) : n \in \omega\}} \mathcal{C}_\alpha = \bigcup_{n \in \omega} \mathcal{C}_{\alpha(n)}$ is countable. Then $\mathcal{K}$ is point-countable.

Claim 3.3. *Y has a point-countable compact k-network.*

Proof. Let $M_1 = \bigoplus \mathcal{K}$. Then M_1 is a locally compact metric space. Let $g : M_1 \to Y$ be the obvious map. Then g is a compact-covering s-map since $\mathcal{P}$ is a k-network. Let $\mathcal{B}$ be a σ-discrete base of M_1 refining $\mathcal{K}$. Then $\{f(\overline{B}) : B \in \mathcal{B}\}$ is a point-countable compact k-network.

Corollary 3.4. *Let a Hausdorff space Y be a compact-covering s-image of metric space. If Y has a point-countable k-network $\mathcal{P}$ with $\overline{P}$ compact for each $P \in \mathcal{P}$, then Y has a point-countable compact k-network.*

Proof. Let M be a metric space and $f : M \to Y$ be an onto compact-covering s-map. Let $\mathcal{B} = \bigcup_n \mathcal{B}_n$ be a σ-locally finite base of M. Pick an $x \in M$. Let $f(x) = y$. Let $\{B_n : n \in \omega\}$ be a decreasing open neighborhood base of x, and let $\mathcal{P}_y = \{P \in \mathcal{P} : y \in P\} = \{P_n : n \in \omega\}$.

Case 1. If $f^{-1}(y)$ contains an open neighborhood of x in X, then there is a $B_n \in \mathcal{B}_x$ with $x \in B_n \subset f^{-1}(y)$. So $f(B_n) = \{y\} \subset P_n$ for some $P_n \in \mathcal{P}_y$.

Case 2. There is a sequence S which converges to x with $S \subset M \setminus f^{-1}(y)$. Suppose that there is a $y_n \in f(B_n) \setminus \bigcup_{i \leq n} P_i \neq \emptyset$ for each $n \in \omega$. Then there is an $x_n \in B_n$ with $f(x_n) = y_n$ for each $n \in \omega$. Since $\{x_n : n \in \omega\}$ converges to x, $S = \{y_n : n \in \omega\}$ converges to $f(x) = y$. So there is an $n \in \omega$ such that $\bigcup_{i \leq n} P_i$ contains S eventually, a contradiction. So there is an n with $\overline{f(B_n)} \subset \overline{\bigcup_{i \leq n} P_i}$.

Then for each $x \in M$, there is a $B \in \mathcal{B}$ such that $x \in B$ and $\overline{f(B)}$ is compact. Then there is a base $\mathcal{B}' \subset \mathcal{B}$ with $\overline{f(B)}$ compact for each $B \in \mathcal{B}'$. Since $f(\mathcal{B}')$ is a point-countable strong k-network by Theorem 2.2, Y has a point-countable compact k-network by Theorem 3.1.

We would like to ask whether or not the condition "compact-covering s-image" can be changed to "quotient s-image" even if the separation axiom is strengthened to regular T_1. So the following question is raised.

Question 3.5. *Let a regular T_1 space Y be a quotient s-image of a metric space. Does Y have a point-countable compact k-network if Y has a point-countable k-network $\mathcal{P}$ with $\overline{P}$ compact for each $P \in \mathcal{P}$?*

We can prove that Question 3.5 above is equivalent to Problem 1.2 in the Introduction.

Proposition 3.6. *Let a regular T_1 space Y be a quotient s-image of a metric space. Then the following are equivalent:*

1. *Y has a point-countable k-network $\mathcal{P}$ with $\overline{P}$ compact for each $P \in \mathcal{P}$.*
2. *Every first countable closed subspace of Y is locally compact.*

Proof. $(2 \Rightarrow 1)$. Let M be a metric space and $f : M \to Y$ be an onto quotient s-map. Let $\mathcal{B} = \bigcup_n \mathcal{B}_n$ be a σ-locally finite base of M and $\{B_n : n \in \omega\}$ be a decreasing open neighborhood base of x for $x \in M$. Suppose that there is an $x \in X$ such that $\overline{f(B_n)}$ is not compact for each $n \in \omega$. Then $\overline{f(B_n)}$ is not countably compact by Theorem 4.1 in [3] since Y is a quotient s-image of a metric space. So there is an infinitely countable discrete closed subset $D_n \subset \overline{f(B_n)}$. Let $D = (\bigcup_{n \in \omega} D_n) \cup \{f(x)\}$.

