SOME REMARKS ON SPREADING MODELS
AND MIXED TSIRELSON SPACES

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ABSTRACT. We prove that if a Banach space with a bimonotone shrinking basis does not contain \( \ell_1^\omega \) spreading models but every block sequence of the basis contains a further block sequence which is a \( c - \ell_1^\omega \) spreading model for every \( n \in \mathbb{N} \), then every subspace has a further subspace which is arbitrarily distortable. We also prove that a mixed Tsirelson space \( T[(S_n, \theta_n)_n] \), such that \( \theta_n \downarrow 0 \), does not contain \( \ell_1^{\omega_2} \) spreading models.

INTRODUCTION

A Banach space \( X \) with a basis \((e_i)\) is an asymptotic \( \ell_1 \) space if there exists \( c > 0 \) such that for all \( n \) and all \( e_n < x_1 < \ldots < x_n \),

\[
\left\| \sum_{i=1}^{n} x_i \right\| \geq c \sum_{i=1}^{n} \| x_i \| .
\]

The first non-trivial example of an asymptotic \( \ell_1 \) space was discovered by Tsirelson [17]. Recent results [6], [7], [15], have shown the necessity of studying the higher ordinal structure of an asymptotic \( \ell_1 \) Banach space in order to obtain results on the global structure of its infinite dimensional subspaces. A normalized sequence \((x_n)_n\) in a Banach space \( X \) is said to be a \( c - \ell_1^\xi \) spreading model if

\[
\| \sum_{n \in F} \alpha_n x_n \| \geq c \sum_{n \in F} \| \alpha_n \| \quad \forall F \in S_\xi, \ (\alpha_n)_{n \in F} \subset \mathbb{R},
\]

where \( S_\xi, \xi < \omega_1 \), are the generalized Schreier families defined in [1].

It is well known that if a separable Banach space \( X \) does not contain \( \ell_1 \), then there exists \( \xi < \omega_1 \), such that \( X \) does not contain an \( \ell_1^\xi \) spreading model. A complete classification of normalized weakly null sequences, in connection with spreading models, has been provided in [7].

Spreading models is a basic tool for the study of the asymptotic structure of a Banach space. The structure of the spreading models may even determine the geometry of the space [14]. Spreading models have been employed in [10] to prove the existence of strictly singular non-compact operators in certain Hereditarily Indecomposable mixed Tsirelson spaces.
The idea of investigating the geometry of a Banach space by studying its asymptotic finite-dimensional subspaces emerged naturally in recent studies related to problems of distortion, i.e., the stabilization of equivalent norms on infinite-dimensional subspaces [2, 3, 5, 9, 15].

A Banach space \((X, \| \cdot \|)\) is said to be \(\lambda\)-distortable if there exists an equivalent norm \(| \cdot |\) on \(X\) so that
\[
\inf_Y \sup \left\{ \frac{|x|}{|y|} : x, y \in S(Y, \| \cdot \|) \right\} \geq \lambda,
\]
where the infimum is taken over all infinite-dimensional subspaces \(Y\) of \(X\). \(X\) is arbitrarily distortable if it is \(\lambda\)-distortable for all \(\lambda > 1\). In section 2, we prove the following:

**Theorem.** Let \(X\) be a Banach space with a bimonotone shrinking basis \((e_i)\) such that

1. \(X\) does not contain an \(\ell_1^\ell\) spreading model.
2. For every \(n \in \mathbb{N}\), every block sequence of \((e_i)\) contains a further block sequence which is a \(c - \ell_1^\ell\) spreading model.

Then every subspace of \(X\) contains a further subspace which is arbitrarily distortable.

The relation of the distortion problem with spreading models has been studied in [15], [9]. In [9] a criterion has been provided related to \(\ell_1^\ell\) spreading models, which implies the distortion of certain asymptotic \(\ell_1\) Banach spaces. The motivation for this theorem was the space constructed in [6], where an example of a mixed Tsirelson space \(X\) was given which has \(c - \ell_1^\ell\) spreading models in every block subspace but has no \(\ell_1^\ell\) spreading model. The norm of \(X\) satisfies, for an appropriate sequence \((k_j, \theta_j)\), the implicit equation
\[
|\!|\!|x|\!|\!| = \max \left\{ \|x\|_\infty, \sup \left\{ \sum_{k=1}^{n} \|x|_{n, +\infty}\|_{j_k} : n \in \mathbb{N}, j_1 < j_2 < \ldots < j_n \right\} \right\},
\]
where \(|\!|\!|x|\!|\!| = \sup \{\theta_j \sum_{i \in F} \|E_i x\| : (E_i x)_{i \in F} \in S_{\theta_j}\text{-admissible}\}.

To prove the theorem we use some results proved by E.Odell, N.Tomczak-Jaegermann and R.Wagner [15]. In this paper, for a Banach space \(X\) with basis \((e_i)\), certain indices \((\delta_\alpha(x_i))_{\alpha < \omega_1}\), for a block sequence \((x_i)\), have been introduced and studied. Roughly speaking, the indices \((\delta_\alpha(x_i))_{\alpha < \omega_1}\) measure the strong presence of \(\ell_1\) in the subspace \(\langle x_i \rangle\) in connection with the families \((S_\alpha)_{\alpha < \omega_1}\). The notion of \(\Delta\)-spectrum, \(\Delta(X)\), is also introduced. Roughly, \(\Delta(X)\) is the set of all \(\gamma = (\gamma_\alpha)_{\alpha < \omega_1}\), where \(\gamma_\alpha\) is the stabilization of \(\delta_\alpha(y_i)\) for some block basis \((y_i)\) of \((e_i)\). Using the stabilization result from [15], we prove that every block subspace has a further subspace \(Y\) such that: For every \(n \in \mathbb{N}\) there exist two asymptotic sets \(A_n\) and \(B_n\) in \(Y\) and a subset \(A_n^*\) of \(X^*\) such that the equivalent norm
\[
|\!|\!|x|\!|\!| = \gamma_n |\!|\!|x|\!|\!| + \sup \{x^*(x) : x^* \in A_n^* \}
\]
is a \(\approx \frac{1}{\gamma_n}\) distortion for \(Y\). Since \(\gamma_\omega = 0\), from the continuity of the indices \((\gamma_\alpha)_{\alpha < \omega_1}\) [15], we have that \(Y\) is arbitrarily distortable.

