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ON SCHWARZ TYPE INEQUALITIES

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Abstract. We show Schwarz type inequalities and consider their converses. A continuous function $f : [0,\infty) \to [0,\infty)$ is said to be semi-operator monotone on (a,b) if $\{f(t^{\frac{1}{2}})\}^2$ is operator monotone on (a^2,b^2). Let T be a bounded linear operator on a complex Hilbert space $\mathcal{H}$ and $T = U|T|$ be the polar decomposition of T. Let $0 \le A, B \in B(\mathcal{H})$ and $\|Tx\| \le \|Ax\|, \|T^*y\| \le \|By\|$ for $x, y \in \mathcal{H}$. (1) If a non-zero function f is semi-operator monotone on $(0,\infty)$, then $|\langle Tx, y\rangle| \le \|f(A)x\|\|g(B)y\|$ for $x, y \in \mathcal{H}$, where $g(t) = t/f(t)$. (2) If f, g are semi-operator monotone on $(0,\infty)$, then $|\langle Uf(|T|)g(|T|)x, y\rangle| \le \|f(A)x\|\|g(B)y\|$ for $x, y \in \mathcal{H}$. Also, we show converses of these inequalities, which imply that semi-operator monotonicity is necessary.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. Furuta [3] extended the Heinz-Kato [5], [6] inequality, which is an extension of the Schwarz inequality.

Proposition 1 (Heinz-Kato-Furuta). *Let $T = U|T|$ be the polar decomposition of $T \in B(\mathcal{H}), 0 \le A, B \in B(\mathcal{H})$ and $\|Tx\| \le \|Ax\|, \|T^*y\| \le \|By\|$ for $x, y \in \mathcal{H}$. Then*

$$|\langle T|T|^{\alpha+\beta-1}x, y\rangle| \le \|A^\alpha x\|\|B^\beta y\|$$

for $x, y \in \mathcal{H}, \alpha, \beta \in [0,1]$ with $1 \le \alpha + \beta$.

The Heinz-Kato inequality is the case $\alpha + \beta = 1$. Recently, M. Uchiyama [7] extended this result as follows.

Proposition 2 (M. Uchiyama). *Let $f, g : [0,\infty) \to [0,\infty)$ be continuous operator monotone functions. Let $T = U|T|$ be the polar decomposition of $T \in B(\mathcal{H}), 0 \le A, B \in B(\mathcal{H})$ and $\|Tx\| \le \|Ax\|, \|T^*y\| \le \|By\|$ for $x, y \in \mathcal{H}$. Then*

$$|\langle Uf(|T|)g(|T|)x, y\rangle| \le \|f(A)x\|\|g(B)y\| \quad \text{for } x, y \in \mathcal{H}.$$

In this paper, we introduce semi-operator monotonicity which assures the conclusion of Proposition 2.

Definition 3. A continuous function $f : [0,\infty) \to [0,\infty)$ is said to be semi-operator monotone on (a,b) if $\{f(t^{\frac{1}{2}})\}^2$ is operator monotone on (a^2,b^2).

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Proposition 4. *Let $f : [0,\infty) \to [0,\infty)$ be a continuous function. Then f is semi-operator monotone on (a,b) if and only if f has an analytic continuation for $\Pi_1 = \left\{ z \in \mathbb{C} \,\middle|\, 0 < \arg z < \frac{\pi}{2} \right\}$ with $f(\Pi_1) \subset \Pi_1$.*

Proof. Let f be semi-operator monotone on (a,b). Then $g(t) = \{f(t^{\frac{1}{2}})\}^2$ is operator monotone on (a^2,b^2). Hence $g(t)$ has analytic continuation $g(z) = \{f(z^{\frac{1}{2}})\}^2$ for $\Pi_+ = \{z \in \mathbb{C} | 0 < \arg z < \pi\}$ with $g(\Pi_+) \subset \Pi_+$. Hence $f(z^{\frac{1}{2}})$ is analytic for $z \in \Pi_+$ and $f(z^{\frac{1}{2}}) \in \Pi_1$. This implies that f has an analytic continuation for Π_1 with $f(\Pi_1) \subset \Pi_1$. The converse is clear. □