Let O be open in Y with $y \in O$. Then there is an $n \in \omega$ such that $\overline{f(B_i)} \subset O$ for each $i > n$ since Y is regular. So $\bigcup_{i > n} D_i \cup \{y\} \subset O$. Then D is first countable and has only one cluster point y in Y. So D is closed in Y.

But D is not locally compact since each neighborhood of y contains an infinitely countable discrete closed subset, a contradiction.

So there is a $B_n \in \mathcal{B}_x$ with $\overline{f(B_n)}$ compact for each $\mathcal{B}_x$. Then there is a base $\mathcal{B}' \subset \mathcal{B}$ with $\overline{f(B)}$ compact for each $B \in \mathcal{B}'$. Since f is a quotient s-map, it is easy to check that $f(\mathcal{B}')$ is a point-countable k-network by Theorem 6.1 and Proposition 2.1 of [3].

$(1 \Rightarrow 2)$. Let $B \subset Y$ be a first countable closed subspace. Pick a $y \in B$. Let $\mathcal{P}(y) = \{P \in \mathcal{P} : y \in P\} = \{P_n : n \in \omega\}$ and $B = O_1 \supset O_2 \supset ...$ be a neighborhood base of y in B. If, for each $n \in \omega$, there is a $y_n \in O_n \setminus (\bigcup_{i \leq n} P_i)$, then $S = \{y_n : n \in \omega\}$ converges to y. Since $S \cup \{y\}$ is compact, then $\bigcup_{i \leq m} P_i$ eventually contains S. So there is a $P_i \in \mathcal{P}(y)$ such that $P_i \cap S$ is infinite, a contradiction. This implies that there is an n with $O_n \subset \bigcup_{i \leq n} P_i$. So B is locally compact since $\overline{\bigcup_{i \leq n} P_i}$ is compact.

4. Counterexamples

L. Foged in [2] presented the following example.

Example 4.1. There is a completely regular space X which has a point-countable base but no point-countable closed k-network.

Claim 4.2. *Any base $\mathcal{B}$ of a regular T_1 space is a strong k-network.*

Proof. Let K be compact and U open with $K \subset U$. For each $x \in K \subset U$, there is a $B_x \in \mathcal{B}$ with $x \in B_x \subset U$. Since X is regular, B_x contains a closed neighborhood of x in X. So there is an open set U_x with $x \in U_x \subset \overline{U_x} \subset B_x$. Since $\bigcup_{x\in K} U_x$ contains K, there is a finite subset $\{x_i : i \leq n\}$ of K with $K \subset \bigcup_{i\leq n} U_{x_i} \subset \bigcup_{i\leq n} \overline{U_{x_i}} \subset \bigcup_{i\leq n} B_{x_i} \subset U$. Here $U_{x_i} \subset \overline{U_{x_i}} \subset B_{x_i} \subset U$ for each $i \leq n$. So $\mathcal{B}$ is also a strong k-network.

It follows from Claim 4.2 that Example 4.1 is a space with a point-countable strong k-network which does not have a point-countable closed k-network. We give an example of a regular T_1 space with a point-countable k-network which does not have a point-countable strong k-network. It gives a negative answer to Problem 1.1 and Problem 1.2 in the Introduction. For this reason, we give a set theoretical assuption.

Definition 4.3. A subset W of the space R of real numbers with the usual topology is called a σ'-set if and only if for each G_δ-set G of R, there is an F_σ-set F of R with $G \cap W \subset F \subset G$.

Proposition 4.4. *Each Sierpinski set is a σ'-set and each σ'-set is a σ-set.*

Proof. Assume that W is a Sierpinski set. Then one can prove that W is a σ'-set just as in the proof of Theorem 4.1 of [12]. It is easy to see that each σ'-set is a σ-set.

Example 4.5. Suppose that there is an uncountable σ'-set. Then there is a quotient s-image X of a metric space such that:

1. X is a regular T_1 sequential space and is the union of countably many compact metric subsets of X.
2. X has a point-countable k-network $\mathcal{P}$ with $\overline{P}$ compact for each $P \in \mathcal{P}$.
3. Every first countable closed subspace of X is locally compact.
4. X is not a compact-covering s-image of any metric space.
5. X does not have a point-countable strong k-network.

Construction. Let i, j, l, m and n be the members of ω. Let $[a, b) = \{r : r$ is a real with $a \leq r < b\}$ and $A \times B = \{\langle a, b\rangle : a \in A$ and $b \in B\}$.

