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EVERY SET OF FINITE HAUSDORFF MEASURE IS A COUNTABLE UNION OF SETS WHOSE HAUSDORFF MEASURE AND CONTENT COINCIDE

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Abstract. A set $E \subseteq \mathbb{R}^n$ is h-straight if E has finite Hausdorff h-measure equal to its Hausdorff h-content, where $h : [0, \infty) \to [0, \infty)$ is continuous and non-decreasing with $h(0) = 0$. Here, if h satisfies the standard doubling condition, then every set of finite Hausdorff h-measure in $\mathbb{R}^n$ is shown to be a countable union of h-straight sets. This also settles a conjecture of Foran that when $h(t) = t^s$, every set of finite s-measure is a countable union of s-straight sets.

1. Introduction

In [7], Foran introduced the notion of an s-straight set, that is, a set whose Hausdorff s-measure and Hausdorff s-content are equal (see Definitions 1 and 2 below). In [1] and [2], we continued the first analysis of such sets, among other results proving that in $\mathbb{R}^2$ a quarter circle is the countable union of 1-straight sets, verifying a conjecture of Foran. In [3] and [4], by different detailed arguments we extended that result, proving that in $\mathbb{R}^2$ the graphs of any convex, continuously differentiable, absolutely continuous, or increasing continuous functions, as well as regular sets of finite 1-measure, all consist of such countable unions.

Here we prove the more general result that if a function h satisfies the standard doubling condition (Condition 1), then every set of finite h-measure in $\mathbb{R}^n$ is a countable union of h-straight sets (Definition 2, Theorem 4, and Theorem 5). This also settles the conjecture of Foran [7] that every set of finite s-measure is a countable union of s-straight sets.

Let d be the standard distance function on $\mathbb{R}^n$ where $n \geq 1$. The diameter of an arbitrary nonempty set $U \subseteq \mathbb{R}^n$ is defined by $|U| = \sup\{d(x, y) : x, y \in U\}$, with $|\emptyset| = 0$. Given $0 < \delta \leq \infty$, let C^n_δ represent the collection of subsets of $\mathbb{R}^n$ with diameter less than δ. Let $h : [0, \infty) \to [0, \infty)$ be continuous and non-decreasing, with $h(0) = 0$. (In Condition 1 we restrict $h(t)$ further.)

Definition 1 ([6, 2.10.1] or [8, p. 60])**.** For $E \subseteq \mathbb{R}^n$, let

$$\mathcal{H}^h_\delta(E) = \inf\left\{\sum h(|E_i|) : E \subseteq \bigcup E_i \text{ where } E_i \in C^n_\delta \text{ for } i = 1, 2, ...\right\}.$$

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Then $\mathcal{H}^h_\infty(E)$ is called the Hausdorff h-content of E, and $\mathcal{H}^h(E) = \sup_{\delta>0} \mathcal{H}^h_\delta(E)$ is called the Hausdorff h-measure, or just the h-measure of E.

Note that Hausdorff h-measure is Borel regular [6, 2.10.2 (1)]. Also [8, p. 54], since for any $0 < \alpha < \beta \leq \infty$ it follows that $\mathcal{H}^h_\beta(E) \leq \mathcal{H}^h_\alpha(E)$, we always have the inequality

$$\mathcal{H}^h_\infty(E) \leq \mathcal{H}^h(E).$$

Definition 2. Define a set $E \subseteq \mathbb{R}^n$ to be h-straight if

$$\mathcal{H}^h_\infty(E) = \mathcal{H}^h(E) < \infty.$$

A set which is the countable union of h-straight sets is called σh-straight. (When $h(t) = t^s$, the terms are s-straight and σs-straight, [1], [2].)

In [7], Foran proves the following theorem for the case of Hausdorff s-measure providing a useful equivalent definition of an s-straight set. We make the easy extension of his proof to h-straight sets.

