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INFINITE APPROXIMATE PEANO DERIVATIVES

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Abstract. In this paper we introduce approximate Peano derivatives with infinite values allowed, and we show that these derivatives are Baire one, and possess the Darboux and Denjoy-Clarkson properties. Also we show that if they are bounded from above or below on an interval, then the corresponding ordinary derivatives exist and equal the approximate Peano derivatives.

1. Introduction

In [7], the authors introduced Peano derivatives with infinite values allowed. They showed that even in this more general setting, Peano derivatives still possess many properties of finite Peano derivatives; namely, belonging to Baire class one and having the Darboux property and the Denjoy-Clarkson property. They also showed that if the n-th Peano derivative, f_n, of a real-valued function, f, is bounded from above or below on an interval, then the n-th ordinary derivative, $f^{(n)}$, exists and it is equal to f_n. (*The so-called monotonicity property.*) In this paper we introduce approximate Peano derivatives with infinite values allowed, and we show that all the properties of Peano derivatives mentioned above also hold for approximate Peano derivatives with infinite values allowed.

2. Definitions and notation

Throughout this paper, n denotes a fixed positive integer, λ denotes Lebesgue measure, and f is a real-valued function. It will be convenient to use the same notation, (a, b), for an open interval even if $a > b$.

Definition 1. We say that f is n-times approximately Peano differentiable at x if there are numbers $f_1(x), ..., f_n(x)$, and a measurable set V_x of density 1 at x such that

$$\lim_{h\to 0,\, x+h\in V_x} \frac{f(x+h) - f(x) - hf_1(x) - \cdots - \frac{h^n}{n!}f_n(x)}{h^n} = 0.$$

Equivalently

$$f(x+h) = \sum_{s=0}^{n} \frac{h^s}{s!} f_s(x) + h^n \epsilon_x(h)$$

where $f_0(x) = f(x)$ and $\lim\limits_{h\to 0,\, x+h\in V_x} \epsilon_x(h) = 0$. The number $f_n(x)$ is called the *n-th approximate Peano derivative*. In the case where $n = 0$ we also say that

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f is approximately continuous, while for $n = 1$ we say that f is approximately differentiable.

Although in the definition above no further restrictions on the set V_x are assumed, it will be convenient to assume that for all $x + h \in V_x$, $|\epsilon_x(h)| < 1$, and we will do so. It is evident that if the n-th approximate Peano derivative exists at x, then it is unique. Also, for $0 \le k \le n-1$, the k-th approximate Peano derivative exists and it is equal to $f_k(x)$.

We could also define the n-th Peano derivative as the limit

$$\lim_{h\to 0} \frac{1}{n!}\frac{f(x+h) - f(x) - hf_1(x) - \cdots - \frac{h^{n-1}}{(n-1)!}f_{n-1}(x)}{h^n}.$$

This way we could include n-th Peano derivatives with infinite values by simply requiring the existence, finite or infinite, of this limit. For $n = 1$, that is exactly how one defines infinite derivatives. But for $n > 1$ we have to be a little more careful because, otherwise, the numbers $f_k(x)$ for $1 \le k \le n-1$ need not be unique as is evident from the following example. Let

$$f(x) = \begin{cases} 1, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then $\lim\limits_{h\to 0} \frac{1}{2!}\frac{f(h)-f(0)-hf_1(0)}{h^2} = +\infty$, where $f_1(0)$ can be assigned any finite number. One way to insure the uniqueness of $f_k(x)$ for $1 \le k \le n-1$ in a definition of n-th Peano derivatives with infinite values allowed is to require that $f_{n-1}(x)$ exists and is finite. This is exactly what the authors did in [7], and in this paper we will follow their approach. Here we should stress that although for finite derivatives the definitions of the first derivative and of the first Peano derivative are identical; if one allows infinite values, then the ordinary derivative is more general since it may be defined even if its primitive is not continuous. Below we give a definition of an $(n+1)$-th approximate Peano derivative.