Every operator monotone function is semi-operator monotone. But the converse does not hold. For example, let $f(t) = \{\log(1+t^2)\}^{\frac{1}{2}} : [0,\infty) \to [0,\infty)$. Then $\{f(t^{\frac{1}{2}})\}^2 = \log(1+t)$ is operator monotone on $(0,\infty)$. But its analytic continuation $f(z) = \{\log(1+z^2)\}^{\frac{1}{2}}$ is singular at $z = i$. Hence $f(t)$ is not operator monotone on $(0,\infty)$. Thus the class of all semi-operator monotone functions strictly includes the class of all operator monotone functions. Also, this example shows that $g(t) = \log(1+t)$ is operator monotone, but $\{g(t^2)\}^{\frac{1}{2}}$ is not operator monotone. Non-constant semi-operator monotone functions are strictly increasing.

J. S. Aujla [1] and J. I. Fujii, M. Fujii [2] studied semi-operator monotone function on $(0,\infty)$ and gave several characterizations. Moreover J. I. Fujii and M. Fujii [2] studied n-operator monotone function on $(0,\infty)$, which generalize semi-operator monotonicity.

Since every operator monotone function $f : [0,\infty) \to [0,\infty)$ is semi-operator monotone on $(0,\infty)$, we can give a simple proof of Uchiyama's result as the following Schwarz type inequality.

Theorem 5. *Let $f, g : [0,\infty) \to [0,\infty)$ be continuous functions. Let $T = U|T|$ be the polar decomposition of $T \in B(\mathcal{H})$, $0 \le A, B \in B(\mathcal{H})$ and $\|Tx\| \le \|Ax\|$, $\|T^*y\| \le \|By\|$ for $x, y \in \mathcal{H}$. Then, if f, g are semi-operator monotone on $(0,\infty)$, we have*

$$|\langle U f(|T|) g(|T|) x, y\rangle| \le \|f(A)x\| \|g(B)y\| \quad \text{for } x, y \in \mathcal{H}. \tag{1}$$

Conversely, if f, g satisfy the conclusion (1) and $f(t)g(t) \not\equiv 0$, then f, g are semi-operator monotone on $(0,\infty)$.

Proof. Let f, g be semi-operator monotone on $(0,\infty)$. Since $|T|^2 \le A^2$ and $|T^*|^2 \le B^2$, we have $\{f(|T|)\}^2 \le \{f(A)\}^2$ and $\{g(|T^*|)\}^2 \le \{g(B)\}^2$. Hence

$$\begin{aligned} |\langle U f(|T|) g(|T|) x, y\rangle| &= |\langle g(|T^*|) U f(|T|) x, y\rangle| \\ &= |\langle U f(|T|) x, g(|T^*|) y\rangle| \\ &\le \|f(|T|)x\| \|g(|T^*|)y\| \\ &\le \|f(A)x\| \|g(B)y\| \quad \text{for } x, y \in \mathcal{H}. \end{aligned}$$

Conversely, take a point $a \in (0,\infty)$ such that $f(a)g(a) > 0$. Let (c,d) be the maximal open interval including a such that $0 < f(t), g(t)$ for $t \in (c,d)$. First we show that f, g are semi-operator monotone on (c,d). Let $\tilde{f}(t) = \{f(t^{\frac{1}{2}})\}^2, \tilde{g}(t) = \{g(t^{\frac{1}{2}})\}^2$ for $t \in (c^2,d^2)$. Let $C \le D, \sigma(C), \sigma(D) \subset (c^2,d^2)$. Then $C, D, g(C^{\frac{1}{2}})$ are invertible. Let $T = C^{\frac{1}{2}}, A = D^{\frac{1}{2}}, B = C^{\frac{1}{2}}$. Then condition (1) implies

$$\begin{aligned} \|f(D^{\frac{1}{2}})x\| \|g(C^{\frac{1}{2}})y\| &\ge |\langle f(C^{\frac{1}{2}}) g(C^{\frac{1}{2}}) x, y\rangle| \\ &= |\langle f(C^{\frac{1}{2}}) x, g(C^{\frac{1}{2}}) y\rangle| \quad \text{for } x, y \in \mathcal{H}. \end{aligned}$$

Hence $\|f(C^{\frac{1}{2}})x\| \leq \|f(D^{\frac{1}{2}})x\|$ for $x \in \mathcal{H}$ and $\tilde{f}(C) = \{f(C^{\frac{1}{2}})\}^2 \leq \{f(D^{\frac{1}{2}})\}^2 = \tilde{f}(D)$. Thus $f(t)$ is semi-operator monotone on (c,d). Similarly we can show that $g(t)$ is semi-operator monotone on (c,d).

