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TRANSFINITE SEQUENCES OF CONTINUOUS AND BAIRE CLASS 1 FUNCTIONS

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Abstract. The set of continuous or Baire class 1 functions defined on a metric space X is endowed with the natural pointwise partial order. We investigate how the possible lengths of well-ordered monotone sequences (with respect to this order) depend on the space X.

Introduction

Any set $\mathcal{F}$ of real valued functions defined on an arbitrary set X is partially ordered by the pointwise order; that is, $f \le g$ iff $f(x) \le g(x)$ for all $x \in X$. Then, $f < g$ iff $f \le g$ and $g \not\le f$; equivalently, $f(x) \le g(x)$ for all $x \in X$ and $f(x) < g(x)$ for at least one $x \in X$. Our aim will be to investigate the possible lengths of the increasing or decreasing well-ordered sequences of functions in $\mathcal{F}$ with respect to this order. A classical theorem (see Kuratowski [7], §24.III, Theorem 2′) asserts that if $\mathcal{F}$ is the set of Baire class 1 functions (that is, pointwise limits of continuous functions) defined on a Polish space X (that is, a complete separable metric space), then there exists a monotone sequence of length ξ in $\mathcal{F}$ iff $\xi < \omega_1$. P. Komjáth [5] proved that the corresponding question concerning Baire class α functions for $2 \le \alpha < \omega_1$ is independent of *ZFC*.

In the present paper we investigate what happens if we replace the Polish space X by an arbitrary metric space.

Section 1 considers chains of continuous functions. We show that for any metric space X, there exists a chain in $C(X, \mathbb{R})$ of order type ξ iff $|\xi| \le d(X)$. Here, $|A|$ denotes the cardinality of the set A, while $d(X)$ denotes the density of the space X, that is,

$$d(X) = \max(\min\{|D| : D \subseteq X \ \&\ \overline{D} = X\},\ \omega).$$

In particular, for separable X, every well-ordered chain has countable length, just as for Polish spaces.

Section 2 considers chains of Baire class 1 functions on separable metric spaces. Here, the situation is entirely different from the case of Polish spaces, since on some

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separable metric spaces there are well-ordered chains of every order type less than ω_2. Furthermore, the existence of chains of type ω_2 and longer is independent of $ZFC + \neg CH$. Under MA, there are chains of all types less than $\mathfrak{c}^+$, whereas in the Cohen model, all chains have type less than ω_2.

We note here that instead of examining well-ordered sequences, which is a classical problem, we could try to characterize all the possible order types of linearly ordered subsets of the partially ordered set $\mathcal{F}$. This problem was posed by M. Laczkovich, and is considered in detail in [3].

1. Sequences of continuous functions

Lemma 1.1. *For any topological space X: If there is a well-ordered sequence of length ξ in $C(X,\mathbb{R})$, then $\xi < d(X)^+$.*

Proof. Let $\{f_\alpha : \alpha < \xi\}$ be an increasing sequence in $C(X,\mathbb{R})$, and let $D \subseteq X$ be a dense subset of X such that $d(X) = \max(|D|, \omega)$. By continuity, the $f_\alpha \restriction D$ are all distinct; so, for each $\alpha < \xi$, choose a $d_\alpha \in D$ such that $f_\alpha(d_\alpha) < f_{\alpha+1}(d_\alpha)$. For each $d \in D$ the set $E_d = \{\alpha : d_\alpha = d\}$ is countable, because every well-ordered subset of $\mathbb{R}$ is countable. Since $\xi = \bigcup_{d \in D} E_d$, we have $|\xi| \le \max(|D|, \omega) = d(X)$. $\square$

The converse implication is not true in general. For example, if X has the countable chain condition (ccc), then every well-ordered chain in $C(X,\mathbb{R})$ is countable (because $X \times \mathbb{R}$ is also ccc). However, the converse is true for metric spaces:

Lemma 1.2. *If (X, ϱ) is any non-empty metric space and $\prec$ is any total order of the cardinal $d(X)$, then there is a chain in $C(X,\mathbb{R})$ which is isomorphic to $\prec$.*

Proof. First, note that every countable total order is embeddable in $\mathbb{R}$, so if $d(X) = \omega$, then the result follows trivially using constant functions. In particular, we may assume that X is infinite, and then fix $D \subseteq X$ which is dense and of size $d(X)$. For each $n \in \omega$, let D_n be a subset of D which is maximal with respect to the property $\forall d, e \in D_n\, [d \neq e \rightarrow \varrho(d,e) \ge 2^{2-n}]$. Then $\bigcup_n D_n$ is also dense, so we may assume that $\bigcup_n D_n = D$. We may also assume that $\prec$ is a total order of the set D. Now, we shall produce $f_d \in C(X,\mathbb{R})$ for $d \in D$ such that $f_d < f_e$ whenever $d \prec e$.

