BELL REPRESENTATIONS OF FINITELY CONNECTED
PLANAR DOMAINS

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Abstract. In this paper, we solve a conjecture of S. Bell (1992) affirmatively. Actually, we prove that every non-degenerate $n$-connected planar domain $\Omega$, where $n > 1$, is representable as $\Omega = \{|f| < 1\}$ with a suitable rational function $f$ of degree $n$. This result is considered as a natural generalization of the classical Riemann mapping theorem for simply connected planar domains.

1. Introduction and the main theorem

Recently, S. Bell posed the following problem ([B1] and [B2]).

Problem 1.1. Can every non-degenerate $n$-connected planar domain with $n > 1$ be mapped biholomorphically onto a domain of the form

$$\left\{ \left| z + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k} \right| < r \right\}$$

with complex numbers $a_k$ and $b_k$, and a positive $r$?

Here and in the sequel, a non-degenerate $n$-connected planar domain is a subdomain $\Omega$ of the Riemann sphere $\hat{\mathbb{C}}$ such that $\hat{\mathbb{C}} - \Omega$ consists of exactly $n$ connected components each of which contains more than one point. In this note, we solve this problem affirmatively. Actually, we give a proof of the following assertion.

Theorem 1.2. Every non-degenerate $n$-connected planar domain with $n > 1$ is mapped biholomorphically onto a domain defined by

$$\left\{ \left| z + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k} \right| < 1 \right\}$$

with suitable complex numbers $a_k$ and $b_k$.

Recall that every domain defined as in Theorem 1.2 has algebraic kernel functions. See Theorem 4.4 in [B1]. This is one of the reasons why we consider such domains.

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Remark 1.3. It is well known (cf. for instance [1]) that the reduced Teichmüller space $T(\Omega)$ of a non-degenerate $n$-connected planar domain $\Omega$ can be identified with the Fricke space of a Fuchsian model $G$ of $\Omega$. Since $G$ is a free real Möbius group with $n - 1$ hyperbolic generators, $T(\Omega)$ is real $(3n - 6)$-dimensional.

Such a Bell representation as in Theorem 1.2 contains $2n - 2$ complex, i.e. $4n - 4$ real, parameters. The reason why we need many more number of parameters in a Bell representation than Teichmüller parameters for $T(\Omega)$ is that every Bell representation of a domain is actually associated with an $n$-sheeted branched covering of the unit disk by $\Omega$.

Remark 1.4. Such a space as $H_{0,n}$ consisting of all branched coverings of $\hat{\mathbb{C}}$ induced by rational functions of degree $n$ with $n > 1$ is called a Hurwitz space ([N]). This space $H_{0,n}$ is parametrized by $2n - 2$ critical values (the images of critical points), and hence is complex $(2n - 2)$-dimensional.

Actually, we show that every Ahlfors map on $\Omega$ can be considered as the restriction of a rational function of degree $n$ to a suitable domain which can be identified with $\Omega$. On the other hand, the set of all branched coverings of $\hat{\mathbb{C}}$ induced from Bell representations can be considered as a subdomain of $H_{0,n}$. Thus to solve the following problem would be interesting.

Problem 1.5. Find the sublocus $A_{0,n}$ of $H_{0,n}$ which corresponds to the set of all Bell representations of non-degenerate $n$-connected planar domains such that the restrictions of the associated rational functions give Ahlfors maps.

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2. Proof of Theorem 1.2

First let a non-degenerate $n$-connected planar domain $\Omega$ be given. Then by using the classical Riemann mapping theorem $n$ times if necessary, we can assume that the boundary of $\Omega$ consists of exactly $n$ smooth simple closed curves. Fix a point $a$ in $\Omega$, and let $f_a$ be the Ahlfors map associated to the pair $(\Omega, a)$. Here for the definition and properties of the Ahlfors maps, see for instance, [B]. In particular, $f_a$ maps $\Omega$ properly and holomorphically onto the unit disk $U$. Moreover, $f_a$ can be extended to a continuous map of the closure $\overline{\Omega}$ of $\Omega$ onto the closed unit disk so that every component $\gamma_j$ of the boundary of $\Omega$, where $j = 1, \cdots, n$, is mapped homeomorphically onto the unit circle.

Lemma 2.1. There is a compact Riemann surface $R$ (without boundary) of genus 0 and a holomorphic injection $\iota$ of $\Omega$ into $R$ such that

$$f_a \circ \iota^{-1}$$

can be extended to a meromorphic function, say $F$, on $R$.

Proof. Since there are only a finite number of zeros of $f_a'$, there is a positive constant $\rho$ such that $\rho < 1$ and that

$$D = \{\rho < |\zeta| < 1\},$$

where $\zeta$ is the complex coordinate on the target plane of the map $f_a$, and contains no critical values (i.e. no images of the zeros of $f_a'$ by $f_a$). Hence every component
$W_j$, where $j = 1, \cdots, n$, of $f_a^{-1}(D)$ is mapped biholomorphically onto $D$ by the restriction $f_a|_{W_j}$ of $f_a$ to $W_j$.

