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BELL REPRESENTATIONS OF FINITELY CONNECTED PLANAR DOMAINS

MOONJA JEONG AND MASAHIKO TANIGUCHI

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Abstract. In this paper, we solve a conjecture of S. Bell (1992) affirmatively. Actually, we prove that every non-degenerate n-connected planar domain Ω, where $n > 1$ is representable as $\Omega = \{|f| < 1\}$ with a suitable rational function f of degree n. This result is considered as a natural generalization of the classical Riemann mapping theorem for simply connected planar domains.

1. Introduction and the main theorem

Recently, S. Bell posed the following problem ([B1] and [B2]).

Problem 1.1. Can every non-degenerate n-connected planar domain with $n > 1$ be mapped biholomorphically onto a domain of the form

$$\left\{\left|z + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k}\right| < r\right\}$$

with complex numbers a_k and b_k, and a positive r?

Here and in the sequel, a *non-degenerate n-connected planar domain* is a subdomain Ω of the Riemann sphere $\hat{\mathbb{C}}$ such that $\hat{\mathbb{C}} - \Omega$ consists of exactly n connected components each of which contains more than one point. In this note, we solve this problem affirmatively. Actually, we give a proof of the following assertion.

Theorem 1.2. *Every non-degenerate n-connected planar domain with $n > 1$ is mapped biholomorphically onto a domain defined by*

$$\left\{\left|z + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k}\right| < 1\right\}$$

with suitable complex numbers a_k and b_k.

Recall that every domain defined as in Theorem 1.2 has algebraic kernel functions. See Theorem 4.4 in [B1]. This is one of the reasons why we consider such domains.

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Remark 1.3. It is well known (cf. for instance [IT]) that the reduced Teichmüller space $T(\Omega)$ of a non-degenerate n-connected planar domain Ω can be identified with the Fricke space of a Fuchsian model G of Ω. Since G is a free real Möbius group with $n-1$ hyperbolic generators, $T(\Omega)$ is real $(3n-6)$-dimensional.

Such a Bell representation as in Theorem 1.2 contains $2n-2$ complex, i.e. $4n-4$ real, parameters. The reason why we need many more number of parameters in a Bell representation than Teichmüller parameters for $T(\Omega)$ is that every Bell representation of a domain is actually associated with an n-sheeted branched covering of the unit disk by Ω.

Remark 1.4. Such a space as $H_{0,n}$ consisting of all branched coverings of $\hat{\mathbb{C}}$ induced by rational functions of degree n with $n > 1$ is called a *Hurwitz space* ([N]). This space $H_{0,n}$ is parametrized by $2n-2$ critical values (the images of critical points), and hence is complex $(2n-2)$-dimensional.

Actually, we show that every Ahlfors map on Ω can be considered as the restriction of a rational function of degree n to a suitable domain which can be identified with Ω. On the other hand, the set of all branched coverings of $\hat{\mathbb{C}}$ induced from Bell representations can be considered as a subdomain of $H_{0,n}$. Thus to solve the following problem would be interesting.

Problem 1.5. Find the sublocus $A_{0,n}$ of $H_{0,n}$ which corresponds to the set of all Bell representations of non-degenerate n-connected planar domains such that the restrictions of the associated rational functions give Ahlfors maps.

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2. Proof of Theorem 1.2

First let a non-degenerate n-connected planar domain Ω be given. Then by using the classical Riemann mapping theorem n times if necessary, we can assume that the boundary of Ω consists of exactly n smooth simple closed curves. Fix a point a in Ω, and let f_a be the Ahlfors map associated to the pair (Ω, a). Here for the definition and properties of the Ahlfors maps, see for instance, [B]. In particular, f_a maps Ω properly and holomorphically onto the unit disk U. Moreover, f_a can be extended to a continuous map of the closure $\overline{\Omega}$ of Ω onto the closed unit disk so that every component γ_j of the boundary of Ω, where $j = 1, \cdots, n$, is mapped homeomorphically onto the unit circle.

Lemma 2.1. *There is a compact Riemann surface R (without boundary) of genus 0 and a holomorphic injection ι of Ω into R such that*

$$f_a \circ \iota^{-1}$$

can be extended to a meromorphic function, say F, on R.

Proof. Since there are only a finite number of zeros of f_a', there is a positive constant ρ such that $\rho < 1$ and that

$$D = \{\rho < |\zeta| < 1\},$$

where ζ is the complex coordinate on the target plane of the map f_a, and contains no critical values (i.e. no images of the zeros of f_a' by f_a). Hence every component

W_j, where $j = 1, \cdots, n$, of $f_a^{-1}(D)$ is mapped biholomorphically onto D by the restriction $f_a|_{W_j}$, of f_a to W_j.