Definition 2. We say that f is $(n+1)$-times approximately Peano differentiable with infinite values allowed at a point x if f has a finite n-th approximate Peano derivative and

$$\lim_{h\to 0,\, x+h\in V_x} \frac{1}{(n+1)!}\frac{f(x+h) - f(x) - hf_1(x) - \cdots - \frac{h^n}{n!}f_n(x)}{h^{n+1}}$$

exists, finite or infinite. We denote this limit by $f_{n+1}(x)$. Equivalently

$$f(x+h) = \sum_{s=0}^{n} \frac{h^s}{s!} f_s(x) + h^n \epsilon_x(h)$$

where $f_0(x) = f(x)$ and

$$\lim_{h\to 0,\, x+h\in V_x} \frac{1}{(n+1)!}\frac{\epsilon_x(h)}{h} = f_{n+1}(x).$$

We will also call these derivatives, derivatives in the extended sense to stress that they are allowed to take on $\pm\infty$. Notice that the existence of $f_n(x)$ implies that $\lim\limits_{h\to 0,\, x+h\in V_x} \epsilon_x(h) = 0$.

The case $n = 0$ requires special attention. From this definition, it follows that

$$f'_{ap} = f_1(x) = \lim_{h\to 0,\, x+h\in V_x} \frac{f(x+h) - f(x)}{h},$$

where f is assumed to be approximately continuous. In general, when we allow infinite values in the definition of ordinary approximate derivatives it is not assumed that the primitive is approximately continuous. In fact, it turns out that the primitive of an approximate derivative with infinite values allowed does not have to be even a Baire one or a Darboux function. But even in this case it is known that the derivative is Baire one and that the monotonicity property holds. (See [9], [10], and [11].)

3. Decomposition of finite approximate Peano derivatives

In [4], the author showed that if the n-th approximate Peano derivative exists at every point of the real line R, then there is a decomposition of R into a countable collection of closed sets such that relative to the sets, f_n is the derivative of f_{n-1}. In this section we extend this result by constructing a countable collection of sets, $\{A_N\}_{N=1}^{\infty}$, such that $R = \bigcup A_N$, and such that for every $0 \le s \le n-1$, relative to $\overline{A_N}$, f_n is the $(n-s)$-th Peano derivative of f_s. Here the sets A_N are much simpler than the sets used in [4].

Theorem 3. *Suppose f is n-times approximately Peano differentiable on R. Let*

$$A_N = \left\{ x \middle| \ \frac{\lambda(V_x \cap (x, x+h))}{|h|} > 1 - \frac{1}{2(2n+3)} \quad \textit{for } |h| < 1/N \right\}.$$

Then, for $x \in \overline{A_N}$ and $0 \le s \le n-1$,

$$\lim_{y \to x, y \in \overline{A_N}} \frac{\left(f_s(y) - \sum\limits_{i=s}^{n} \frac{(y-x)^{i-s}}{(i-s)!} f_i(x) \right)}{|y-x|^{n-s}} = 0.$$

Before we prove this result we state and prove the following two lemmas..

Lemma 4. *Let $0 < a < 1$ and $x \in R$. Let $\delta(x) > 0$ be such that if $0 < |h| < \delta(x)$, then $\frac{\lambda(V_x \cap (x,x+h))}{|h|} > 1 - \frac{1}{2}a$. If I is an interval inside $(x, x+h)$ with $\lambda(I) \ge a|h|$, then $\lambda(V_x \cap I) > \frac{1}{2}\lambda(I)$.*

Proof. Indeed we have that $|h| - \lambda(I) + \lambda(V_x \cap I) \ge \lambda(V_x \cap (x, x+h)) > |h| - \frac{1}{2}a|h|$. Hence $\lambda(V_x \cap I) > \lambda(I) - \frac{1}{2}a|h| \ge \lambda(I) - \frac{1}{2}\lambda(I) = \frac{1}{2}\lambda(I)$. □

The identity in the next lemma is from Theorem 1.1.17 in [3]. For the sake of completeness we will include the proof of this lemma.