Theorem 1. *Let $E \subseteq \mathbb{R}^n$ have finite h-measure. Then, E is h-straight if and only if for each $A \subseteq E$*

$$\mathcal{H}^h(A) \leq h(|A|).$$

In particular, sets of zero h-measure are h-straight.

Proof. (Based on [7, p. 733].) On the one hand, suppose for each $A \subseteq E$ that $\mathcal{H}^h(A) \leq h(|A|)$. Then

$$\begin{aligned} \mathcal{H}^h(E) &\geq \mathcal{H}^h_\infty(E) = \inf\left\{\sum h(|E_i|) : E = \bigcup E_i\right\} \\ &\geq \inf\left\{\sum \mathcal{H}^h(E_i) : E = \bigcup E_i\right\} \\ &\geq \mathcal{H}^h(E), \end{aligned}$$

where the infima are over countable covers $\{E_i\}$ of E. Since E has finite h-measure, it then follows that $\mathcal{H}^h_\infty(E) = \mathcal{H}^h(E) < \infty$. Hence E is h-straight. Conversely, suppose $\mathcal{H}^h_\infty(E) = \mathcal{H}^h(E) < \infty$. If there were a subset $A \subseteq E$ such that $\mathcal{H}^h(A) > h(|A|)$, then, since h is non-decreasing and $\mathcal{H}^h_\infty(E\backslash A) \leq \mathcal{H}^h(E\backslash A) \leq \mathcal{H}^h(E) < \infty$, we have

$$\begin{aligned} \mathcal{H}^h(E) &= \mathcal{H}^h(A) + \mathcal{H}^h(E\backslash A) \\ &> h(|A|) + \mathcal{H}^h_\infty(E\backslash A) \\ &\geq \mathcal{H}^h_\infty(A) + \mathcal{H}^h_\infty(E\backslash A) \\ &\geq \mathcal{H}^h_\infty(E), \end{aligned}$$

contradicting the assumption that $\mathcal{H}^h_\infty(E) = \mathcal{H}^h(E)$. $\square$

Theorem 2 is proved in [1], [2], for s-straight sets using a standard exhaustion argument. The proof of the generalization here to h-straight sets is omitted.

Theorem 2. *Let $E \subseteq \mathbb{R}^n$ have finite h-measure. Then, every $\mathcal{H}^h$-measurable subset of positive h-measure of E contains an h-straight set of positive h-measure if and only if E is σh-straight.*

2. Main result

To prepare for the proof of the main result, Theorem 4, we begin with a well-known definition.

Definition 3 ([5, p. 21]). Let $E \subseteq \mathbb{R}^n$ have finite h-measure, and $x \in E$. The upper convex density of E at x is defined to be

$$\overline{D}_c^h(E,x) = \lim_{\eta\to 0} \sup \left\{ \frac{\mathcal{H}^h(E\cap S)}{h(|S|)} : x \in S \text{ and } 0 < |S| \le \eta \right\}$$

where the supremum is over all convex sets $S \subseteq \mathbb{R}^n$.

Note that since any set is contained in a convex set of the same diameter, Definition 3 may be interpreted as taking the supremum over *all* sets $S \subseteq \mathbb{R}^n$. We also assume h satisfies the following standard doubling condition.

Condition 1. *For some $c < \infty$, $h(2t) \le c \cdot h(t)$ for all $t > 0$.*

Theorem 3 is the restriction we need here of a result of Federer, [6, 2.10.18 (3)].

Theorem 3 (Corollary to [6, 2.10.18 (3)]). *Let $E \subseteq \mathbb{R}^n$ have finite h-measure, where h satisfies Condition 1. Then for $\mathcal{H}^h$-almost all $x \in E$ it follows that*

$$\overline{D}_c^h(E,x) \le 1.$$

Lemma 1 is then a straightforward consequence of Theorem 3. The proof, though a standard argument, is included for completeness.