We show $c = 0$ and $d = \infty$. If $d < \infty$, then $f(d) = 0$ or $g(d) = 0$. But f, g are positive and semi-operator monotone on (c,d). Hence $0 < f(d)$ and $0 < g(d)$. This is a contradiction. Hence $d = \infty$. Assume $0 < c$. Then $f(c) = 0$ or $g(c) = 0$. In this case $f(0)g(0) = 0$. Because if $f(0)g(0) > 0$, then $f(t) > 0, g(t) > 0$ for $t \in (0,\infty)$ by the preceding argument. Let

$$A = B = \begin{pmatrix} \sqrt{2}c & 0 \\ 0 & c \end{pmatrix}, T = \frac{2c}{\sqrt{3}} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Then $0 \leq A, B, T$ and $T^2 \leq A^2 = B^2$. Let $x = y = \binom{0}{1}$. Then

$$\langle Uf(|T|)g(|T|)x, y\rangle = \frac{1}{2} f\left(\frac{2c}{\sqrt{3}}\right) g\left(\frac{2c}{\sqrt{3}}\right) \neq 0$$

and

$$\|f(A)x\|\|g(B)y\| = \left\| \begin{pmatrix} 0 \\ f(c) \end{pmatrix} \right\| \left\| \begin{pmatrix} 0 \\ g(c) \end{pmatrix} \right\| = 0.$$

This is a contradiction. Thus $c = 0$ and f, g are semi-operator monotone on $(0,\infty)$. $\square$

Next we show a direct extension of the Heinz-Kato inequality.

Theorem 6. *Let $T \in B(\mathcal{H}), 0 \leq A, B \in B(\mathcal{H})$ and $\|Tx\| \leq \|Ax\|, \|T^*y\| \leq \|By\|$ for $x, y \in \mathcal{H}$. Then, if a non-zero function f is semi-operator monotone on $(0,\infty)$, then we have*

$$|\langle Tx, y\rangle| \leq \|f(A)x\|\|g(B)y\| \tag{2}$$

for $x, y \in \mathcal{H}$, where $g(t) = t/f(t)$.

Conversely, if continuous functions $f, g : [0,\infty) \to [0,\infty)$ satisfy conclusion (2) *and $f(t)g(t) = t$ for $t \in (0,\infty)$, then f, g are semi-operator monotone on $(0,\infty)$.*

Proof. Let f be semi-operator monotone on $(0,\infty)$. Then $\left\{f(t^{\frac{1}{2}})\right\}^2$ is operator monotone on $(0,\infty)$ and $0 < f(t^{\frac{1}{2}})$ for $t \in (0,\infty)$. Hence $\left\{g(t^{\frac{1}{2}})\right\}^2 = t/\left\{f(t^{\frac{1}{2}})\right\}^2$ is operator monotone on $(0,\infty)$ by [4, Corollary 2.6]. Thus g is semi-operator monotone on $(0,\infty)$. Hence

$$\begin{aligned} |\langle Tx, y\rangle| &= |\langle U|T|x, y\rangle| \\ &= |\langle Uf(|T|)g(|T|)x, y\rangle| \\ &\leq \|f(A)x\|\|g(B)y\| \quad \text{for } x, y \in \mathcal{H} \end{aligned}$$

by Theorem 5. The converse is easy from Theorem 5. $\square$

Remark 7. For example, $f(t) = \sqrt{1+t^2}$ is semi-operator monotone on $(0,\infty)$. Hence, if $\|Tx\| \leq \|Ax\|, \|T^*y\| \leq \|By\|$ for $x, y \in \mathcal{H}$, then

$$|\langle Tx, y\rangle| \leq \left\|\sqrt{1+A^2}x\right\| \left\|\frac{B}{\sqrt{1+B^2}}y\right\| \quad \text{for } x, y \in \mathcal{H}.$$

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