In section 3, we prove that in the mixed Tsirelson spaces \(T[(S_n, \theta_n)_n]\) the following holds.
Theorem. Let \( X = T[(S_n, \theta_n)_n] \) such that \( \theta_{n+m} \geq \theta_n \theta_m \), \( \lim_n \theta_n = 0 \). Then the space \( X \) does not contain \( \ell_0^2 \) spreading models.

In \([5]\) it has been proved that, assuming \( \lim_n \theta_n^{1/n} = 1 \), the space \( T[(S_n, \theta_n)_n] \) contains an \( \ell_0^2 \) spreading model. The existence of \( \ell_0^2 \) spreading models in these spaces is based on the disjoint representability of \( c_0 \) in these spaces \([4]\). Another approach to the existence of \( \ell_0^2 \) in certain mixed Tsirelson spaces has been provided in \([10]\). The key point for the proof of this theorem is to produce for every normalized block sequence \((x_n)\) of the basis, a vector in the linear span of \((x_n)\), whose norm is arbitrarily small yet its support with respect to \((x_n)\) belongs to \( S_{n,2} \). The dual of the original Tsirelson’s space \([17]\) contains no \( \ell_0^2 \) spreading model. This is due to the fact that every block sequence is equivalent to a subsequence of the basis.

1. Preliminaries

Notation. Let \((e_i)_{i=1}^{\infty}\) be a basic sequence. For \( x = \sum_{i=1}^{\infty} a_i e_i \) the support of \( x \) w.r.t. \((e_i)\) is the set \( \text{supp} x = \{ i \in \mathbb{N} : a_i \neq 0 \} \). The range of \( x \), written \( \text{range} (x) \), is the smallest interval of \( \mathbb{N} \) containing the support of \( x \). For finite subsets \( E, F \) of \( \mathbb{N} \), \( E < F \) means \( \max E < \min F \) or either \( E \) or \( F \) is empty. For \( n \in \mathbb{N}, E \subset \mathbb{N}, n < E \) (resp. \( E < n \)) means \( n < \min E \) (resp. \( \max E < n \)). For \( x, y \in \mathbb{N}_0 \), \( x < y \) means \( \text{supp} x < \text{supp} y \). For \( n \in \mathbb{N}, x \in \mathbb{N}_0 \), we write \( n < x \) (resp. \( x < n \)) if \( n < \text{supp} x \) (resp. \( \text{supp} x < n \)). We say that the sets \( E_i \subset \mathbb{N}, i = 1, \ldots, n \), are successive if \( E_1 < E_2 < \ldots < E_n \). Similarly, the vectors \( x_i, i = 1, \ldots, n \), are successive if \( x_1 < x_2 < \ldots < x_n \). If \((x_i)\) is a block sequence of \((e_i)\) we write \((x_i) \prec (e_i)\). For \( x = \sum_{i=1}^{\infty} a_i e_i \) and \( E \) a subset of \( \mathbb{N} \), we denote by \( Ex \) the vector \( Ex = \sum_{i \in E} a_i e_i \).

For an infinite subset \( M \) of \( \mathbb{N} \) we denote by \([M]\) the class of infinite subsets of \( M \) and by \([M]^{<\omega}\) the class of finite subsets of \( M \).

The generalized Schreier families \( \{S_\xi\}_{\xi<\omega_1}\), introduced in \([1]\), are defined by transfinite induction as follows:

\[ S_0 = \{\{n\} : n \in \mathbb{N}\} \cup \emptyset. \]

Suppose that the families \( S_\alpha \) have been defined for all \( \alpha < \xi \).

If \( \xi = \zeta + 1 \), we set

\[ S_\xi = \{F \in [\mathbb{N}]^{<\omega} : F = \bigcup_{i=1}^{n} F_i, n \in \mathbb{N}, \forall i \leq n F_i \in S_\zeta \text{ and } n \leq F_1 < \ldots < F_n \cup \emptyset\}. \]

If \( \xi \) is a limit ordinal, let \((\xi_n + 1)_n \) be a sequence of successor ordinals which strictly increases to \( \xi \). We set

\[ S_\xi = \{F \in \mathbb{N}^{<\omega} : \text{ for some } n \in \mathbb{N}, n \leq \min F \text{ and } F \in S_{\xi_n+1}\}. \]

If \( N = (n_i)_i \) is an infinite subset of \( \mathbb{N} \), then we define

\[ S_\xi[N] = \{F : F \subset N, F \in S_\xi\} \quad \text{and} \quad S_\xi(N) = \{(n_i)_{i \in F} : F \in S_\xi\}. \]

Proposition 1.1. (a) \([2]\) Let \( N \in [\mathbb{N}] \). Then there exists \( L = (\ell_i) \in [\mathbb{N}] \) so that for all \( \alpha < \omega_1 \),

\[ (\ell_i)_{i \in F} \in S_\alpha \Rightarrow (\ell_i)_{i \in F \setminus \{\min F\}} \in S_\alpha(N). \]