Lemma 5. *Let $x, y, h \in R$. Suppose that f is n-times approximately Peano differentiable at x and y. Then*

$$\sum_{s=0}^{n} \frac{h^s}{s!} \left(f_s(y) - \sum_{i=s}^{n} \frac{(y-x)^{i-s}}{(i-s)!} f_i(x) \right) = (y-x+h)^n \epsilon_x(y-x+h) - h^n \epsilon_y(h). \tag{1}$$

Proof. This identity is obtained by writing $f(y+h)$ two ways as follows. First, we have $f(y+h) = \sum\limits_{s=0}^{n} \frac{h^s}{s!} f_s(y) + h^n \epsilon_y(h)$ by expanding about the point y. Then write

$y+h$ as $x+(y-x+h)$ and expand about x to get

$$\begin{aligned} f(y+h) &= f(x+y-x+h) \\ &= \sum_{i=0}^{n} \frac{(y-x+h)^i}{i!} f_i(x) + (y-x+h)^n \epsilon_x(y-x+h) \\ &= \sum_{i=0}^{n} \sum_{s=0}^{i} \frac{(y-x)^{i-s} h^s}{s!(i-s)!} f_i(x) + (y-x+h)^n \epsilon_x(y-x+h) \\ &= \sum_{s=0}^{n} \frac{h^s}{s!} \left(\sum_{i=s}^{n} \frac{(y-x)^{i-s}}{(i-s)!} f_i(x) \right) + (y-x+h)^n \epsilon_x(y-x+h). \end{aligned}$$

Equating these two expansions gives the desired result. □

Proof of Theorem 3. Let $x \in \overline{A_N}$. Let $1 > \epsilon > 0$ be given. Since x is a density point of V_x, and $\lim\limits_{h\to 0, x+h\in V_x} \epsilon_x(h) = 0$, there is $\delta_1 > 0$ such that for $x+h \in V_x$, $|\epsilon_x(h)| < \epsilon^n$. Let $\delta_2(x)$ from Lemma 4 correspond to $a = \frac{\epsilon}{(1+\epsilon)(2n+3)}$. Let $0 < \delta < \min(\frac{1}{N}, \frac{1}{(1+\epsilon)}\delta_1, \frac{1}{(1+\epsilon)}\delta_2(x))$.

Now let $y \in A_N$, with $|y-x| < \delta$. From $V_x \cap V_y \cap (y, y+\epsilon|y-x|)$ we select $(n+1)$ points $y+h_0, y+h_1, \dots, y+h_n$ such that for $i \neq j$, $|h_i - h_j| \geq \frac{\epsilon|y-x|}{2n+3}$ as follows. Divide the interval $[y, y+\epsilon|y-x|]$ into $(2n+3)$ subintervals of equal length. Let I denote one of these intervals. If $y > x$, set $h = (1+\epsilon)|y-x|$, otherwise $h = -|y-x|$. Then I is inside $(x, x+h)$ and $\lambda(I) = \frac{\epsilon|y-x|}{(2n+3)} \geq a|h|$. Hence by Lemma 4, $\lambda(V_x \cap I) > 1/2\lambda(I)$. Also I is inside $(y, y+\epsilon|y-x|)$; so by the definition of A_N and Lemma 4, applied with $a = \frac{1}{2n+3}$, we obtain $\lambda(V_y \cap I) > 1/2\lambda(I)$. Combining these inequalities we obtain $\lambda(V_x \cap V_y \cap I) \neq \emptyset$. Now, it is enough to select the points from every other interval, for in that case for $i \neq j$ we will have $|h_i - h_j| \geq \lambda(I) = \frac{\epsilon|y-x|}{(2n+3)}$.

By substituting h in (5) with h_j for $j = 0, 1, \dots, n$, we obtain a system of linear equations

$$\left\{ \sum_{s=0}^{n} (h_j)^s X_s = b_j \mid 0 \leq j \leq n \right\}$$

where

$$X_s = \frac{1}{s!} \left(f_s(y) - \sum_{i=s}^{n} \frac{(y-x)^{i-s}}{(i-s)!} f_i(x) \right)$$

and

$$b_j = (y-x+h_j)^n \epsilon_x(y-x+h_j) - (h_j)^n \epsilon_y(h_j).$$

Let

$$\Delta = \begin{bmatrix} 1 & h_0 & \cdots & (h_0)^n \\ 1 & h_1 & \cdots & (h_1)^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & h_n & \cdots & (h_n)^n \end{bmatrix}$$

and let Δ_s be the matrix obtained by replacing the s-th column of Δ with the values $b_0, ..., b_n$. Then

$$|\det(\Delta)| = \left| \prod_{i>j} (h_i - h_j) \right| \geq \left(\frac{\epsilon\,|y-x|}{2n+3} \right)^{\frac{n(n+1)}{2}} > 0.$$

