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MULTI-QUADRATIC MAPPINGS IN BANACH SPACES

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Abstract. We prove the stability of multi-quadratic functional equations in Banach spaces.

1. Stability of multi-quadratic functional equations in Banach spaces

Let E_1 and E_2 be Banach spaces with norms $||\cdot||$ and $\|\cdot\|$, respectively. Consider $f : E_1 \to E_2$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$. Assume that there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(||x||^p + ||y||^p)$$

for all $x, y \in E_1$. Th.M. Rassias [2] showed that there exists a unique $\mathbb{R}$-linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p} ||x||^p$$

for all $x \in E_1$.

In [1], the author showed the stability of multilinear functional equations in Banach modules over a unital C^*-algebra.

Throughout this paper, let $\mathcal{B}_i$ be normed spaces for $i = 1, \cdots, d$. Let $\mathcal{D}$ be a Banach space with norm $\|\cdot\|$.

The main purpose of this paper is to prove the stability of multi-quadratic functional equations in Banach spaces.

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For a given mapping $f:\prod_{i=1}^d \mathcal{B}_i \to \mathcal{D}$, we set

$$\begin{aligned} Df(x_1,y_1,\cdots,x_d,y_d) :&= \sum_{i=1}^d f(x_1,\cdots,x_{i-1},x_i+y_i,x_{i+1},\cdots,x_d) \\ &+ \sum_{i=1}^d f(x_1,\cdots,x_{i-1},x_i-y_i,x_{i+1},\cdots,x_d) \\ -2d\, f(x_1,\cdots,x_i,\cdots,x_d) &- \sum_{i=1}^d 2f(x_1,\cdots,x_{i-1},y_i,x_{i+1},\cdots,x_d) \end{aligned}$$

for all $(x_1,\cdots,x_d)$, $(y_1,\cdots,y_d)\in\prod_{i=1}^d \mathcal{B}_i$.

Theorem 1. *Let $f:\prod_{i=1}^d \mathcal{B}_i\to\mathcal{D}$ be a mapping for which there exists a function $\varphi:\prod_{i=1}^d \mathcal{B}_i^2\to[0,\infty)$ such that*

$$\begin{aligned} \widetilde{\varphi}(x_1,y_1,\cdots,x_d,y_d) := \sum_{j=0}^\infty\sum_{i=1}^d \frac{1}{4^{i+jd}}\varphi(2^{j+1}x_1,0,\cdots,2^{j+1}x_{i-1},0,& \\ 2^jx_i,2^jy_i,2^jx_{i+1},0,\cdots,2^jx_d,0)<\infty,& \qquad \text{(i)} \end{aligned}$$

$$\|Df(x_1,y_1,\cdots,x_d,y_d)\| \le \varphi(x_1,y_1,\cdots,x_d,y_d) \qquad \text{(ii)}$$

for all $(x_1,\cdots,x_d)$, $(y_1,\cdots,y_d)\in\prod_{i=1}^d \mathcal{B}_i$. Assume that $f(x_1,\cdots,x_d)=0$ if $x_i=0$ for any $i=1,\cdots,d$. Then there exists a unique multi-quadratic mapping $M:\prod_{i=1}^d \mathcal{B}_i\to\mathcal{D}$ such that

$$\|f(x_1,\cdots,x_d)-M(x_1,\cdots,x_d)\| \le \widetilde{\varphi}(x_1,x_1,\cdots,x_d,x_d) \qquad \text{(iii)}$$

for all $(x_1,\cdots,x_d)\in\prod_{i=1}^d \mathcal{B}_i$.

Proof. For each fixed i, let $y_1=\cdots=y_{i-1}=y_{i+1}=\cdots=y_d=0$ and $y_i=x_i$ in (ii). Then we get

$$\begin{aligned} \|f(x_1,\cdots,x_{i-1},2x_i,x_{i+1},\cdots,x_d)-4f(x_1,\cdots,x_{i-1},x_i,x_{i+1},\cdots,x_d)\| & \\ \le \varphi(x_1,0,\cdots,x_{i-1},0,x_i,x_i,x_{i+1},0,\cdots,x_d,0)& \end{aligned}$$

for all $(x_1,\cdots,x_d)\in\prod_{i=1}^d \mathcal{B}_i$. So one can obtain

$$\begin{aligned} \|f(x_1,\cdots,x_{i-1},x_i,x_{i+1},\cdots,x_d)-\frac{1}{4}f(x_1,\cdots,x_{i-1},2x_i,x_{i+1},\cdots,x_d)\| & \\ \le \frac{1}{4}\varphi(x_1,0,\cdots,x_{i-1},0,x_i,x_i,x_{i+1},0,\cdots,x_d,0)& \end{aligned}$$