(b) \([15]\) Let \( \beta < \alpha < \omega_1 \). There exists \( n_0 \in \mathbb{N} \) such that

\[ n_0 < F \in S_\beta \Rightarrow F \in S_\alpha. \]

(c) \([15]\) Let \( \beta < \alpha < \omega_1 \). There exists \( M \in [\mathbb{N}] \) such that \( S_\alpha[S_\beta](M) \subset S_{\beta+\alpha} \).
We next pass to the definition of the \textit{repeated averages hierarchy} introduced in [7]. We let \( (e_n) \) denote the standard basis of \( c_{00} \). For every countable ordinal \( \xi \) and every \( M \in [\mathbb{N}] \), we define a convex block sequence \( \left( \xi_n^M \right)_{n=1}^\infty \) of \( (e_n) \) by transfinite induction on \( \xi \) in the following manner:

If \( \xi = 0 \) and \( M = (m_n)_{n=1}^\infty \), then \( \xi_n^M = e_{m_n} \), for all \( n \in \mathbb{N} \).

Assume that \( \left( \xi_n^M \right)_{n=1}^\infty \) has been defined for all \( \zeta < \xi \) and \( M \in [\mathbb{N}] \). Let \( \xi = \zeta + 1 \). We set

\[
\xi_1^M = \frac{1}{m_1} \sum_{i=1}^{m_1} \xi_i^M
\]

where \( m_1 = \min M \). Suppose that \( \xi_1^M < \ldots < \xi_n^M \) have been defined. Let

\[
M_n = \{ m \in M : m > \max \supp \xi_n^M \} \quad \text{and} \quad k_n = \min M_n.
\]

Set

\[
\xi_{n+1}^M = \frac{1}{k_n} \sum_{i=1}^{k_n} \xi_i^M = \xi_n^M.
\]

If \( \xi \) is a limit ordinal, let \( (\xi_n + 1)_n \) be the sequence of ordinals associated to \( \xi \), and also let \( M \in [\mathbb{N}] \). Define

\[
\xi_1^M = [\xi_{m_1} + 1]_1^M
\]

where \( m_1 = \min M \). Suppose that \( \xi_1^M < \ldots < \xi_n^M \) have been defined. Let

\[
M_n = \{ m \in M : m > \max \supp \xi_n^M \} \quad \text{and} \quad k_n = \min M_n.
\]

Set

\[
\xi_{n+1}^M = [\xi_{k_n} + 1]_1^M.
\]

The inductive definition of \( \left( \xi_n^M \right)_{n=1}^\infty, M \in [\mathbb{N}] \), is now complete. We note that \( \supp \xi_n^M \in \mathcal{S}_\xi \), for all \( M \in [\mathbb{N}] \), \( \xi < \omega_1 \) and \( n \in \mathbb{N} \).

**Definition 1.2.** (a) Let \( k \in \mathbb{N} \). A finite sequence \( \left( E_i \right)_{i=1}^m \) of successive subsets of \( \mathbb{N} \) is said to be \( \mathcal{S}_k \)-admissible if \( \{ \min E_i \}_{i=1}^m \in \mathcal{S}_k \). A finite block sequence \( \left( x_i \right)_{i=1}^m \) in \( c_{00} \) is said to be \( \mathcal{S}_k \)-admissible if \( \{ \supp x_i \}_{i=1}^m \) is \( \mathcal{S}_k \)-admissible. If \( \left( y_i \right) \) is a block sequence of the basis of \( c_{00} \), \( \left( x_i \right) \) is a block sequence of \( \left( y_i \right) \), and \( \tilde{E}_i \) is the range of \( x_i \) with respect to (w.r.t.) the basic sequence \( \left( y_i \right) \), then the block sequence \( \left( x_i \right) \) is \( \mathcal{S}_n \)-admissible w.r.t. \( (y_i) \) if \( \{ \min \tilde{E}_i \}_{i=1}^n \in \mathcal{S}_n \).

(b) Let \( \{ k_n \}_n \) be an increasing sequence of integers and \( \{ \theta_n \} \subset (0, 1) \) such that \( \theta_n \searrow 0 \). The mixed Tsirelson space \( X = T \left( \left( \mathcal{S}_{k_n}, \theta_n \right)_{n=1}^\infty \right) \) is the completion of \( c_{00} \) under the norm which satisfies the implicit equation

\[
\| x \| = \max \{ \| x \|_{\infty}, \sup_n \theta_n \{ \sup \sum_{i=1}^m \| E_i(x) \| \} \},
\]

where the inside supremum is taken over all \( \mathcal{S}_{k_n} \)-admissible families \( \{ E_i \}_{i=1}^m, m \in \mathbb{N} \). The space \( X \) is a reflexive Banach space, and the sequence \( \left( e_i \right) \) is a basis on \( X \).

An essential role in our proofs is played by the following special vectors.

**Definition 1.3.** Let \( (x_n) \) be a normalized block sequence of \( (e_n) \), \( \varepsilon > 0 \) and \( \zeta < \xi < \omega_1 \). Set \( m_n = \min \supp x_n \) and \( M = (m_n)_n \).
An \((\varepsilon, \xi, \zeta)\) basic special convex combination (basic s.c.c.) for \(M\) is any vector of the form \(\xi^F = \sum_n \xi^F_n(m_n)e_{m_n}, \ L \in [M]\) and \(\|\xi^F\|_\zeta < \varepsilon\), where \(\|\sum \alpha_i e_i\|_\zeta = \sup\{\sum_{i \in F} |\alpha_i| : F \in \mathcal{S}_\zeta\}\).

An \((\varepsilon, \xi, \zeta)\) special convex combination, (s.c.c.), of \((x_n)\) is any vector of the form \(\sum_n \xi^F_n(m_n)x_n\), such that \(\sum_n \xi^F_n(m_n)e_{m_n}\) is an \((\varepsilon, \xi, \zeta)\)-basic s.c.c.