Pick an $n \in \omega$. For each $m < 2^n$, let $l(n, m) = [m/2^n, (m+1)/2^n] \times \{1/2^{2n}\}$ and $x(n, m) = \langle (2m + 1)/2^{n+1}, 1/2^{2n+1}\rangle$. Let $T(n, m) \subset [0, 1] \times [0, 1]$ be the triangle with side $l(n, m)$ and vertex $x(n, m)$.

Let $l'(n, m) = [m/2^n, (m + 1)/2^n]$. Let $\mathcal{T}_n = \{T(n, m) : m < 2^n\}$ and $\mathcal{K}_1 = \bigcup\{\mathcal{T}_n : n \in \omega\}$. Then $\mathcal{K}_1$ is a collection of compact subsets. Let $C_1 = \{x(n, m) : n \in \omega$ and $m < 2^n\} \cup ([0, 1] \times \{0\}) \subset [0, 1] \times [0, 1]$. Then C_1 is compact. Let $\mathcal{K}_2 = \{C_1\}$.

Let W be an uncountable σ'-set of $[0, 1]$ with $W \cap Q = \emptyset$. Here Q is the set of all rational numbers. Pick a $y \in W$. Then uniquely there is a sequence

$l'(1,m_1) \supset l'(2,m_2) \supset \dots \supset l'(n,m_n) \supset \dots$ with $\bigcap_{n\in\omega} l'(n,m_n) = \{y\}$. Let $p(y,n) = \langle y, 1/2^{2n}\rangle$ and $p(y,\omega) = \langle y, 0\rangle$. Then $p(y,n) \in l(n,m) \subset T(n,m)$. Let $V(y,n)$ be the segment with endpoints $x(n,m_n)$ and $p(y,n)$. Then $V(y,n) \subset T(n,m)$ is a compact set. Let $L'_y = \{p(y,\omega)\} \cup \bigcup_{n\in\omega} V(y,n) \subset [0,1]\times[0,1]$. Then L'_y is compact. Let $L_y = L'_y \setminus \{x(n,m_n) : n \in \omega\}$ and $\mathcal{K}_3 = \{L_y : y \in W\}$. Let $\mathcal{K} = \bigcup_{i\le 3} \mathcal{K}_i$ and let $X = \bigcup \mathcal{K}$ as a set.

Let $M = \bigoplus \mathcal{K}$. Then M is a metric space. Let X have the quotient topology induced by the obvious map $f : M \to X$. Then f is a two-to-one quotient map and X is a Hausdorff sequential space. So X has a point-countable k-network since $f(\mathcal{K})$ is a point-countable cover which determines the topology of X, by Proposition 2.7 in [3] and by the implication (1.5)$\longrightarrow$(1.4) in Diagram I of [3].

Let $\mathcal{C} = \{C = \overline{f(K)} : K \in \mathcal{K}\}$.

Claim 4.6. *$\mathcal{C}$ is a collection of compact subsets of X.*

Proof. Case 1. $K = C_1$ or $K = T(n,m)$. Then K is compact in M. So $f(K) = \overline{f(K)}$ is compact.

Case 2. $K = \overline{f(L_y)} = L'_y$. Let A be an infinite subset of K. If $A \cap V(y,n)$ is infinite for some n, then $A \cap V(y,n)$ has a cluster in the compact set $V(y,n)$. If $A \cap V(y,n)$ is finite for each n, then $p(y,\omega)$ is a cluster of A in $f(L_y)$ since $p(y,\omega)$ is in $f(L_y)$. So $K = \overline{f(L_y)} = L'_y$ is compact by Theorem 4.1 of [3].

Claim 4.7. *A set O is open if and only if $O \cap C$ is open for each $C \in \mathcal{C}$.*

Proof. If O is open, then $O \cap C$ is open for each $C \in \mathcal{C}$. On the other hand, if O is not open, then there is a convergence sequence $S \cup \{x\}$ such that $(S \cup \{x\}) \cap O$ is not open in $S \cup \{x\}$ since X is a Hausdorff sequential space. Then there is a finite subcollection $\mathcal{F} \subset f(\mathcal{K})$ with $S \cup \{x\} \subset \bigcup \mathcal{F}$ by Proposition 2.1 of [3]. Then there is a $f(K) \in \mathcal{F}$ with $\overline{f(K)} \cap O$ not open in $\overline{f(K)}$ since $C = \overline{f(K)}$ is compact for each $f(K) \in \mathcal{F}$.

