

ON A PROPOSED CHARACTERIZATION OF SCHATTEN-CLASS COMPOSITION OPERATORS

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ABSTRACT. For an analytic function φ which maps the open unit disc D to itself, let C_φ be the operator of composition with φ on the Bergman space $L_a^2(D, dA)$. It has been a longstanding problem to determine whether or not the membership of C_φ in the Schatten class \mathcal{C}_p , $1 < p < \infty$, is equivalent to the condition that the function $z \mapsto \{(1 - |z|^2)/(1 - |\varphi(z)|^2)\}^p$ has a finite integral with respect to the Möbius-invariant measure $d\lambda(z) = (1 - |z|^2)^{-2}dA(z)$ on D . We show that the answer is negative when $2 < p < \infty$.

1. INTRODUCTION

Let D be the open unit disc $\{z \in \mathbf{C} : |z| < 1\}$ in the complex plane and let dA be the area measure on D normalized in such a way that $A(D) = 1$. Throughout the paper, $d\lambda$ denotes the Möbius-invariant measure on D , i.e.,

$$(1.1) \quad d\lambda(z) = (1 - |z|^2)^{-2}dA(z).$$

Recall that the Bergman space $L_a^2(D, dA)$ is defined to be $\{f \in L^2(D, dA) : f \text{ is analytic on } D\}$. For an analytic function $\varphi : D \rightarrow D$, let C_φ denote the operator of composition with φ on $L_a^2(D, dA)$. That is,

$$(C_\varphi f)(z) = f(\varphi(z)), \quad f \in L_a^2(D, dA).$$

It is a consequence of Littlewood's subordination theorem that the operator C_φ is bounded on $L_a^2(D, dA)$ for any analytic function $\varphi : D \rightarrow D$ [7, Theorem 10.3.2].

Recall that, for any $1 \leq p < \infty$, the Schatten p -class \mathcal{C}_p consists of operators T satisfying the condition $\|T\|_p < \infty$, where the p -norm is defined by the formula

$$\|T\|_p = \{\operatorname{tr}(|T|^p)\}^{1/p} = \{\operatorname{tr}((T^*T)^{p/2})\}^{1/p}.$$

The main interest of this paper concerns a proposed characterization of the membership of C_φ in \mathcal{C}_p . Let us briefly review the relevant background for the benefit of the reader, even though the problem itself is well known. In [5], D. Luecking and K. Zhu proved that $C_\varphi \in \mathcal{C}_p$ if and only if the function $z \mapsto \{\log(1/|z|)\}^{-2}N_{\varphi,2}(z)$ belongs to $L^{p/2}(D, d\lambda)$, where $d\lambda$ is given by (1.1) and $N_{\varphi,2}(z) = \sum_{w \in \varphi^{-1}\{z\}} \{\log(1/|w|)\}^2$, which is a generalized Nevanlinna counting function for φ . But such a characterization of the membership $C_\varphi \in \mathcal{C}_p$ was not considered to be completely satisfactory

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(see [7, page 226]) because φ enters in an indirect way, namely through the counting function $N_{\varphi,2}$. An alternate criterion for $C_\varphi \in \mathcal{C}_p$, one which not only involves φ directly but also appears to be aesthetically more pleasing, was proposed and debated in [4, 5, 7]. More precisely, the following problem arose (in chronological order) on page 363 of [4], on pages 226-228 of [7], and in Section 7 of [5].

Problem 1. *Let $1 < p < \infty$ and let $\varphi : D \rightarrow D$ be an analytic function. Is it true that the composition operator $C_\varphi : L^2_a(D, dA) \rightarrow L^2_a(D, dA)$ belongs to the Schatten class \mathcal{C}_p if and only if*

$$(1.2) \quad \int_D \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^p d\lambda(z) < \infty?$$

There are several reasons for proposing this problem. First of all,

$$\{(1 - |z|^2)/(1 - |\varphi(z)|^2)\}^p = \{\langle C_\varphi C_\varphi^* k_z, k_z \rangle\}^{p/2},$$

where $k_z(w) = (1 - |z|^2)/(1 - \bar{z}w)^2$ (see [7, Proposition 10.3.4]) and the function $z \mapsto \langle C_\varphi C_\varphi^* k_z, k_z \rangle = \|C_\varphi^* k_z\|^2$ is the *Berezin symbol* of $C_\varphi C_\varphi^*$. Secondly, it is known that C_φ is compact if and only if

$$\lim_{|z| \uparrow 1} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0$$

(see [6] or [1, Theorem 3.22]). Thirdly, (1.2) is sufficient for $C_\varphi \in \mathcal{C}_p$ when $1 < p \leq 2$ and necessary for $C_\varphi \in \mathcal{C}_p$ when $2 \leq p < \infty$ (see [5, Section 7]). In particular, $C_\varphi \in \mathcal{C}_2$ if and only if (1.2) holds with $p = 2$. Thus, when $2 < p < \infty$, (1.2) appears to be the natural “interpolation” between the compactness criterion and the Hilbert-Schmidt-class criterion for C_φ . Furthermore, Zhu recently proved the following:

Theorem 2 ([8, Theorem 1.1]). *Let $2 \leq p < \infty$. Let $\varphi : D \rightarrow D$ be an analytic function which has bounded valence, i.e., there is a positive integer N such that $\text{card}\{w \in D : \varphi(w) = z\} \leq N$ for every $z \in D$. Then $C_\varphi \in \mathcal{C}_p$ if and only if (1.2) holds.*

The purpose of this paper is to report that, however pleasing or natural (1.2) may appear, this condition in fact is *not* sufficient for the membership $C_\varphi \in \mathcal{C}_p$ when $2 < p < \infty$. Thus Luecking’s comment on page 363 of [4] turns out to be prophetic after all. The following is our main result:

Theorem 3. *For any $2 < p < \infty$, there exists an analytic function $\varphi : D \rightarrow D$ such that*

$$(1.3) \quad \int_D \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^p d\lambda(z) < \infty$$

and such that the composition operator $C_\varphi : L^2_a(D, dA) \rightarrow L^2_a(D, dA)$ does not belong to the Schatten class \mathcal{C}_p .

This result leads to the following contrast between the Berezin symbol of $C_\varphi^* C_\varphi$ and the Berezin symbol of $C_\varphi C_\varphi^*$ which seems to be interesting. Consider the conditions

$$(a) \quad \int_D \|C_\varphi k_z\|^p d\lambda(z) < \infty, \quad (b) \quad \int_D \|C_\varphi^* k_z\|^p d\lambda(z) < \infty.$$

Because $\|C_\varphi k_z\|^2$ is the Berezin symbol of $C_\varphi^* C_\varphi$ and because $C_\varphi^* C_\varphi$ is a Toeplitz operator as defined on page 106 of [7], condition (a) implies $C_\varphi \in \mathcal{C}_p$ [8, Lemmas 2.1 and 2.2]. On the other hand, as we have already mentioned, the Berezin symbol of $C_\varphi C_\varphi^*$ equals $\|C_\varphi^* k_z\|^2 = (1 - |z|^2)^2 / (1 - |\varphi(z)|^2)^2$. Theorem 3 tells us that condition (b) does not imply $C_\varphi \in \mathcal{C}_p$ when $2 < p < \infty$. Thus conditions (a) and (b) are not equivalent.

The proof of Theorem 3 is technical but self-contained. In fact, the only thing the reader needs to know about $L_a^2(D, dA)$ is that $\{(\ell + 1)^{1/2} z^\ell : \ell = 0, 1, 2, \dots\}$ is an orthonormal basis. To ensure that (1.3) holds, we only need to control the modulus $|\varphi|$ of φ . Thanks to a new lower bound for $\text{tr}((C_\varphi^* C_\varphi)^{p/2})$ (see (3.4) below), our proof that $C_\varphi \notin \mathcal{C}_p$ also involves the modulus $|\varphi|$ only. But the main obstacle in the proof of Theorem 3 lies in the fact that it is difficult to prescribe the modulus of an analytic function. In other words, the analyticity requirement is a severe handicap for the construction of φ . Fortunately, by carefully exploiting the distribution of the Poisson kernel, we are able to find just enough control on D to make such a construction possible.

The rest of the paper is organized as follows. In Section 2 we construct the required φ and establish the estimates which are necessary for the proof of Theorem 3. The proof itself is completed in Section 3.

2. CONSTRUCTION AND ESTIMATES

The construction of the desired φ begins with the intervals

$$T_n = (2^{-(n+1)}, 2^{-n}], \quad S_n = ((4/3)2^{-(n+1)}, (5/3)2^{-(n+1)})$$

in \mathbf{R} , $n = 1, 2, \dots$. That is, S_n is the middle third of T_n . Let $t_n = (4/3)2^{-(n+1)}$, which is the left end-point of S_n , $n \in \mathbf{N}$.

Let $2 < p < \infty$ be given. We choose a rational number ϵ such that

$$(2.1) \quad 0 < \epsilon < p^{-1}.$$

The requirement $\epsilon < p^{-1}$ ensures $\lim_{k \rightarrow \infty} 2^{-(p^{-1} - \epsilon)k} = 0$. This allows us to choose a strictly increasing sequence $k(1) < \dots < k(n) < \dots$ of positive integers such that

$$(2.2) \quad 2^{-(p^{-1} + \epsilon)k(n)} \cdot 2 \cdot 2^{2\epsilon k(n)} \leq (1/3)2^{-(n+1)}$$

for every $n \in \mathbf{N}$ and such that every $\epsilon k(n)$ is an integer. While the requirement that both inequalities in (2.1) be strict is indispensable to the proof of Theorem 3, the stipulation that ϵ be rational, which makes it possible for $\epsilon k(n)$ to be an integer, is not. But the fact that $\epsilon k(n)$ is chosen to be an integer does allow us to avoid certain unnecessary trivialities.