Lemma 1. *Let $E \subseteq \mathbb{R}^n$ have finite h-measure, where h satisfies Condition 1. Let $\{\varepsilon_j\}_{j=0}^{\infty}$ be a positive decreasing sequence of real numbers such that $\lim_{j\to\infty} \varepsilon_j = 0$. Then there exists $A \subseteq E$ with positive h-measure satisfying the condition that for each $j \ge 0$ there is $\rho_j > 0$ so that for all $S \subseteq \mathbb{R}^n$ with $0 < |S| \le \rho_j$ we have*

$$\mathcal{H}^h(A \cap S) \le (1+\varepsilon_j)\cdot h(|S|).$$

Proof. Suppose h is such a function, $E \subseteq \mathbb{R}^n$ satisfies $\mathcal{H}^h(E) < \infty$, and $\{\varepsilon_j\}_{j=0}^{\infty}$ is a positive decreasing sequence of real numbers such that $\lim_{j\to\infty} \varepsilon_j = 0$. By Theorem 3, for $\mathcal{H}^h$ -almost all $x \in E$ it follows that

$$\overline{D}_c^h(E,x) = \lim_{\eta\to 0} \sup \left\{ \frac{\mathcal{H}^h(E\cap S)}{h(|S|)} : x \in S \text{ and } 0 < |S| \le \eta \right\} \le 1.$$

So given $\varepsilon_j > 0$ there exists a corresponding $\eta_j(x) > 0$ such that

$$\sup \left\{ \frac{\mathcal{H}^h(E\cap S)}{h(|S|)} : x \in S \text{ and } 0 < |S| \le \eta_j(x) \right\} \le 1 + \varepsilon_j.$$

Then for each $S \subseteq \mathbb{R}^n$ with $x \in E \cap S$ and $0 < |S| \le \eta_j(x)$ we have

$$\mathcal{H}^h(E\cap S) \le (1+\varepsilon_j)\cdot h(|S|).$$

Now let $\{\rho_j\}_{j=0}^{\infty}$ be a positive decreasing sequence of real numbers such that $\lim_{j\to\infty} \rho_j = 0$. Define

$$A_j = \{x \in \mathbb{R}^n : \text{ If } x \in S \text{ and } 0 < |S| \le \rho_j, \text{ then } \mathcal{H}^h(E\cap S) \le (1+\varepsilon_j)\cdot h(|S|)\}.$$

Then A_j is a Borel (in fact, closed) set, hence $\mathcal{H}^h$-measurable. Since $\overline{D}_c^h(E,x) \le 1$ for $\mathcal{H}^h$-almost all $x \in E$, we may further choose ρ_j small enough so that given δ satisfying $0 < \delta < \mathcal{H}^h(E)$, we have $\mathcal{H}^h(E \setminus A_j) < \frac{\delta}{2^{j+1}}$. Hence, $\sum_{j=0}^{\infty} \mathcal{H}^h(E \setminus A_j) <$

$\sum_{j=0}^{\infty} \frac{\delta}{2^{j+1}} = \delta$. For such choices of ρ_j, define $A = \bigcap_{j=0}^{\infty} A_j$. By the definition of A as an intersection, for each $j \geq 0$ there is $\rho_j > 0$, so for all $S \subseteq \mathbb{R}^n$ with $0 < |S| \leq \rho_j$ we have that A satisfies

$$\mathcal{H}^h(A \cap S) \leq \mathcal{H}^h(E \cap S) \leq (1 + \varepsilon_j) \cdot h(|S|).$$

Also, A has positive h-measure since

$$\begin{aligned} \mathcal{H}^h(A) &= \mathcal{H}^h\left(\bigcap_{j=0}^{\infty} A_j\right) = \mathcal{H}^h(E) - \mathcal{H}^h\left(\bigcup_{j=0}^{\infty} (E \setminus A_j)\right) \\ &\geq \mathcal{H}^h(E) - \sum_{j=0}^{\infty} \mathcal{H}^h(E \setminus A_j) > \mathcal{H}^h(E) - \delta > 0. \end{aligned}$$

So, $A \subseteq E$ is the desired set. □

The argument used in the proof of Theorem 4 below was developed from a suggestion of Preiss [10] for sets of finite s-measure made after reading a preprint of [3].