Claim 4.8. *X is a regular Lindelöf space.*

Proof. Since $\mathcal{K}_1 \cup \mathcal{K}_2$ is a countable cover of X by compact sets, X is σ-compact, hence every open cover has a countable subcover. It suffices to show that X is a regular space. Let $x \in X$ and U be a neighborhood of x in X. Let $I_0 = [0,1]\times\{0\} \subset X$. Since every point of $X \setminus I_0$ in X has the usual Euclidean neighborhoods, we may assume that $x \in I_0$. Take a closed interval J of I_0 with $x \in J \subset U$ such that x is in the interior of the interval J. Then the pair $\langle X \cap ([0,1]\times J), J\rangle$ can be considered as similar to the pair $\langle X, I_0\rangle$. Thus to show that X is a regular space, let O be an open set of X such that $I_0 \subset O$. If we can prove that there is an open neighborhood U with $I_0 \subset U \subset \overline{U} \subset O$, then X is regular T_1. Let $B = X \setminus O$. Then $B \cap I_0 = \emptyset$. Pick a $p(y,\omega) \in I_0$ for each $y \in W$. Then there is an $n(y) \in \omega$ with $\{p(y,\omega)\} \cup (\bigcup_{n \ge n(y)} V(y,n)) \subset O$ since $L'_y = \overline{f(L_y)}$ is compact by Claim 4.6. Let $A_n = \{p(y,\omega) : V(y,n+i) \subset O \text{ for each } i \in \omega\}$. Then $A_1 \subset A_2 \subset \dots \subset A_n \subset \dots$ and $W = \bigcup_{n\in\omega} A_n$. Since $I_0 \subset O$ and $C_1 = \{x(n,m) : n \in \omega \text{ and } m < 2^n\} \cup ([0,1]\times\{0\}) \subset [0,1]\times[0,1]$ is compact, then there is an $n(O) \in \omega$ with $D_1 = \{x(n,m) : \text{ for each } n > n(O) \text{ and } m < 2^n\} \subset O$. Giving an $n > n(O)$, take an open neighborhood $O(x(n,m))$ of $x(n,m)$ in X with

$O(x(n,m)) \subset T(n,m) \cap O$ and $O(x(n,m)) \cap l(n,m) = \emptyset$ for each $m < 2^n$. Let

$$g_n : \bigcup_{m<2^n} (T(n,m) \setminus O(x(n,m))) \to [0,1] \times \{1/2^{2n}\} = I_n$$

such that $g_n(V(y,n) \setminus O(x(n,m))) = \{y\} \times \{1/2^{2n}\}$ for each $y \in [0,1]$. Then g_n is a perfect map. Then $g_n((\bigcup_{m<2^n} T(n,m)) \cap B) = B_n \subset I_n$ is compact.

Let $f_n : I_n \to [0,1] \times \{0\} = I_0$ with $f_n(\langle y, 1/2^{2n}\rangle) = \langle y, 0\rangle$. If $p(y,\omega) \in A_n$, then $V(y, n{+}i) \subset O$ and $p(y, n{+}i) \in I_{n+i} \setminus B_{n+i} = O_{n+i}$. Then $G_n = \bigcap_{i\in\omega} f_{n+i}(O_{n+i}) \subset I_0$ is a G_δ-set containing A_n. Since W is a σ'-set, then there is a collection $\{K_{nm} : m \in \omega\}$ of compact subsets of I_0 with $G_n \cap W \subset \bigcup\{K_{nm} : m \in \omega\} \subset G_n$. So $\bigcup\{K_{nm} : m + n \leq l,$ and $n > n(O)\} \subset f_l(O_l)$ for each $l > n(O)$. Then $f_l^{-1}(\bigcup\{K_{nm} : m + n \leq l,$ and $n > n(O)\}) \subset O_l = I_l \setminus B_l$ for each $l > n(O)$. Let $T_l = \bigcup_{m<2^l} T(l,m)$, and let $K_l = \bigcup\{V(x,l) : x \in f_l^{-1}(\bigcup\{K_{nm} : m{+}n \leq l$ and $n > n(O)\})\} \cup \{x(l,m) : m < 2^l\}$. Then K_l is a compact subset with $K_l \subset O \cap T_l$. Then there is an open neighborhood U_l of K_l in X with $U_l \subset \overline{U_l} \subset O \cap T_l$ since T_l is a closed open compact subset of X.

Let $U = I_0 \cup \bigcup_{n>n(O)} U_n$.