**Proposition 1.4** ([15]). For every \(M \in [N]\), \(\varepsilon > 0\) and all ordinals \(\zeta < \xi < \omega_1\), there exists \(N \in [M]\) such that \(\|\xi^F\|_\zeta < \varepsilon\) for all \(L \in [N]\).

It is not hard to see that the average \(n^F_{1}\) is a \((\frac{3}{\min L}, n, n - 1)\)-basic s.c.c. for every \(L \in [M]\).

**Definition 1.5.** Let \(\xi < \omega_1\) and \(\delta > 0\). A normalized sequence \((x_n)\) in a Banach space is an \(\delta - \ell_1^\xi\) spreading model if
\[
\|\sum_{i \in F} \alpha_i x_i\| \geq \delta \sum_{i \in F} |\alpha_i|
\]
for every \(F \in \mathcal{S}_\xi\) and all choices of scalars \((\alpha_i)_{i \in F}\).

\((x_n)\) is called an \(\ell_1^\xi\) spreading model if it is a \(\delta - \ell_1^\xi\) spreading model for some \(\delta > 0\).

2.

**Theorem 2.1.** Let \(X\) be a Banach space with a bimonotone shrinking basis \((e_i)\) such that

1. \(X\) does not contain an \(\ell_1^\xi\) spreading model.
2. There exists \(c > 0\) such that for every \(n \in \mathbb{N}\), every block sequence of \((e_i)\) contains a further block sequence which is a \(c - \ell_1^n\) spreading model.

Then every subspace of \(X\) contains a further subspace which is arbitrarily distortable.

Our proof is based on the following results of E.Odell, N. Tomczak-Jaegermann and R.Wagner [15].

**Definition 2.2** ([15]). Let \(\mathcal{F}\) be a regular set of finite subsets of \(\mathbb{N}\). For a basic sequence \((x_i)\) in \(X\) we define
\[
\delta_\mathcal{F} = \sup\{\delta \geq 0 : \|\sum_{i=1}^k y_i\| \geq \delta \sum_{i=1}^k \|y_i\| \text{ whenever } (y_i)_1^k < (x_i)\}
\]
is \(\mathcal{F}\)-admissible w.r.t. \((x_i)\).

For \(\alpha < \omega_1\) we set \(\delta_\alpha(x_i) = \delta_{S_\alpha}(x_i)\) and \(\delta_\alpha(X) = \delta_{S_\alpha}(X)\).

**Definition 2.3** ([15]). Let \(X\) be a Banach space with basis \((e_i)\), and let \(\gamma = (\gamma_\alpha)_{\alpha < \omega_1} \subset \mathbb{R}\). We say that a basic sequence \((x_i)\) in \(X\) \(\Delta\)-stabilizes \(\gamma\), if there exists \(\varepsilon_n \setminus 0\) so that for every \(\alpha < \omega_1\) there exists \(m \in \mathbb{N}\) such that for every \(n \geq m\) if \((y_i) < (x_i)_m\), then \(|\delta_\alpha(y_i) - \gamma_\alpha| < \varepsilon_n\).

The \(\Delta\)-spectrum of \(X\), \(\Delta(X)\), is defined to be the set of all \(\gamma\)'s so that there exists \((x_i) < (e_i)\) such that \((x_i)\) \(\Delta\)-stabilizes \(\gamma\).

**Proposition 2.4** ([15] Proposition 4.11]). (1) Let \(X\) be a Banach space with a basis \((e_i)\). Then there exists \(\gamma = (\gamma_\alpha)_{\alpha < \omega_1}\) and \((x_i)\) block sequence of \((e_i)\) so that \((x_i)\) \(\Delta\)-stabilizes \(\gamma\).

(2) \((\gamma_\alpha)_{\alpha < \omega_1}\) is a continuous function of \(\alpha\).
In the above definitions, the admissibility refers with respect to the block basis \((x_i)\) itself. It has been proved in [2] that, if we consider reference level for admissibility fixed the basis, then these two concepts of spectrum actually coincide.

Proof of Theorem 2.1. Let \(W\) be a block subspace of \(X\). Then there exists a block sequence \((y_i)\) in \(W\) and \((\gamma_\alpha)_{\alpha \in \omega_1}\) such that \((y_i)\) \(\Delta\)-stabilizes \((\gamma_\alpha)_{\alpha \in \omega_1}\). Let \(\epsilon_n \downarrow 0\) be the sequence in the stabilization of \((\gamma_\alpha)_{\alpha \in \omega_1}\) by \((y_i)\), i.e. for every \(\alpha < \omega_1\) there exists \(m \in \mathbb{N}\) such that for every \(n \geq m\) if \((x_i) \prec (y_i)\), then \(|\delta_\alpha (x_i) - \gamma_\alpha| < \epsilon_n\).