for all $(x_1,\cdots,x_d)\in\prod_{i=1}^d \mathcal{B}_i$. Hence

$$\begin{aligned} \|\frac{1}{4^{i-1}}f(2x_1,\cdots,2x_{i-1},x_i,\cdots,x_d)-\frac{1}{4^i}f(2x_1,\cdots,2x_i,x_{i+1},\cdots,x_d)\| & \\ \le \frac{1}{4^i}\varphi(2x_1,0,\cdots,2x_{i-1},0,x_i,x_i,x_{i+1},0,\cdots,x_d,0)& \end{aligned}$$

for all $(x_1, \cdots, x_d) \in \prod_{i=1}^d \mathcal{B}_i$. Thus

$$\|f(x_1, \cdots, x_d) - \frac{1}{4^d} f(2x_1, \cdots, 2x_d)\| \leq \sum_{i=1}^{d} \frac{1}{4^i} \varphi(2x_1, 0, \cdots, 2x_{i-1}, 0, x_i, x_i, x_{i+1}, 0, \cdots, x_d, 0)$$

for all $(x_1, \cdots, x_d) \in \prod_{i=1}^d \mathcal{B}_i$. We also get

$$\|\frac{1}{4^{jd}} f(2^j x_1, \cdots, 2^j x_d) - \frac{1}{4^{(j+1)d}} f(2^{j+1} x_1, \cdots, 2^{j+1} x_d)\| \leq \sum_{i=1}^{d} \frac{1}{4^{i+jd}} \varphi(2^{j+1} x_1, 0, \cdots, 2^{j+1} x_{i-1}, 0, 2^j x_i, 2^j x_i, 2^j x_{i+1}, 0, \cdots, 2^j x_d, 0)$$

for all $(x_1, \cdots, x_d) \in \prod_{i=1}^d \mathcal{B}_i$. So

$$\|f(x_1, \cdots, x_d) - \frac{1}{4^{nd}} f(2^n x_1, \cdots, 2^n x_d)\| \leq \sum_{j=0}^{n-1} \sum_{i=1}^{d} \frac{1}{4^{i+jd}} \times \varphi(2^{j+1} x_1, 0, \cdots, 2^{j+1} x_{i-1}, 0, 2^j x_i, 2^j x_i, 2^j x_{i+1}, 0, \cdots, 2^j x_d, 0) \tag{1}$$

for all $(x_1, \cdots, x_d) \in \prod_{i=1}^d \mathcal{B}_i$.

For each $i = 1, \cdots, d$, let x_i be an element in $\mathcal{B}_i$. For positive integers n and m with $n > m$,

$$\|\frac{1}{4^{md}} f(2^m x_1, \cdots, 2^m x_d) - \frac{1}{4^{nd}} f(2^n x_1, \cdots, 2^n x_d)\| \leq \sum_{j=m}^{n-1} \sum_{i=1}^{d} \frac{1}{4^{i+jd}} \times \varphi(2^{j+1} x_1, 0, \cdots, 2^{j+1} x_{i-1}, 0, 2^j x_i, 2^j x_i, 2^j x_{i+1}, 0, \cdots, 2^j x_d, 0),$$

which tends to zero as $m \to \infty$ by (i). So $\{\frac{1}{4^{nd}} f(2^n x_1, \cdots, 2^n x_d)\}$ is a Cauchy sequence for all $(x_1, \cdots, x_d) \in \prod_{i=1}^d \mathcal{B}_i$. Since $\mathcal{D}$ is complete, $\{\frac{1}{4^{nd}} f(2^n x_1, \cdots, 2^n x_d)\}$ converges for all $(x_1, \cdots, x_d) \in \prod_{i=1}^d \mathcal{B}_i$. We can define a mapping $M : \prod_{i=1}^d \mathcal{B}_i \to \mathcal{D}$ by

$$M(x_1, \cdots, x_d) = \lim_{j \to \infty} \frac{1}{4^{jd}} f(2^j x_1, \cdots, 2^j x_d) \tag{2}$$

for all $(x_1, \cdots, x_d) \in \prod_{i=1}^d \mathcal{B}_i$.