We can prove that U is an open set in Y. It is sufficient to prove that $U \cap C$ is open in C for each $C \in \mathcal{C}$ by Claim 4.7. To do it pick a $C \in \mathcal{C}$.

Case 1. $C = C_1 = \{x(n,m) : n \in \omega$ and $m < 2^n\} \cup ([0,\ 1] \times \{0\})$. Then $C \cap U = D_1 \cup I_0$ is open in C.

Case 2. $C = T(n,m) \subset T_n$. Then $C \cap U = U_n \cap T(n,m)$ is open in $T(n,m)$ since U_n is open in X. So it is open in C.

Case 3. $C = L'_y = \{p(y,\omega)\} \cup \bigcup_{n\in\omega} V(y,n)$. By Case 2, it suffices to show that if $p(y,\omega) \in C \cap U$, then $C \cap U$ is a neighborhood of $p(y,\omega)$ in C. Then there is an $m \in \omega$ with $p(y,\omega) \in K_{n(y)m}$. Then $C \cap U \supset \bigcup_{l\geq n(y)+m} U_l \supset \bigcup_{l\geq n(y)+m} K_l \supset \bigcup_{l\geq n(y)+m} V(y,n)$. Hence $C \cap U$ is a neighborhood of $p(y,\omega)$ in C, which implies that $C \cap U$ is open in $C = L'_y$.

Notice that $\overline{U} = \overline{I_0 \cup (\bigcup_{n>n(O)} U_n)} = I_0 \cup (\bigcup_{n>n(O)} \overline{U_n}) \subset O$ since $I_0 \subset O$ and $\overline{U_n} \subset O \cap T_n \subset O$ for each $n > n(O)$. Then X is regular T_1.

Proof of 1–5 *in Example* 4.5. We have already proved properties 1 and 2. Property 3 follows from 2 and Proposition 3.6. To show 4 and 5, it suffices to show that X does not have a point-countable compact k-network. Indeed, suppose that 4 (resp. 5) is not true. Then by property 2, Theorem 2.2 and Theorem 3.1 (resp. by property 2 and Theorem 3.1), X has a point-countable compact k-network.

Now assume that X has a point-countable compact k-network $\mathcal{P}$. Let

$$D = \{x(n,m) : n \in \omega \text{ and } m < 2^n\} = C_1 \setminus [0,1] \times \{0\}.$$

Let $\mathcal{P}_1 = \{P \in \mathcal{P} : P \cap D \neq \emptyset\}$ and $\mathcal{P}_2 = \{P \in \mathcal{P} : P \cap D = \emptyset\}$. Then $\mathcal{P}_1$ is countable since D is countable and $\mathcal{P}$ is point-countable. Let $\mathcal{P}_1 = \{P_n : n \in \omega\}$. Pick an $L'_y \in \mathcal{C}$. Then there is a finite subcollection $\mathcal{P}_y$ of $\mathcal{P}$ with $L'_y \subset \bigcup \mathcal{P}_y$. Let $\mathcal{P}_{y1} = \mathcal{P}_y \cap \mathcal{P}_1$. Since $D \cap L'_y \subset \bigcup \mathcal{P}_{y1}$ and $p(y,\omega) \in \overline{D \cap L'_y}$, we have $p(y,\omega) \in \overline{\bigcup \mathcal{P}_{y1}}$. Furthermore, since $\bigcup(\mathcal{P}_y \setminus \mathcal{P}_1)$ is a compact set missing D, there is a sequence $S_y \subset L'_y \cap (\bigcup \mathcal{P}_{y1}) \setminus D$ converging to $p(y,\omega)$. Since $\mathcal{P}_1$ is countable and W is uncountable, there is a finite subcollection $\mathcal{F}$ of $\mathcal{P}$ and a countably infinite subset W' of W such that $\mathcal{P}_{y1} = \mathcal{F}$ for each $y \in W'$. Let $W' = \{y_n : n \in \omega\}$. Note that $\{S_{y_n} : n \in \omega\}$ is a disjoint collection. Now we can take a sequence $\{z_n : n \in \omega\}$

such that $z_n = \langle a_n, b_n \rangle \in S_{y_n}$ and b_n converges to 0. Let $S = \{z_n : n \in \omega\}$. Then we can show that:

1. S is a discrete closed set since $S \cap C$ is finite for each $C \in \mathcal{C} = \{C = \overline{f(K)} : K \in \mathcal{K}\}$, and

2. S has at least a cluster point since S is an infinite subset of a compact set $\bigcup \mathcal{F}$.

It is a contradiction.

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