Now we construct a compact Riemann surface R by using the Ahlfors map f_a to attach disks to the exterior of Ω along each boundary curve. More precisely, we consider the disjoint union $\mathbf{R}$ of Ω and n copies V_j $(j = 1, \cdots, n)$ of

$$V = \{\rho < |\zeta|\} \cup \{\infty\}.$$

Identify every subdomain W_j of Ω with the subdomain D_j of V_j corresponding to D by the biholmorphic map corresponding to $f_a|_{W_j}$. Then the resulting set, which we denote by $R = \mathbf{R}/f_a$, has a natural complex structure induced from those on Ω and on every V_j, and hence is a Riemann surface. Here the natural inclusion map ι of Ω into R is a holomorphic injection, and using the complex coordinate ζ_j on the copy V_j corresponding to ζ on V, we have

$$f_a \circ \iota^{-1}(\zeta_j) = \zeta$$

on D_j by the definition.

Now, since topologically R is obtained from Ω by attaching a disk along each boundary curve of Ω, R is a simply connected compact Riemann surface without boundary, and hence in particular, is of genus 0. Also we can extend $F = f_a \circ \iota^{-1}$ to a meromorphic function on the whole R by setting $F(\zeta_j) = \zeta$ and $F(\infty) = \infty$ on the whole V_j for every j. □

Here the following uniformization theorem (which is also called the generalized Riemann mapping theorem) is classical and well-known. As references, we cite for instance [FK] and [IT].

Proposition 2.2 (Klein, Koebe and Poincaré)**.** *Every simply connected Riemann surface is mapped biholomorphically onto one of*

- *the unit disk U,*
- *the complex plane $\mathbb{C}$, and*
- *the Riemann sphere $\hat{\mathbb{C}}$.*

Corollary 2.3. *There is a biholomorphic map h of the above Riemann surface R onto the Riemann sphere $\hat{\mathbb{C}}$, and hence $F \circ h^{-1}$ is a rational function.*

Proof. Since R is compact, R is mapped by a biholomorphic map h onto the Riemann sphere. Set $f = F \circ h^{-1}$. Then f is meromorphic on the whole $\hat{\mathbb{C}}$, which implies that f is a rational function. □

Here and in the sequel, we may assume that

$$f(\infty) = \infty$$

by applying to f the pre-composition of a Möbius transformation S which sends ∞ to a pole of f, i.e. by replacing h by $S^{-1} \circ h$, if necessary.

Lemma 2.4. *Let w be the complex variable of the above rational function f. Then f has the following partial fraction decomposition:*

$$f(w) = Cw + D + \sum_{k=1}^{n-1} \frac{A_k}{w - B_k}.$$

Here A_k, B_k, C and D are complex constants, every A_k and C are non-zero, and $\{B_k\}$ are mutually distinct.

Proof. Since f has exactly n simple poles, as is seen from the construction, and one of them is ∞ by the above assumption, f is of degree exactly n and has $n-1$ finite, mutually distinct, simple poles, say $B_1, \cdots, B_{n-1}$. Hence we can write $f(w)$ as

$$f(w) = \frac{P(w)}{Q(w)}$$

with polynomials $P(w)$ of degree exactly n and

$$Q(w) = (w - B_1) \cdots (w - B_{n-1}).$$

Thus it is easy to see that the partial fraction decomposition of f is as claimed. □

Proof of Theorem 1.2. We replace the complex variable w of f by

$$z = T(w) = Cw + D$$

by applying to f the precomposition by an affine transformation T. Further set

$$a_k = CA_k, \quad b_k = CB_k + D$$

for every k. Then we conclude that

$$f \circ T^{-1}(z) = z + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k}.$$

Thus the Bell representation

$$\left\{ \left| z + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k} \right| < 1 \right\}$$

is the subdomain

$$\{|f \circ T^{-1}(z)| < 1\} = \{|F \circ h^{-1} \circ T^{-1}(z)| < 1\} = (T \circ h \circ \iota)(\Omega)$$

of $\hat{\mathbb{C}}$, which is mapped biholomorphically onto Ω by the holomorphic injection $(T \circ h \circ \iota)^{-1}$. □

References

[B] S. R. Bell, *The Cauchy Transform, Potential Theory, and Conformal Mapping,* CRC Press, Boca Raton, 1992. MR **94k:**30013

[B1] S. R. Bell, *Finitely generated function fields and complexity in potential theory in the plane,* Duke Math. J., **98** (1999), 187–207. MR **2000i:**30015

[B2] S. R. Bell, *A Riemann surface attached to domains in the plane and complexity in potential theory,* Houston J. Math., **26**, (2000), 277–297. MR **2001m:**30009

[FK] H. M. Farkas and I. Kra, Riemann Surfaces, Springer-Verlag, 1980. MR **82c:**30067

[IT] Y. Imayoshi and M. Taniguchi, *An Introduction to Teichmüller Spaces,* Springer-Verlag, Tokyo, 1992. MR **94b:**32031

[N] S. M. Natanzon, *Hurwitz spaces,* Topics on Riemann surfaces and Fuchsian Groups, LMS Lecture Notes, **287** (2001), 165–177.

Department of Mathematics, The University of Suwon, Suwon P.O. Box 77, Kyungkido, 440-600, Korea

E-mail address: mjeong@mail.suwon.ac.kr

Department of Mathematics, Graduate school of Science, Kyoto University, Kyoto 606, Japan

E-mail address: tanig@kusm.kyoto-u.ac.jp