Theorem 4. *If $E \subseteq \mathbb{R}^n$ has finite h-measure where h satisfies Condition 1, then there exists $B \subseteq E$ with positive h-measure such that B is h-straight.*

Proof. Let $E \subseteq \mathbb{R}^n$ have finite h-measure, where h is such a function. Let $\{\varepsilon_j\}_{j=0}^{\infty}$ be a positive, decreasing sequence of real numbers such that $\sum_{j=0}^{\infty} \varepsilon_j < 1$. Hence, $\lim_{j\to\infty} \varepsilon_j = 0$. So, by Lemma 1 there exists $A \subseteq E$ with positive h-measure and corresponding $\rho_j > 0$ such that for all $S \subseteq \mathbb{R}^n$ with $0 < |S| \leq \rho_j$ we have

$$(*) \qquad \mathcal{H}^h(A \cap S) \leq (1 + \varepsilon_j) \cdot h(|S|).$$

For $j \geq 0$ replace as needed the ρ_j with values r_j small enough so that $\lim_{j\to\infty} r_j = 0$, with both $2r_0 < \rho_0$ and

$$2r_{j+1} < \min(r_j, \rho_{j+1}).$$

By the continuity of h we may also assume the r_j small enough so that whenever $r_j \leq t \leq r_{j-1}$, we have

$$h(t + 2r_{j+1}) < (1 + \varepsilon_{j-1}^2) \cdot h(t).$$

We may further assume that

$$0 < \mathcal{H}^h(A) \leq h(r_0)$$

by the continuity of h-measure, since $0 < \mathcal{H}^h(A) \leq \mathcal{H}^h(E) < \infty$.

Now, for each fixed $j \geq 1$, let $\{A_{j,i}\}_i$ be a partition of A into (relatively) Borel subsets such that $|A_{j,i}| < r_{j+1}$ for every i. Then, for $j \geq 1$ let $B_{j,i} \subseteq A_{j,i}$ be slightly smaller in measure, satisfying

$$\mathcal{H}^h(B_{j,i}) = (1 - \varepsilon_{j-1}) \cdot \mathcal{H}^h(A_{j,i}).$$

It then follows for each fixed $j \geq 1$ that since the $A_{j,i}$ partition A,

$$\mathcal{H}^h\left(\bigcup_i B_{j,i}\right) = \sum_i \mathcal{H}^h(B_{j,i}) = (1 - \varepsilon_{j-1}) \cdot \sum_i \mathcal{H}^h(A_{j,i}) = (1 - \varepsilon_{j-1}) \cdot \mathcal{H}^h(A).$$

Define $B = \bigcap_{j\geq 1}\bigcup_i B_{j,i}$. Observe that using the subadditivity of $\mathcal{H}^h$, we have

$$\begin{aligned}\mathcal{H}^h(A\backslash B) &= \mathcal{H}^h\left(\bigcup_{j\geq 1}\left[A\backslash\bigcup_i B_{j,i}\right]\right)\\ &\leq \sum_{j\geq 1}\left[\mathcal{H}^h(A)-\mathcal{H}^h\left(\bigcup_i B_{j,i}\right)\right]\\ &= \left(\sum_{j\geq 1}\varepsilon_{j-1}\right)\cdot\mathcal{H}^h(A) < \mathcal{H}^h(A).\end{aligned}$$

Thus $\mathcal{H}^h(B) = \mathcal{H}^h(A) - \mathcal{H}^h(A\backslash B) > 0$, so B has positive h-measure. Using Theorem 1 we show that B is the desired h-straight subset of E by showing that $\mathcal{H}^h(B\cap S)\leq h(|S|)$ for every $S\subseteq\mathbb{R}^n$.