Inductively choose a strictly increasing sequence \((m(n))\) of integers such that \(\epsilon_{m(n)} < \frac{(\delta_1(y_i))}{2}\) and

\[|\delta_n (x_i) - \gamma_n| < \epsilon_{m(n)}\text{ for all } (x_i) \prec (y_{m(i)})_n\text{.}

Let \(N_0 = (\min \sup y_{m(i)})_i = (n_i)_i\). Passing to a subset \(N = (n_i)_i\), we may assume that for every \(\alpha < \omega_1\), if \((n_i)_i \in \mathcal{S}_\alpha\), then \((n_i)_i \in \mathcal{S}_\alpha(N)\). In particular \((k_i)_i \in \mathcal{S}_\alpha\) we shall prove that the subspace \(Y = \text{span} \{y_{m(k_i)}\}\) is arbitrarily distortable. Let \(y^*_m(k_i) \in B^*_Y\) such that \(y^*_m(k_i) = \|y_{m(k_i)}\|\) and \(\text{range}(y^*_m(k_i)) = \text{range}(y_{m(k_i)})\) for every \(i \in \mathbb{N}\). From the hypothesis we have that \(X\) does not contain an \(\ell^1_\delta\) spreading model. It follows that \(\gamma_\omega = 0\), for otherwise there exists \((x_i) \prec (y_{m(k_i)})\) such that \(\delta_\omega(x_i) > 0\), and therefore \((x_i)\) would be an \(\delta_\omega(x_i) - \ell^1_\delta\) spreading model. Since \((\gamma_\alpha)\) is a continuous function of \(\alpha\) it follows that \(\gamma_n \downarrow 0\). We set

\[A^*_n = \{x^* : x^* = \frac{\gamma_n}{2} \sum_{i \in F} x^*_i, x^*_i \in B^{Y^*}\},
\]

\[y^*_{m(k_i)} \leq (x^*_i)_{i \in F} \text{ is } \mathcal{S}_n\text{-admissible w.r.t. } (y^*_m(k_i))_i\} ;
\]

\[A_n = \{y \in S(Y) : y \text{ is } \frac{1}{6}\text{-normed by } A^*_n\}\text{.}
\]

We observe that \(A^*_n \subset B_{Y^*}\). Indeed, first we observe that if \((x_i) \prec (y_{m(k_i)})_n\), then

\[\frac{1}{2} \delta_n (x_i) \leq \delta_n (x_i) - \epsilon_{m(k_i)} < \gamma_n < \delta_n (x_i) + \epsilon_{m(k_i)} \leq 2\delta_n (x_i)\text{,}
\]

since \(\delta_n (x_i) \geq (\delta_1(y_i))^{n} > 2\epsilon_{m(k_i)}\).

Let \(x^* = \frac{\gamma_n}{2} \sum_{i \in F} x^*_i \in A^*_n\) and for \(i \in F\) we set \(E_i = \text{range}(x^*_i)\). For every \(y \in S_Y\) we have that

\[|x^*(y)| \leq \frac{\gamma_n}{2} \sum_{i \in F} |x^*_i(y)| \leq \frac{\gamma_n}{2} \sum_{i \in F} \|\text{range}(x^*_i)\| \leq \delta_n ((y_{m(k_i)})_n) \sum_{i \in F} \|E_i y\| \leq \|y\|\text{,}
\]

since \((E_i(y))_{i \in F}\) is \(\mathcal{S}_n\)-admissible w.r.t. \((y_{m(k_i)})_n\), and the basis is bimonotone. Also \(A_n\) is an asymptotic set. Indeed, for any block sequence \((x_i) \prec (y_{m(k_i)})_n\) from the stabilization of \(\gamma_n\) we have that \(|\delta_n (x_i) - \gamma_n| < \epsilon_{m(k_i)}\), from which it follows that \(\delta_n (x_i) < \gamma_n + \epsilon_{m(k_i)}\). It follows from the definition of \(\delta_n (x_i)\) that there exists a block sequence \((w_i)_{i \in F}\) of \((x_i)\) which is \(\mathcal{S}_n\)-admissible w.r.t. \((x_i)\) and therefore w.r.t. \((y_{m(i)})\) as well, such that

\[\delta_n (x_i) \sum_{i \in F} \|w_i\| \leq \| \sum_{i \in F} w_i \| < (\gamma_n + \epsilon_{m(k_i)}) \sum_{i \in F} \|w_i\|\text{.}
\]
Let \( x^*_i \in B_Y \) be such that \( x^*_i(w_i) = \|w_i\| \) and \( \text{range}(x^*_i) \subset \text{range}(w_i) \). We set \( x^* = \frac{n_\ell}{2} \sum_{i \in F} x^*_i \in A^*_n \), and \( y = \frac{\sum_{i \in F} w_i}{\| \sum_{i \in F} w_i \|} \). Then we have that

\[
(2.1) \quad x^*(y) = \frac{\gamma_n}{2} \sum_{i \in F} \| w_i \| \geq \frac{\gamma_n}{2(\gamma_n + \varepsilon_{m(k_n)})} \geq \frac{1}{6},
\]

since \( \gamma_n \geq \delta_n(x_i) - \varepsilon_{m(k_n)} > \frac{(\delta_n(x_i))^n}{2} > \varepsilon_{m(k_n)}. \)

From the hypothesis we have that every normalized block sequence has a further block subsequence which is a \( c - \ell^n_1 \) spreading model, \( k \in \mathbb{N} \). It follows that for every \( \varepsilon > 0 \) every block sequence \( (z_i) \subset (y_{m(k_i)}) \) contains \( \frac{1}{2} \)-normalized \( (\varepsilon, n + 1, n) \)-s.c.c. Indeed let \( 0 < \varepsilon < \frac{1}{6} \) and \( (z_i) \) be an \( c - \ell^n_1 \) spreading model, and \( \text{supp}(z_i) = n_i \).