Now we construct a compact Riemann surface $R$ by using the Ahlfors map $f_a$ to attach disks to the exterior of $\Omega$ along each boundary curve. More precisely, we consider the disjoint union $R$ of $\Omega$ and $n$ copies $V_j$ ($j = 1, \cdots, n$) of 

$$V = \{\rho < \|\zeta\|\} \cup \{\infty\}.$$

Identify every subdomain $W_j$ of $\Omega$ with the subdomain $D_j$ of $V_j$ corresponding to $D$ by the biholmorphic map corresponding to $f_a|_{W_j}$. Then the resulting set, which we denote by $R = f_a|_{W_j}$, has a natural complex structure induced from those on $\Omega$ and on every $V_j$, and hence is a Riemann surface. Here the natural inclusion map $\iota$ of $\Omega$ into $R$ is a holomorphic injection, and using the complex coordinate $\zeta_j$ on the copy $V_j$ corresponding to $\zeta$ on $V$, we have

$$f_a \circ \iota^{-1}(\zeta_j) = \zeta$$
on $D_j$ by the definition.

Now, since topologically $R$ is obtained from $\Omega$ by attaching a disk along each boundary curve of $\Omega$, $R$ is a simply connected compact Riemann surface without boundary, and hence in particular, is of genus 0. Also we can extend $F = f_a \circ \iota^{-1}$ to a meromorphic function on the whole $R$ by setting $F(\zeta_j) = \zeta$ and $F(\infty) = \infty$ on the whole $V_j$ for every $j$. $\square$

Here the following uniformization theorem (which is also called the generalized Riemann mapping theorem) is classical and well-known. As references, we cite for instance [FK] and [IT].

**Proposition 2.2** (Klein, Koebe and Poincaré). Every simply connected Riemann surface is mapped biholomorphically onto one of

- the unit disk $U$,
- the complex plane $\mathbb{C}$, and
- the Riemann sphere $\hat{\mathbb{C}}$.

**Corollary 2.3.** There is a biholomorphic map $h$ of the above Riemann surface $R$ onto the Riemann sphere $\hat{\mathbb{C}}$, and hence $F \circ h^{-1}$ is a rational function.

**Proof.** Since $R$ is compact, $R$ is mapped by a biholomorphic map $h$ onto the Riemann sphere. Set $f = F \circ h^{-1}$. Then $f$ is meromorphic on the whole $\hat{\mathbb{C}}$, which implies that $f$ is a rational function. $\square$

Here and in the sequel, we may assume that

$$f(\infty) = \infty$$

by applying to $f$ the pre-composition of a Möbius transformation $S$ which sends $\infty$ to a pole of $f$, i.e. by replacing $h$ by $S^{-1} \circ h$, if necessary.

**Lemma 2.4.** Let $w$ be the complex variable of the above rational function $f$. Then $f$ has the following partial fraction decomposition:

$$f(w) = Cw + D + \sum_{k=1}^{n-1} \frac{A_k}{w - B_k}.$$

Here $A_k, B_k, C$ and $D$ are complex constants, every $A_k$ and $C$ are non-zero, and $\{B_k\}$ are mutually distinct.
Proof. Since \( f \) has exactly \( n \) simple poles, as is seen from the construction, and one of them is \( \infty \) by the above assumption, \( f \) is of degree exactly \( n \) and has \( n - 1 \) finite, mutually distinct, simple poles, say \( B_1, \ldots, B_{n-1} \). Hence we can write \( f(w) \) as

\[
f(w) = \frac{P(w)}{Q(w)}
\]

with polynomials \( P(w) \) of degree exactly \( n \) and

\[
Q(w) = (w - B_1) \cdots (w - B_{n-1}).
\]

Thus it is easy to see that the partial fraction decomposition of \( f \) is as claimed. \( \square \)

Proof of Theorem 1.2. We replace the complex variable \( w \) of \( f \) by

\[
z = T(w) = Cw + D
\]

by applying to \( f \) the precomposition by an affine transformation \( T \). Further set

\[
a_k = CA_k, \quad b_k = CB_k + D
\]

for every \( k \). Then we conclude that

\[
f \circ T^{-1}(z) = z + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k}.
\]

Thus the Bell representation

\[
\left\{ \left| z + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k} \right| < 1 \right\}
\]

is the subdomain

\[
\{ |f \circ T^{-1}(z)| < 1 \} = \{ |F \circ h^{-1} \circ T^{-1}(z)| < 1 \} = (T \circ h \circ \iota)(\Omega)
\]

of \( \hat{C} \), which is mapped biholomorphically onto \( \Omega \) by the holomorphic injection \((T \circ h \circ \iota)^{-1}\). \( \square \)

References