For each n, if $c \in D_n$, define $\varphi^n_c(x) = \max(0,\ 2^{-n} - \varrho(x,c))$. For each $d \in D$, let $\psi^n_d = \sum\{\varphi^n_c : c \in D_n \,\&\, c \prec d\}$. Since every $x \in X$ has a neighborhood on which all but at most one of the φ^n_c vanish, we have $\psi^n_d \in C(X, [0, 2^{-n}])$, and $\psi^n_d \le \psi^n_e$ whenever $d \prec e$. Thus, if we let $f_d = \sum_{n<\omega} \psi^n_d$, we have $f_d \in C(X, [0,2])$, and $f_d \le f_e$ whenever $d \prec e$. But also, if $d \in D_n$ and $d \prec e$, then $\psi^n_d(d) = 0 < 2^{-n} = \psi^n_e(d)$, so actually $f_d < f_e$ whenever $d \prec e$. $\square$

Putting these lemmas together, we have:

Theorem 1.3. *Let (X, ϱ) be a metric space. Then there exists a well-ordered sequence of length ξ in $C(X,\mathbb{R})$ iff $\xi < d(X)^+$.*

Corollary 1.4. *A metric space (X, ϱ) is separable iff every well-ordered sequence in $C(X,\mathbb{R})$ is countable.*

2. Sequences of Baire class 1 functions

If we replace continuous functions by Baire class 1 functions, then Corollary 1.4 becomes false, since on some separable metric spaces we can get well-ordered sequences of every type less than ω_2. To prove this, we shall apply some basic facts about $\subset^*$ on $\mathcal{P}(\omega)$. As usual, for $x, y \subseteq \omega$, we say that $x \subseteq^* y$ iff $x \backslash y$ is finite. Then $x \subset^* y$ iff $x \backslash y$ is finite and $y \backslash x$ is infinite. This $\subset^*$ partially orders $\mathcal{P}(\omega)$.

Lemma 2.1. *If $X \subset \mathcal{P}(\omega)$ is a chain in the order $\subset^*$, then on X (viewed as a subset of the Cantor set $2^\omega \cong \mathcal{P}(\omega)$) there is a chain of Baire class 1 functions which is isomorphic to $(X, \subset^*)$.*

Proof. Note that for each $x \in X$,

$$\{y \in X : y \subseteq^* x\} = \bigcup_{m \in \omega} \{y \in X : \forall n \geq m\, [y(n) \leq x(n)]\},$$

which is an F_σ set in X. Likewise, the sets $\{y \in X : y \supseteq^* x\}$, $\{y \in X : y \subset^* x\}$, and $\{y \in X : y \supset^* x\}$, are all F_σ sets in X, and hence also G_δ sets. It follows that if $f_x : X \to \{0, 1\}$ is the characteristic function of $\{y \in X : y \subset^* x\}$, then $f_x : X \to \mathbb{R}$ is a Baire class 1 function. Then $\{f_x : x \in X\}$ is the required chain. $\square$

Lemma 2.2. *For any infinite cardinal κ, suppose that $(\mathcal{P}(\omega), \subset^*)$ contains a chain $\{x_\alpha : \alpha < \kappa\}$ (i.e., $\alpha < \beta \to x_\alpha \subset^* x_\beta\}$). Then $(\mathcal{P}(\omega), \subset^*)$ contains a chain X of size κ such that every ordinal $\xi < \kappa^+$ is embeddable into X.*

Proof. Let $S = \bigcup_{1 \leq n < \omega} \kappa^n$. For $s = (\alpha_1, \ldots, \alpha_{n-1}, \alpha_n) \in S$, let

$$s^+ = (\alpha_1, \ldots, \alpha_{n-1}, \alpha_n + 1).$$

Starting with the $x_{(\alpha)} = x_\alpha$, choose $x_s \in \mathcal{P}(\omega)$ by induction on length(s) so that

$$x_s = x_{s^\frown 0} \subset^* x_{s^\frown \alpha} \subset^* x_{s^\frown \beta} \subset^* x_{s^+}$$

whenever $s \in S$ and $0 < \alpha < \beta < \kappa$. Let $X = \{x_s : s \in S\}$. Then, whenever $x, y \in X$ with $x \subset^* y$, the ordinal κ is embeddable in $(x, y) = \{z \in X : x \subset^* z \subset^* y\}$. From this, one easily proves by induction on $\xi < \kappa^+$ (using $\mathrm{cf}(\xi) \leq \kappa$) that ξ is embeddable in each such interval (x, y). $\square$

Since $\mathcal{P}(\omega)$ certainly contains a chain of type ω_1, these two lemmas yield:

Theorem 2.3. *There is a separable metric space X on which, for every $\xi < \omega_2$, there is a well-ordered chain of length ξ of Baire class 1 functions.*