Let \( [n + 1]\ell^n_1 = \sum_{i \in F} \alpha_i e_{n_i} \) be an \( (n + 1) \)-average of a subset \( L \) of \( (n_i) \) with \( l_{\min} F > 3/\varepsilon \). Then by the remark following Proposition 1.3 \( \sum_{i \in F} \alpha_i e_{n_i} \) is an \( (\varepsilon, n + 1, n) \)-basic s.c.c., \( (n_i)_{i \in F} \in \mathcal{S}_{n + 1} \), and by the properties of the set \( N \), we have that \( (l_i)_{i \in G \setminus \min(F)} \in \mathcal{S}_{n + 1} \). Let \( G = \{i : i \in F\} \) and set \( x = \sum_{j \in G} \alpha_j z_j \). Then \( x \) is an \( (\varepsilon, n + 1, n) \) s.c.c., and

\[
\| x \| \geq c \sum_{j \in G \setminus \min(G)} \alpha_j \geq c(1 - \varepsilon),
\]

since \( (z_i) \) is a \( c - \ell^n_1 \) spreading model. We set

\[
B_n = \{b : b \text{ is an } \frac{c}{2} \text{-normalized } (\varepsilon_{m(k_n)}, n + 1, n) \text{-s.c.c.}
\]

of a normalized block sequence of the basis of \( Y \).

From the above it follows that \( B_n \) is an asymptotic set in \( Y \). For every \( b \in B_n \) we have

\[
(2.2) \quad |x^*(b)| \leq 3\gamma_n \text{ for every } x^* \in A^*_n.
\]

Indeed, let \( (z_i) \) be a normalized block sequence of the basis of \( Y \), \( b = \sum_{i \in G} b_i z_i \in B_n \), and \( x^* = \gamma_n/2 \sum_{k \in F} x^*_k \in A^*_n \). Set \( I = \{i \in G : \text{supp} z_i \cap \text{supp} x^*_k \neq \emptyset \} \) for at most one \( k_j \), and \( J = G \setminus I \). Also for every \( i \in J \), let \( K_i = \{k : \text{supp} z_i \cap \text{supp} x^*_k \neq \emptyset\} \).

Then \( (z_i)_{i \in J} \) is the union of at most two \( S_n \) admissible sets, hence

\[
\frac{\gamma_n}{2} \sum_{k} |x^*_k(\sum_{i} b_i z_i)| \leq \frac{\gamma_n}{2} \left( \sum_{i \in I} b_i \|z_i\| + \sum_{i \in J} b_i \sum_{k \in K_i} |x^*_k(z_i)| \right)
\]

\[
\leq \frac{\gamma_n}{2} \left( 1 + \frac{4\varepsilon_{m(k_n)}}{\gamma_n} \max_i \|z_i\| \right) \leq 3\gamma_n.
\]

Combining (2.1) and (2.2), it follows that the equivalent norm \( \|y\| = \gamma_n \|y\| + \sup \{x^*(y) : x^* \in A^*_n \} \) gives a \( \frac{1}{6\gamma_n(1 + 6/\gamma_n)} \) distortion on \( Y \). Since \( \inf \{\gamma_n : n \in \mathbb{N}\} = 0 \) we have that \( Y \) is arbitrarily distortable.

\textbf{Remark.} The arguments of Theorem 2.1 may give us another approach to the distortion of the space \( X \), constructed by E. Odell and Th.Schlumprecht [13], which does not have any \( \ell_p \), \( 1 \leq p \leq \infty \), as a spreading model. Following [15], we have to consider the indices \( \delta_{A_n}(x_i) \) for the families \( A_n = \{F \subset \mathbb{N} : \# F \leq n\} \) and to prove a stabilization result for the sequence \( (\delta_{A_n}(x_i))_n \) for every block subspace. Since \( \ell_1 \) is finitely block representable in \( X \), and \( X \) does not contain \( \ell_1 \) spreading models, we may deduce that every subspace has an arbitrarily distortable subspace.
Theorem [24] should be compared with the following result from [15]: Let $Y = \langle (y_i) \rangle$ be a subspace of $X$, and $(\gamma_\alpha)\alpha$ be stabilized by $(y_i)$. For $\alpha < \omega_1$ let $\hat{\gamma}_\alpha(Y) = \lim_n(\gamma_{\alpha \cdot n}(Y))^{1/n}$. If $\lim_n \gamma_{\alpha \cdot n}(Y)^{-n} = 0$, then $Y$ contains an arbitrarily distortable subspace.

3.

Theorem 3.1. Let $X = T((S_n, \theta_n)\alpha)$ be such that $\theta_{n+m} \geq \theta_n \theta_m$, $\lim_n \theta_n = 0$. Then the space $X$ does not contain an $\ell^2_1$ spreading model.

Proof. On the contrary assume that there exists a normalized block sequence $(x_i)_i$ which is an $2c - \ell^2_1$ spreading model for some constant $2c > 0$. We shall prove that for every $n_0 \in \mathbb{N}$ we have that $c \leq 10 \theta_{n_0}$, which yields that $c = 0$.

By Proposition 1.1(b) choose $k(0) \in \mathbb{N}$ such that if $k(0) \leq F \in S_\omega$, then $F \in S_{\omega^2}$ and $k(n)$ such that if $k(n) \leq F \in S_{\omega+n}$, then $F \in S_{\omega^2}$. Without loss of generality we may assume that if $n \leq F \in S_n$, then $F \in S_\omega$.

Let $n_0 \in \mathbb{N}$ and set $N_0 = \{\min \{\sup \{x\}\}_{i \in \mathbb{N}} = (n_i)_i\}$. Define a sequence $(\alpha_i)$ by the rule $m_1 = n_1$ and $m_{i+1} = n_{m_i}$, and consider the subset $N = (n_{m_i})_{i \in \mathbb{N}}$ of $N_0$. Passing to a further subset of $N$ and relabelling we may assume that $m_{\min N} > \max\{k(0), k(n_0)\}$. $S_{n_0}[S_n](N) \subset S_{\omega+n_0}$ (by Proposition 1.1(c)), and moreover that the following holds:

$$\forall \alpha < \omega_1, \text{ if } (n_{m_i})_{i \in F} \in S_\alpha, \text{ then } (n_{m_i})_{i \in F \setminus \{\min \{F\}\}} \in S_\alpha(N_0),$$

hence $(\alpha_i)_{i \in F \setminus \{\min \{F\}\}} \in S_\alpha$ (by Proposition 1.1(a)).