In the expansion of $\det(\Delta_s)$ about the s-th column, each minor is the sum of $n!$ terms of the form $(\pm)\prod_{j\neq s}(h_j)^{k_j}$ where $\sum_{j\neq s} k_j = \frac{n(n+1)}{2} - s$. Since $|h_j| < \epsilon|y-x|$ and $|\epsilon_x(y-x+h_j)| \leq \epsilon^n, |\epsilon_y(h_j)| < 1$, we have $|b_j| \leq \epsilon^n |y-x|^n((1+\epsilon)^n + 1)$. Hence

$$\begin{aligned} |\det(\Delta_s)| &\leq \sum_{j=0}^{n} |b_j|\, n!(\epsilon|y-x|)^{\frac{n(n+1)}{2}-s} \\ &\leq nn!\epsilon^{\frac{n(n+1)}{2}+n-s}|y-x|^{\frac{n(n+1)}{2}+n-s}((1+\epsilon)^n+1). \end{aligned}$$

By Cramer's Rule

$$\begin{aligned} &\left|\frac{1}{s!}\left(f_s(y) - \sum_{i=s}^{n}\frac{(y-x)^{i-s}}{(i-s)!}f_i(x)\right)\right| \\ =\ & \left|\frac{\det(\Delta_s)}{\det(\Delta)}\right| \leq nn!\epsilon^{n-s}|y-x|^{n-s}(2n+3)^{\frac{n(n+1)}{2}}((1+\epsilon)^n+1) \\ =\ & M\epsilon^{n-s}|y-x|^{n-s}, \end{aligned}$$

where $M = nn!(2n+3)^{\frac{n(n+1)}{2}}((1+\epsilon)^n+1)$. Since ϵ is arbitrarily small, for $s < n$ we obtain that $\lim_{y\to x, y\in A_N} \frac{\left(f_s(y) - \sum_{i=s}^{n}\frac{(y-x)^{i-s}}{(i-s)!}f_i(x)\right)}{|y-x|^{n-s}} = 0$.

Finally, if $y \in \overline{A_N}$ with $|y-x| < \delta$, pick a sequence $\{y_m\} \subset A_N$ converging to y, and such that $|y_m - x| < \delta$. Replacing y with y_m, the calculation above shows that $\left|f_s(y_m) - \sum_{i=s}^{n}\frac{(y_m-x)^{i-s}}{(i-s)!}f_i(x)\right| \leq M\epsilon^{n-s}|y_m - x|^{n-s}$. Replacing x with y, the same formulas yield $\lim_{m\to\infty} f_s(y_m) = f_s(y)$ for $0 \leq s < n$. Letting $m \to \infty$, we obtain that $\left|f_s(y) - \sum_{i=s}^{n}\frac{(y-x)^{i-s}}{(i-s)!}f_i(x)\right| \leq M\epsilon^{n-s}|y-x|^{n-s}$. Since ϵ was arbitrary,

$$\lim_{y\to x, y\in\overline{A_N}} \frac{\left(f_s(y) - \sum_{i=s}^{n}\frac{(y-x)^{i-s}}{(i-s)!}f_i(x)\right)}{|y-x|^{n-s}} = 0.$$

□

There are some interesting consequences of Theorem 3 and the fact that $\bigcup_{N=1}^{\infty} A_N = R$.

Corollary 6. *Suppose that f is a measurable function defined on a measurable set E. Let $F \subset E$ be the set of points at which f is n-times approximately Peano differentiable. Then for almost every $x \in F$ and for all $0 \leq s < n$, f_s is $(n-s)$-times approximately Peano differentiable with $(f_s)_j = f_{s+j}$ for $j = 1, 2, \ldots, n-s$.*

Proof. It is a known fact that the set F is measurable. Let $A_N \subset F$ be the same as in Theorem 3. Then by the proof of Theorem 3, we have that the conclusion of Theorem 3 holds on $\overline{A_N} \cap F$. Then if $x \in \overline{A_N} \cap F$ is a density point of $\overline{A_N} \cap F$, then for all $0 \leq s < n$, f_s is $(n-s)$-times approximately Peano differentiable at x with $(f_s)_j(x) = f_{s+j}(x)$ for $j = 1, 2, \ldots, n-s$. Since $\bigcup_{N=1}^{\infty} A_N = F$, the same is true for almost every point of F. □

Corollary 6 is a generalization of the following result by Marcinkiewicz and Zygmund. (See Theorem 4.26, page 77 in [13].)