Under *CH*, this is best possible, since there will be only $2^\omega = \omega_1$ Baire class 1 functions on a separable metric space, so there could not be a chain of length ω_2. Under $\neg CH$, the existence of longer chains of Baire class 1 functions depends on the model of set theory. It is consistent with $\mathfrak{c} = 2^\omega$ being arbitrarily large that there is a chain in $(\mathcal{P}(\omega), \subset^*)$ of type $\mathfrak{c}$; for example, this is true under *MA* (see [2]). In this case, there will be a separable X with well-ordered chains of all lengths less than $\mathfrak{c}^+$. However, in the Cohen model, where $\mathfrak{c}$ can also be made arbitrarily large, we never get chains of type ω_2. We shall prove this by using the following lemma, which relates it to the rectangle problem:

Lemma 2.4. *Suppose that there is a separable metric space Y with an ω_2-chain of Borel subsets, $\{B_\alpha : \alpha < \omega_2\}$ (so, $\alpha < \beta \to B_\alpha \subsetneqq B_\beta$). Then in $\omega_2 \times \omega_2$, the well-order relation $<$ is in the σ-algebra generated by the set of all rectangles, $\{S \times T : S, T \in \mathcal{P}(\omega_2)\}$.*

Proof. Each B_α has some countable Borel rank. Since there are only ω_1 ranks, we may, by passing to a subsequence, assume that the ranks are bounded. Say each B_α is a Σ^0_μ set for some fixed $\mu < \omega_1$.

Let $J = \omega^\omega$, and let $A \subseteq Y \times J$ be a universal Σ^0_μ set; that is, A is Σ^0_μ in $Y \times J$ and every Σ^0_μ subset of Y is of the form $A^j = \{y : (y, j) \in A\}$ for some $j \in J$ (see [7], §31). Now, for $\alpha, \beta < \omega_2$, fix $y_\alpha \in B_{\alpha+1} \backslash B_\alpha$, and fix $j_\beta \in J$ such that $A^{j_\beta} = B_\beta$. Then $\alpha < \beta$ iff $(y_\alpha, j_\beta) \in A$. Thus, $\{(y_\alpha, j_\beta) : \alpha < \beta < \omega_2\}$ is a Borel subset of $\{y_\alpha : \alpha < \omega_2\} \times \{j_\beta : \beta < \omega_2\}$, and is hence in the σ-algebra generated by open rectangles, so $<$, as a subset of $\omega_2 \times \omega_2$, is in the σ-algebra generated by rectangles. □

Theorem 2.5. *Assume that the well-order relation $<$ on ω_2 is not in the σ-algebra generated by the set of all rectangles. Then no separable metric space can have a chain of length ω_2 of Baire class 1 functions.*

Proof. Suppose that $\{f_\alpha : \alpha < \omega_2\}$ is a chain of Baire class one functions on the separable metric space X. Let $B_\alpha = \{(x, r) \in X \times \mathbb{R} : r \le f_\alpha(x)\}$. Then the B_α form an ω_2-chain of Borel subsets of the separable metric space $X \times \mathbb{R}$, so we have a contradiction by Lemma 2.4. □

Finally, we point out that the hypothesis of this theorem is consistent, since it holds in the extension $V[G]$ formed by adding $\ge \omega_2$ Cohen reals to a ground model V which satisfies *CH*. This fact was first proved in [6]. It also follows from the more general principle $HP_2(\omega_2)$ of Brendle, Fuchino, and Soukup [1]. They define this principle, prove that it holds in Cohen extensions (and in a number of other forcing extensions), and show the following:

Lemma 2.6. *$HP_2(\kappa)$ implies that if R is any relation on $\mathcal{P}(\omega)$ which is first-order definable over $H(\omega_1)$ from a fixed element of $H(\omega_1)$, then there is no $X \subseteq \mathcal{P}(\omega)$ such that $(X; R)$ is isomorphic to $(\kappa; <)$.*

These matters are also discussed in [4], which indicates how such statements are verified in Cohen extensions. Here, $H(\omega_1)$ denotes the set of hereditarily countable sets.

Lemma 2.7. *$HP_2(\omega_2)$ implies that in $\omega_2 \times \omega_2$, the well-order relation $<$ is not in the σ-algebra generated by the set of all rectangles, $\{S \times T : S, T \in \mathcal{P}(\omega_2)\}$.*

Proof. Suppose that $<$ were in this σ-algebra. Then we would have fixed $K_n \subseteq \omega_2$ for $n < \omega$ such that $<$ is in the σ-algebra generated by all the $K_m \times K_n$. For each α, let $u_\alpha = \{n \in \omega : \alpha \in K_n\}$. There is then a formula $\varphi(x, y, z)$ and a fixed $w \in H(\omega_1)$ such that for all $\alpha, \beta < \omega_2$, $\alpha < \beta$ iff $H(\omega_1) \models \varphi(u_\alpha, u_\beta, w)$; here, w encodes the particular countable boolean combination used to get $<$ from the K_n. Now, if $X = \{u_\alpha : \alpha < \omega_2\}$, then φ defines a relation R on $H(\omega_1)$ such that $(X; R)$ is isomorphic to $(\omega_2; <)$, contradicting Lemma 2.6. □

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