Let $L_0 = (m_{n_i})_{m_0} \subset N$ be such that $\min \{L_0\} > \min \{n_0, \frac{3}{\theta_{n_0}}\}$. Let $n_1 > n_0 + 10$. Passing to further subset $L_1 = (n_{m_i})_{M_1}$ of $L_0$ we may assume that $m_{\min L_1} > \min \{n_1, \frac{3}{\theta_{n_1}}\}$.

Let $[n_{m_1}]_{L_1} = \sum_{j \in F_1} \alpha_{m_j} e_{m_j}$ be the first $n_1$-average of $L_1$, where $(e_i)$ denotes the unit vector basis of the space $X$. We have that $[n_{m_1}]_{L_1}$ is a $(\theta_{n_1}^2, n_1, n_1 - 1)$-basic s.c.c. The set $(n_{m_1})_{j \in F_1} \in S_{n_1}$ and therefore by the property of the set $N$ we have that $G_1 = \{m_1 : i \in F_1 \setminus \{\min \{F_1\}\}\} \in S_{n_1}$. We set $J_1 = \{m_1 : i \in F_1\}$, i.e. $J_1 = G_1 \cup \{\min \{F_1\}\}$. Then $[n_{m_1}]_{L_1} = \sum_{j \in J_1} \alpha_j x_j$.

Set $y_1 = \sum_{j \in J_1} \alpha_j x_j$. Then $y_1$ is a $(\theta_{n_1}^2, n_1, n_1 - 1)$-s.c.c. of $(x_i)$ with $c \leq \|y_1\|$.

Indeed, since $\{\min \sup \{x\} : j \in J_1\} = \{n_{m_j} : j \in F_1\}$ and $[n_{m_1}]_{L_1} = \sum_{j \in J_1} \alpha_{m_j} e_{m_j}$, we have that $y_1$ is an $(\theta_{n_1}^2, n_1, n_1 - 1)$ s.c.c. Also since $G_1 \supseteq n_1$ and $G_1 \in S_{n_1}$, we have that $G_1 \in S_\omega$. Also from the choice of $L_0$, $G_1 > k(0)$. Therefore $G_1 \in S_{\omega^2}$, and

$$\|y_1\| \geq \sum_{j \in G_1} \alpha_j x_j \geq 2c \sum_{j \in G_1} \alpha_j \geq 2c \sum_{j \in F_1 \setminus \{\min \{F_1\}\}} \alpha_{m_j} \geq c,$$

since $(x_i)_i$ is a $2c - \ell^2_1$ spreading model.

Assume that we have chosen $y_1 < y_2 < \ldots < y_t$ and $n_1 < n_2 < \ldots < n_t$ such that

1. Each $y_r = \sum_{j \in J_r} \alpha_j x_j$ is a c-normalized $(\theta_{n_r}^2, n_r, n_r - 1)$ s.c.c. of $(x_i)$, for $r = 1, \ldots, t$.
2. $n_r < J_r \setminus \min \{J_r\} = G_r \in S_{n_r}$ for $r = 1, \ldots, t$.
3. $\|y_r\|_{\ell_1} \leq \frac{\theta_{n_r}}{\overline{\sigma}_{n_r+1}}$ for $r = 1, \ldots, t - 1$ where $\| \cdot \|_{\ell_1}$ denotes the norm of the space $\ell_1$. 

Then we choose \( n_{\ell+1} \) such that \( \| y_{\ell} \|_{\ell} \leq \frac{\theta_{n_1}}{\sigma_{n_{\ell+1}}} \) and \( 2 < \theta_{n_1}/\theta_{n_{\ell+1}} \). We also choose a subset \( L_{\ell+1} = (n_m)_i \) of \( L \setminus \bigcup_{r=1}^{\ell} \text{supp}[n_r]_{1}^{L} \) such that \( \min_{L} L_{\ell+1} \geq \min \{ n_{\ell+1}, \frac{3}{n_{\ell+1}} \} \).

Let \( \{ n_{\ell+1} \}_{\ell=1}^{\ell+1} = \sum_{j \in F_{\ell+1}} \alpha_{m_j} e_{n_{m_j}} \) be the first \( n_{\ell+1} \)-average of the set \( L_{\ell+1} \). Then \( \{n_{\ell+1}\}_{\ell=1}^{\ell+1} \) is \( (\theta_{2}^{2}, n_{\ell+1}, n_{\ell+1} - 1) \) basic s.c.c. Also \( (n_{m_j})_{j \in F_{\ell+1}} \in S_{n_{\ell+1}} \) and therefore by the property of the set \( N \) we have that the set \( G_{\ell+1} = \{ m_j : j \in F_{\ell+1} \setminus \min(F_{\ell+1}) \} \in S_{n_{\ell+1}} \). We set \( J_{\ell+1} = \{ m_j : j \in F_{\ell+1} \} \). By the choice of \( L_{\ell+1} \), we have that \( n_{\ell+1} \leq G_{\ell+1} \), and therefore \( G_{\ell+1} \in S_{\omega} \). We also have that \( k(0) \leq G_{\ell+1} \), so we deduce that \( G_{\ell+1} \in S_{\omega, 2} \).

We set \( y_{\ell+1} = \sum_{j \in J_{\ell+1}} \alpha_{j} x_{j} \). With the same arguments as for \( y_{1} \) we have that \( y_{\ell+1} \) is a \( (\theta_{n_{\ell+1}}, n_{\ell+1}, n_{\ell+1} - 1) \)-s.c.c. of \( (x_{i}) \) with \( \| y_{\ell+1} \| \geq c \).