Theorem 7 (Marcinkiewicz-Zygmund). *If f is n-times Peano differentiable on a set E of positive measure, then for almost every $x \in E$ and for all $1 \leq j \leq n$, $f_j(x)$ is the approximate derivative of $f_{j-1}(x)$.*

The second interesting consequence of Theorem 3 deals with the so-called class $[\Delta']$. In [1] the authors introduced the class of functions $[\Delta']$, as those real-valued functions, f, for which there is a decomposition of the real line into a countable collection of closed sets, $\{A_N\}$, and a sequence of differentiable functions, $\{g_N\}$, such that $f = g_N'$ on A_N. In that paper they showed that approximate derivatives are in $[\Delta']$. One natural generalization of this class to higher order derivatives is the following definition.

Definition 8. We say that a real-valued function $g \in [\Delta^n]$, if there are a countable collection of closed sets, $\{H_N\}$, and a sequence of n-times Peano differentiable functions, $\{g_N\}$, such that on H_N, g agrees with the n-th Peano derivative of g_N.

Corollary 9. *Suppose that f is n-times approximately Peano differentiable on R. Then $f_n \in [\Delta^n]$.*

Proof. From Theorem 3, it follows that relative to the closed sets $\overline{A_N}$, for $0 \leq s < n$, f_s is $(n-s)$-times Peano differentiable with for $0 \leq j \leq (n-s)$, the j-th Peano derivative of f_s equals f_{s+j}. Thus the restriction, $f_{|\overline{A_N}}$, of f to the set $\overline{A_N}$ satisfies the conditions of Theorem 3.3 from [5], and by that theorem there is an n-times Peano differentiable extension, g_N, of $f_{|\overline{A_N}}$. Therefore, $f \in [\Delta^n]$. □

It would be interesting to study this class, but that is not our immediate objective here.

4. Approximate Peano derivatives in the extended sense

Now we turn our attention to approximate Peano derivatives with infinite values allowed. Our first goal is to show that f_{n+1} is a Baire one function. Since the existence of f_{n+1} in the extended sense, implies finite existence of f_n, we will be able to use Theorem 3 from the previous section. But first we will need to show that under this additional assumption, that is, the existence of f_{n+1} in the extended sense, f_n is continuous on the sets $\overline{A_N}$ from Theorem 3.

Theorem 10. *Suppose f is $(n+1)$-times approximately Peano differentiable on R with infinite values allowed. Let A_N denote the sets from Theorem 3. Then, f_n is continuous on $\overline{A_N}$.*

Before we prove this theorem we remind the reader that the remainder $\epsilon_x(h)$ in the definition of infinite approximate Peano derivatives satisfies the following equations:

$$(2) \qquad \lim_{h\to 0,\, x+h\in V_x} \epsilon_x(h) = 0 \quad \text{and} \quad \lim_{h\to 0,\, x+h\in V_x} \frac{1}{(n+1)!}\frac{\epsilon_x(h)}{h} = f_{n+1}(x).$$

The reader is also reminded that the statement of Theorem 10 is correct only for approximate Peano derivatives, and that it is false for ordinary approximate derivatives with infinite values allowed. This follows from the fact that in the definition of first order approximate Peano derivatives with infinite values allowed, the primitive f is approximately continuous (since $\lim_{h\to 0,\, x+h\in V_x} \epsilon_x(h) = 0$), which is not the case for ordinary approximate derivatives with infinite values allowed. (See the case $n=0$ following Definition 2.)