By (i) and (2), we get

$$\begin{aligned} &\|DM(x_1, 0, \cdots, x_{i-1}, 0, x_i, y_i, x_{i+1}, 0, \cdots, x_d, 0)\| \\ &= \lim_{j \to \infty} \frac{1}{4^{jd}} \|Df(2^j x_1, 0, \cdots, 2^j x_{i-1}, 0, 2^j x_i, 2^j y_i, 2^j x_{i+1}, 0, \cdots, 2^j x_d, 0)\| \\ &\leq \lim_{j \to \infty} \frac{1}{4^{jd}} \varphi(2^j x_1, 0, \cdots, 2^j x_{i-1}, 0, 2^j x_i, 2^j y_i, 2^j x_{i+1}, 0, \cdots, 2^j x_d, 0) \\ &= 0 \end{aligned}$$

for all $(x_1, \cdots, x_d) \in \prod_{i=1}^d \mathcal{B}_i$ and all $y_i \in \mathcal{B}_i$. Hence

$$DM(x_1, 0, \cdots, x_{i-1}, 0, x_i, y_i, x_{i+1}, 0, \cdots, x_d, 0) = 0$$

for all $(x_1, \cdots, x_d) \in \prod_{i=1}^d \mathcal{B}_i$ and all $y_i \in \mathcal{B}_i$, which implies that M is quadratic for each $i = 1, \cdots, d$. Moreover, by passing to the limit in (1) as $n \to \infty$, we get the inequality (iii).

Now let $L : \prod_{i=1}^d \mathcal{B}_i \to \mathcal{D}$ be another multi-quadratic mapping satisfying

$$\|f(x_1, \cdots, x_d) - L(x_1, \cdots, x_d)\| \le \widetilde{\varphi}(x_1, x_1, \cdots, x_d, x_d)$$

for all $(x_1, \cdots, x_d) \in \prod_{i=1}^d \mathcal{B}_i$. Then

$$\begin{aligned}\|M(x_1, \cdots, x_d) - L(x_1, \cdots, x_d)\| =& \frac{1}{4^{jd}}\|M(2^j x_1, \cdots, 2^j x_d) - L(2^j x_1, \cdots, 2^j x_d)\| \\ \le& \frac{1}{4^{jd}}\|M(2^j x_1, \cdots, 2^j x_d) - f(2^j x_1, \cdots, 2^j x_d)\| \\ &+ \frac{1}{4^{jd}}\|f(2^j x_1, \cdots, 2^j x_d) - L(2^j x_1, \cdots, 2^j x_d)\| \\ \le& \frac{2}{4^{jd}}\widetilde{\varphi}(2^j x_1, 2^j x_1, \cdots, 2^j x_d, 2^j x_d),\end{aligned}$$

which tends to zero as $j \to \infty$ by (i). Thus $M(x_1, \cdots, x_d) = L(x_1, \cdots, x_d)$ for all $(x_1, \cdots, x_d) \in \prod_{i=1}^d \mathcal{B}_i$. This proves the uniqueness of M. This completes the proof. □

Theorem 3. *Let $f : \prod_{i=1}^d \mathcal{B}_i \to \mathcal{D}$ be a mapping for which there exists a function $\varphi : \prod_{i=1}^d \mathcal{B}_i^2 \to [0, \infty)$ such that*

$$\begin{aligned}\widetilde{\varphi}(x_1, y_1, \cdots, x_d, y_d) := \sum_{j=0}^{\infty}\sum_{i=1}^{d} 4^{i-1+jd}\varphi(&\frac{1}{2^{j+1}}x_1, 0, \cdots, \frac{1}{2^{j+1}}x_{i-1}, 0, \\ &\frac{1}{2^{j+1}}x_i, \frac{1}{2^{j+1}}y_i, \frac{1}{2^j}x_{i+1}, 0, \cdots, \frac{1}{2^j}x_d, 0) < \infty,\end{aligned}$$

$$\|Df(x_1, y_1, \cdots, x_d, y_d)\| \le \varphi(x_1, y_1, \cdots, x_d, y_d)$$

for all $(x_1, \cdots, x_d), (y_1, \cdots, y_d) \in \prod_{i=1}^d \mathcal{B}_i$. Assume that $f(x_1, \cdots, x_d) = 0$ if $x_i = 0$ for any $i = 1, \cdots, d$. Then there exists a unique multi-quadratic mapping $M : \prod_{i=1}^d \mathcal{B}_i \to \mathcal{D}$ such that

$$\|f(x_1, \cdots, x_d) - T(x_1, \cdots, x_d)\| \le \widetilde{\varphi}(x_1, x_1, \cdots, x_d, x_d)$$

for all $(x_1, \cdots, x_d) \in \prod_{i=1}^d \mathcal{B}_i$.

Proof. The proof is similar to the proof of Theorem 2. □

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