We continue in the same manner to produce a sequence \( (y_{r})_{r \in \mathbb{N}} \) satisfying properties (1)–(3) for all \( r \).

Now let \( M \subset \{ \min \text{supp} y_{i} : i \in \mathbb{N} \} \subset L \) and let \( [n_{0}]_{1}^{M} = \sum_{j \in G_{0}} \gamma_{m_{j}} e_{n_{m_{j}}} \) be the first \( n_{0} \) average of the set \( M \), such that \( [n_{0}]_{1}^{M} \) is \( (\theta_{2}^{2}, n_{0}, n_{0} - 1) \)-basic s.c.c. Then we have that \( [n_{0}]_{1}^{M} = \sum_{j \in G_{0}} \gamma_{j} e_{n_{j}} \), where \( G = \{ m_j : j \in G_0 \} \), and \( G \setminus \min(G) \in S_{n_0} \). Also we have that \( \bigcup_{G \setminus \min(G)} G_{j} \in S_{n_0}[S_{\omega}] \) and therefore \( \{ n_{i} : i \in G \setminus \min(G) \} \in S_{n_0}[S_{\omega}](N) \subset S_{\omega + n_{0}} \), by the choice of the set \( N \). From the definition of \( m_{i+1} = n_{m_{i}} \) it follows that

\[
(3.1) \quad H = \bigcup_{G \setminus \min(G)} (G \setminus \min G_{j}) \in S_{\omega + n_{0}},
\]

since the families \( (S_{\alpha})_{\alpha} \) are spreading. Indeed, let \( G_{j} = \{ m_{(r)}^{(i)} < m_{(r)}^{(j)} < \ldots < m_{(r)}^{(j)} \} \). Then for every \( k \geq 2 \), \( m_{(r)}^{(j)} = n_{m_{(r)}^{(j)} - 1} \geq n_{m_{(r)}^{(j)}} \) and since \( (n_{i})_{i \in \bigcup_{G \setminus \min(G)} G_{j} \in S_{\omega + n_{0}} \) we have \( \| y_{\ell} \| \geq H \) and therefore \( H \) belongs to \( S_{\omega, 2} \).

Set \( z = \sum_{j \in G} \gamma_{j} y_{j} = \sum_{j \in G} \gamma_{j} \sum_{i \in j} \alpha_{i} x_{i} \). Since we have assumed that \( (x_{i}) \) is a \( 2c - \ell_{j}^{2} \) spreading model, setting \( z_{i} = \gamma_{j} \alpha_{i} x_{i} \), for every \( i \in G_{j} \) and every \( j \in G \), we have that

\[
\| z \| \geq \| \sum_{i \in \bigcup_{G \setminus \min(G)} G_{j}} z_{i} \| \geq 2c \sum_{j \in G \setminus \min(G)} \gamma_{j} \sum_{i \in G \setminus \min(G)} \beta_{i} \geq 2c \sum_{j \in G \setminus \min(G)} \gamma_{j} (1 - \theta_{n_{j}}^{2}) \geq 2c(1 - \theta_{n_{0}}^{2}))(1 - \theta_{n_{0}}^{2}) \geq c .
\]

The vector \( z \) is also an \( (\theta_{n_{0}}^{2}, n_{0}) \)-R.I.s.c.c. That is, \( z \) is of the form \( z = \sum_{j=1}^{k} \gamma_{j} y_{j} \) such that

1. \( y_{j} \) is a \( c \)-normalized \( (\theta_{n_{j}}^{2}, n_{j}, n_{j} - 1) \)-s.c.c.
2. \( n_{0} + 2 < n_{1} < \ldots < n_{k} \), and \( \{ \min \text{supp} y_{j} \}_{j} \in S_{n_0} \).
3. \( \| y_{j} \|_{\ell} \leq \frac{\theta_{n_{j}}}{\sigma_{n_{j+1}}} \) and \( 2 < \theta_{n_{j}}/\theta_{n_{j+1}} \).
For such vectors we have that there exists a constant $M \leq 10$ such that
\begin{equation}
\|z\| \leq M\theta_n.
\end{equation}
We refer to [8] (Corollary 2.15), [4] (Proposition 1.15), and [9] for a proof of this estimation. Therefore we have that $c \leq 10\theta_n$. Since $n_0$ was arbitrarily chosen we have the result.

**Remark.** Let $X = T([S_n, \theta_n])$ be such that $\lim \xi_n = \xi$ is a limit ordinal. If the sequence $(\xi_n, \theta_n)$ is “appropriate” chosen, quoting the above arguments, for appropriate $(\theta_n, \xi_n, \zeta)$-s.c.c., we have that the space $X$ does not contain $\ell_2^2$ spreading models. We do not have a proof of inequality (3.2) for every sequence $(\xi_n) \not\rightarrow \xi$. We refer to [8] for a proof of inequality (3.2) for the appropriate chosen sequence $(\xi_n, \theta_n)$.

The same arguments apply in the case of $p$-spaces, and yield that those spaces contain no $\ell_2^1$ spreading model. A space $X$ is said to be $p$-space if $X = T([A_n, \frac{1}{n^{1/p_n}}])$, where $\frac{1}{p_n} + \frac{1}{q_n} = 1$ and $p_n \notin [1, \infty)$ and $A_n = \{F \subset \mathbb{N} : |F| \leq n\}$ where the norm is defined similarly to the one in Definition 1.2(b). We can also apply the above argument for $A_k[S_1]$ sequences to produce R.L.s.c.c. of length $k$ and in this case we know that the norm is less than or equal to $M_0^{1/p_k}$ [12]. Let us recall that the existence of $\ell_1$ spreading models in Schlumprecht’s space [10] has been established in [11].

**References**


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