Proof of Theorem 10. Without loss of generality, we will assume the following for the sets V_x from Definition 2. If $f_{n+1}(x)>0$, then for all $x+h\in V_x$ we have that $\frac{\epsilon_x(h)}{h}>0$, if $f_{n+1}(x)<0$, then for all $x+h\in V_x$ we have that $\frac{\epsilon_x(h)}{h}<0$, and if $f_{n+1}(x)=0$, then for all $x+h\in V_x$, we have that $|\epsilon_x(h)|<|h|$.

Now let $x\in\overline{A_N}$. Let $\delta_2(x)$ from Lemma 4 correspond to $a=\frac{1}{3}$, and let $0<\delta<\min\left\{\frac{1}{2N},\frac{\delta_2(x)}{3}\right\}$. Let $y\in A_N$, $|y-x|<\delta$ and let $h=3|y-x|$. Then $|h|<\delta_2$ and the intervals $[y-2|y-x|,y-|y-x|]$ and $[y+|y-x|,y+2|y-x|]$ are inside $(x-h,x)$ and $(x,x+h)$ respectively; so by Lemma 4,

$$\lambda(V_x\cap[y-2|y-x|,y-|y-x|])>\frac{1}{2}|y-x|$$

and

$$\lambda(V_x\cap[y+|y-x|,y+2|y-x|])>\frac{1}{2}|y-x|.$$

Also by Lemma 4, the same inequalities hold with V_y in place of V_x. Hence the sets $V_x\cap V_y\cap[y-2|y-x|,y-|y-x|]$ and $V_x\cap V_y\cap[y+|y-x|,y+2|y-x|]$ are not empty. Select points $y+h_1$ from the first set and $y+h_2$ from the second set. Thus $h_1<0$, while $h_2>0$ and for $i=1,2$ we have $\frac{|h_i|}{|y-x|}\geq 1$. Also if $f_{n+1}(y)>0$, then $\epsilon_y(h_1)<0$ and $\epsilon_y(h_2)>0$, while if $f_{n+1}(y)<0$, then $\epsilon_y(h_2)<0$ and $\epsilon_y(h_1)>0$ and if $f_{n+1}(y)=0$, then $|\epsilon_y(h_i)|<|h_i|$ for $i=1,2$. We will use the identity (5) in the form

$$\begin{aligned} &f_n(x)-f_n(y)\\ &\quad=\sum_{s=0}^{n-1}\frac{(y-x)^{n-s}}{h_i^{n-s}}\frac{1}{s!}\frac{\left(f_s(y)-\sum\limits_{i=s}^{n}\frac{(y-x)^{i-s}}{(i-s)!}f_i(x)\right)}{(y-x)^{n-s}}\\ &\qquad-\frac{(y-x+h)^n}{h_i^n}\epsilon_x(y-x+h_i)+\epsilon_y(h_i)\ \text{where } i=1,2. \end{aligned}\tag{3}$$

So in the case $f_{n+1}(y)>0$ equation (3) yields the following inequalities:

$$\begin{aligned} &\sum_{s=0}^{n-1}\frac{(y-x)^{n-s}}{h_2^{n-s}}\frac{1}{s!}\frac{\left(f_s(y)-\sum\limits_{i=s}^{n}\frac{(y-x)^{i-s}}{(i-s)!}f_i(x)\right)}{(y-x)^{n-s}}\\ &\qquad-\frac{(y-x+h_2)^n}{h_2^n}\epsilon_x(y-x+h_2)\\ &\quad\leq f_n(x)-f_n(y)\\ &\quad\leq\sum_{s=0}^{n-1}\frac{(y-x)^{n-s}}{h_1^{n-s}}\frac{1}{s!}\frac{\left(f_s(y)-\sum\limits_{i=s}^{n}\frac{(y-x)^{i-s}}{(i-s)!}f_i(x)\right)}{(y-x)^{n-s}}\\ &\qquad-\frac{(y-x+h_1)^n}{h_1^n}\epsilon_x(y-x+h_1). \end{aligned}\tag{4}$$