First, for any $S\subseteq\mathbb{R}^n$ such that $|S|>r_0$, since $B\cap S\subseteq A$ and h is non-decreasing we immediately have that

$$\mathcal{H}^h(B\cap S)\leq\mathcal{H}^h(A)\leq h(r_0)\leq h(|S|).$$

So, consider all other sets $S\subseteq\mathbb{R}^n$ such that $r_j<|S|\leq r_{j-1}$ for some $j\geq 1$. Define

$$S' = \{x\in\mathbb{R}^n : d(x,S)\leq r_{j+1}\}.$$

So $S\subseteq S'$. By the restrictions on $\{r_k\}_{k=0}^\infty$ we then have that

$$0<|S'|\leq|S|+2r_{j+1}<r_{j-1}+r_j<2r_{j-1}<\rho_{j-1}.$$

By definition for each $j\geq 1$ we have $B\cap A_{j,i}\subseteq B_{j,i}$. Denote by $\bigcup'$ the union and by $\sum'$ the summation over those indices i for which $S\cap B_{j,i}\neq\emptyset$. So for $j\geq 1$ we have $B\cap S\subseteq\bigcup_i' B_{j,i}$. Also, because $|A_{j,i}|<r_{j+1}$ it follows that each $A_{j,i}\subseteq A\cap S'$ for those same indices i. Then, for any fixed $j\geq 1$, we have

$$\begin{aligned}\mathcal{H}^h(B\cap S) &\leq \sum_i{}' \mathcal{H}^h(B_{j,i})\\ &= (1-\varepsilon_{j-1})\cdot\sum_i{}'\mathcal{H}^h(A_{j,i}) = (1-\varepsilon_{j-1})\cdot\mathcal{H}^h\left(\bigcup_i{}' A_{j,i}\right)\\ &\leq (1-\varepsilon_{j-1})\cdot\mathcal{H}^h(A\cap S'). \qquad (**)\end{aligned}$$

Since $r_j<|S|\leq r_{j-1}$ and h is non-decreasing, again by the restrictions on $\{r_k\}_{k=0}^\infty$ we have

$$h(|S'|)\leq h\left(|S|+2r_{j+1}\right)<\left(1+\varepsilon_{j-1}^2\right)\cdot h\left(|S|\right).$$

Since also $0<|S'|<\rho_{j-1}$, by inequality $(*)$ we further have

$$\mathcal{H}^h(A\cap S')\leq(1+\varepsilon_{j-1})\cdot h(|S'|).$$

So, by inequality $(**)$, we finally conclude

$$\begin{aligned}\mathcal{H}^h(B\cap S) &\leq (1-\varepsilon_{j-1})\cdot(1+\varepsilon_{j-1})\cdot h(|S'|)\\ &< \left(1-\varepsilon_{j-1}^2\right)\cdot\left(1+\varepsilon_{j-1}^2\right)\cdot h(|S|)\\ &= \left(1-\varepsilon_{j-1}^4\right)\cdot h(|S|)\\ &< h(|S|).\end{aligned}$$

Therefore in particular if $S \subseteq B$ it follows that $\mathcal{H}^h(S) = \mathcal{H}^h(B \cap S) \leq h(|S|)$. So by Theorem 1, the set B is h-straight with positive h-measure, as desired. □

Theorem 5. *If $E \subseteq \mathbb{R}^n$ has finite h-measure where h satisfies Condition 1, then E is σh-straight.*

Proof. Let $E \subseteq \mathbb{R}^n$ have finite h-measure, where h is such a function. By Theorem 4, every $\mathcal{H}^h$-measurable subset of E of positive h-measure contains an h-straight subset of positive h-measure. By Theorem 2 it then follows that E is σh-straight. □

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