In the case $f_{n+1}(y)<0$ we have the same inequalities but with the roles of h_1 and h_2 interchanged. Finally in the case $f_{n+1}(y)=0$, equation (3) yields the inequality

$$
\begin{aligned}
&|f_n(x) - f_n(y)| \\
&\quad \le \left| \sum_{s=0}^{n-1} \frac{(y-x)^{n-s}}{h_1^{n-s}} \frac{1}{s!} \frac{\left(f_s(y) - \sum_{i=s}^{n} \frac{(y-x)^{i-s}}{(i-s)!} f_i(x) \right)}{(y-x)^{n-s}} \right. \\
&\qquad \left. - \frac{(y-x+h_1)^n}{h_1^n} \epsilon_x(y-x+h_1) \right| + |h_1|.
\end{aligned} \tag{5}
$$

Letting $y \to x$, in inequalities (4) and (5) we obtain

$$\lim_{y \to x, y \in A_N} f_n(y) - f_n(x) = 0. \tag{6}$$

To show that $\lim_{y \to x, y \in \overline{A_N}} f_n(y) - f_n(x) = 0$, let $\epsilon > 0$ be given. Let $\delta > 0$ be such that if $z \in A_N$ with $|z-x| < \delta$, then $|f_n(z) - f_n(x)| < \epsilon$. Let $y \in \overline{A_N}$ with $|y-x| < \delta$. Select a sequence $\{y_m\}$ from A_N that converges to y, and such that $|y_m - x| < \delta$. By replacing x with y and y with y_m in (6), we have $\lim_{m \to \infty} f_n(y_m) = f_n(y)$. Then

$$|f_n(y) - f_n(x)| \le |f_n(x) - f_n(y_m)| + |f_n(y) - f_n(y_m)| \le \epsilon + |f_n(y) - f_n(y_m)|.$$

Letting $m \to \infty$ we obtain that $\lim_{y \to x, y \in \overline{A_N}} f_n(y) - f_n(x) = 0$. □

Now we are ready to prove Baire one property for approximate Peano derivatives.

Theorem 11. *Suppose that $f : R \to R$, is $(n+1)$-times approximately Peano differentiable with infinite values allowed. Then, f_{n+1} is Baire one.*

Proof. It is enough to show that for every integer c, the sets $\{x : f_{n+1}(x) > c\}$ and $\{x : f_{n+1}(x) < c\}$ are F_σ sets. For a positive integer k, we define

$$N_k(x,h) = \{y \in (x-h, x+h) : \frac{1}{(n+1)!} \frac{\epsilon_x(y-x)}{y-x} \ge c + \frac{1}{k}\}.$$

The substitution $h = y - x$ in the second equality in (2) yields

$$\lim_{y \to x,\ y \in V_x} \frac{1}{(n+1)!} \frac{\epsilon_x(y-x)}{y-x} = f_{n+1}(x)$$

where the set V_x has density 1 at x. It follows that if $f_{n+1}(x) > c$, then for some positive integer k we have

$$\lambda(N_k(x,h)) \ge h \quad \text{for all } 0 < h < \frac{1}{k}. \tag{7}$$

Let A_N be the sets from Theorem 3, and let

$$M_{k,N} = \{x \in \overline{A_N} : f_{n+1}(x) > c \text{ and } \lambda(N_k(x,h)) \ge h \text{ for all } 0 < h < \frac{1}{k}\}.$$

We will show that $M_{k,N}$ is a closed set. To that end let x_j be a sequence in $M_{k,N}$ converging to x. Then both the sequence $\{x_j\}_{j=1}^{\infty}$ and x are in $\overline{A_N}$. From theorems

3 and 10 and by recalling Definition 2 we have that for every fixed y the expression

$$\frac{1}{(n+1)!}\frac{\epsilon_x(y-x)}{y-x} = \frac{f(y) - \sum_{s=0}^{n} \frac{(y-x)^s}{s!} f_s(x)}{(y-x)^{n+1}}$$

is continuous on $\overline{A_N}$. So if $y \in \bigcap_{j=1}^{\infty} \bigcup_{s=j}^{\infty} N_k(x_s, h)$, then y is in $N_k(x_j, h)$ for infinitly many j's and thus $\frac{1}{(n+1)!}\frac{\epsilon_x(y-x)}{y-x} \geq c + \frac{1}{k}$. It follows that $\bigcap_{j=1}^{\infty} \bigcup_{s=j}^{\infty} N_k(x_s, h) \subset N_k(x, h)$. On the other hand, since the measure of each set in the nested sequence $\left\{ \bigcup_{s=j}^{\infty} N_k(x_s, h) \right\}_{j=1}^{\infty}$ is greater than or equal to h, we have that $\lambda(N_k(x, h)) \geq h$ for all $0 < h < \frac{1}{k}$. Hence $f_{n+1}(x) > c$, and therefore $x \in M_{k,N}$.

Finally, since

$$\{x : f_{n+1}(x) > c\} = \bigcup_{N=1}^{\infty} \bigcup_{k=1}^{\infty} M_{k,N},$$

we have that the sets $\{x : f_{n+1}(x) > c\}$ are F_σ. The proof that $\{x : f_{n+1}(x) < c\}$ are F_σ is similar. □

5. Darboux and Denjoy-Clarkson properties

We need the Baire one property of f_{n+1} that was established in the previous section, to prove the Darboux and Denjoy-Clarkson properties of f_{n+1}. But first we have to prove the following auxiliary result.

Theorem 12. *If $f_{n+1}(x)$ exists, finite or infinite, for every x in an interval, I, and if f_{n+1} is bounded above or below on I, then the $(n+1)$-th ordinary derivative, $f^{(n+1)}$, exists and the two are equal.*

Proof. We will prove this theorem under the assumption that f_{n+1} is bounded from below; the other case follows from this one by replacing f with $-f$. It is enough to show that under the assumptions of the theorem, the $(n+1)$-st Peano derivative of f, exists and equals f_{n+1}, since then the result follows from Theorem 3.2 in [7]. If $f_{n+1}(x)$ is finite, then the existence and equality of the $(n+1)$-st Peano derivative of f was proved by Lee in [8]; so it remains only to consider the case $f_{n+1}(x) = +\infty$. As in Lee's proof for the finite case, by adding a suitable polynomial of degree less than $n+1$ and by repeatedly applying Theorem ($\mathrm{II}_{\text{n-1}}$) from [8], we can assume that $f_k(x) = 0$ for $k = 0, 1, \ldots, n$, and that f is monotone increasing on I. Under these assumptions it remains to show that

$$\lim_{h \to 0} \frac{f(x+h)}{h^{n+1}} = +\infty. \tag{8}$$

Since $f_{n+1}(x) = +\infty$, the limit in (8) is $+\infty$, as $h \to 0$ and $h \in V_x$. Now for $h \notin V_x$, $h > 0$ select $\frac{h}{2} < h_1 < h$ and $h_1 \in V_x$. Then monotonicity of f implies

$$\frac{f(x+h_1)}{h_1^{n+1}} \leq 2^{n+1} \frac{f(x+h)}{h^{n+1}}. \tag{9}$$

As $h \to 0^+$, the left-hand side of the inequality in (9) is $+\infty$; so (8) is established in case $h \to 0^+$. In case $h \to 0^-$, we consider separately cases $(n+1)$ even and

$(n+1)$ odd. In case $(n+1)$ is even, for sufficiently small h we can select $h_1 \in V_x$ such that $2h < h_1 < h$. The monotonicity of f yields $\frac{f(x+h_1)}{h_1^{n+1}} \le \frac{f(x+h)}{h^{n+1}}$. In case $(n+1)$ is odd, for sufficiently small h we can select $h_1 \in V_x$ such that $h < h_1 < \frac{h}{2}$. The monotonicity of f yields the same inequality as in (9). Taking the limit we get the desired result. □

Theorem 13. *If $f_{n+1}(x)$ exists, finite or infinite, for every x in an interval, then: (1) f_{n+1} has the Darboux property and (2) f_{n+1} has the Denjoy-Clarkson property.*

Proof. The proof of both results is almost identical to the proofs of Theorem 3.4 in [7]. The only necessary adjustments one has to make is to replace the reference to Theorem 1(i) from [12] in the proof of Theorem 3.4 (1), with the reference to Theorem ($\mathrm{II}_{\mathrm{n-1}}$) from [8] which is the approximate version of Theorem 1(i) from [12]. The proof of (2) is identical to the proof of Theorem 3.4 (2) in [7]. □

We end this paper by remarking that the corresponding approximate analogs of the results from [7], namely, Lemma 4.1 and Theorem 4.2, have been established in [6].

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