Next we subdivide S_n . For integers $n \geq 1$ and $1 \leq j \leq 2^{2\epsilon k(n)}$, define the intervals

$$J_{n,j} = (t_n + 2^{-(p^{-1} + \epsilon)k(n)} \cdot 2 \cdot (j - 1), t_n + 2^{-(p^{-1} + \epsilon)k(n)} \cdot 2 \cdot j),$$

$$I_{n,j} = (t_n + 2^{-(p^{-1} + \epsilon)k(n)} \cdot 2 \cdot (j - 1), t_n + 2^{-(p^{-1} + \epsilon)k(n)} \cdot (2j - 1)),$$

where, as we recall, t_n is the left end-point of S_n . Thus, $I_{n,j}$ is the left half of $J_{n,j}$ and the $J_{n,j}$'s are pairwise disjoint. (2.2) ensures that $\bigcup_{j=1}^{2^{2\epsilon k(n)}} J_{n,j} \subset S_n$. Keep in mind that the length of $I_{n,j}$ equals $2^{-(p^{-1} + \epsilon)k(n)}$. We now define a measurable

function u on the unit circle $T = \{\tau \in \mathbf{C} : |\tau| = 1\}$ as follows:

$$u(e^{it}) = 2^{-k(n)} \quad \text{if } t \in \bigcup_{j=1}^{2^{2\epsilon k(n)}} I_{n,j}, \quad n \geq 1,$$

$$u(e^{it}) = 1 \quad \text{if } t \in (-\pi, \pi] \setminus \left\{ \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{2^{2\epsilon k(n)}} I_{n,j} \right\}.$$

The harmonic extension of u to D will be denoted by the same symbol. Finally, define

$$h(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} u(e^{it}) dt,$$

$$(2.3) \quad \varphi(z) = \exp(-h(z)), \quad z \in D.$$

Obviously, $\text{Re}\{h(z)\} = u(z) > 0$ for every $z \in D$. Therefore φ maps D to itself. Our remaining task is to verify that φ satisfies (1.3) and has the property $C_\varphi \notin C_p$. This verification is based on a number of estimates of the modulus $|\varphi(z)|$, which will take up the rest of the section.

We begin with the Poisson kernel $P(z; \tau) = (1 - |z|^2)/|1 - z\bar{\tau}|^2$, $\tau \in T$ and $z \in D$. There exist constants $0 < \alpha < \beta < \infty$ such that if $1/2 \leq r < 1$ and $|\theta - t| \leq 5$, then

$$(2.4) \quad \frac{\alpha(1-r)}{(1-r)^2 + (\theta-t)^2} \leq \frac{1}{2\pi} P(re^{i\theta}; e^{it}) \leq \frac{\beta(1-r)}{(1-r)^2 + (\theta-t)^2}.$$

This follows from the identity $|1 - re^{i(\theta-t)}|^2 = (1-r)^2 + 2r(1 - \cos(\theta-t))$ and the fact that there are $0 < a < b < \infty$ such that $a \leq x^{-2}(1 - \cos x) \leq b$ when $0 < x \leq 5$.

Lemma 4. For any $n \in \mathbf{N}$ and $1 \leq j \leq 2^{2\epsilon k(n)}$, define

$$(2.5) \quad G_{n,j} = \{re^{i\theta} : \theta \in I_{n,j}, 0 < 1-r \leq 2^{-(p^{-1}+\epsilon)k(n)}\}.$$

Then there is a constant C_4 which is independent of n, j such that

$$\int_{G_{n,j}} \frac{(1-|z|)^{p-2}}{(1-|\varphi(z)|)^p} dA(z) \leq C_4 2^{-\epsilon p k(n)}.$$

Proof. Given such a pair of n, j , we have $G_{n,j} = \bigcup_{\nu=0}^{k(n)} G_{n,j}^\nu$, where

$$G_{n,j}^0 = \{re^{i\theta} : \theta \in I_{n,j}, 0 < 1-r \leq 2^{-(1+p^{-1}+\epsilon)k(n)}\},$$

$$G_{n,j}^\nu = \{re^{i\theta} : \theta \in I_{n,j}, 2^{-(1+p^{-1}+\epsilon)k(n)} \cdot 2^{(\nu-1)} < 1-r \leq 2^{-(1+p^{-1}+\epsilon)k(n)} \cdot 2^\nu\}$$

for $1 \leq \nu \leq k(n)$. Since $|\varphi(z)| = e^{-\text{Re}\{h(z)\}} = e^{-u(z)}$, the proof hinges on an estimate of $u(z)$ on each $G_{n,j}^\nu$. For this purpose let us write $I_{n,j} = (a_{n,j}, b_{n,j})$ and $J_{n,j} = (a_{n,j}, c_{n,j})$. Then $c_{n,j} - b_{n,j} = b_{n,j} - a_{n,j} = 2^{-(p^{-1}+\epsilon)k(n)}$, which will be denoted by ρ_n . Let us first consider the case $z \in G_{n,j}^\nu$ where $1 \leq \nu \leq k(n)$. For such a z , write $z = re^{i\theta}$ with $\theta \in I_{n,j}$. Since $u(e^{it}) = 1$ for $t \in J_{n,j} \setminus I_{n,j} = [b_{n,j}, c_{n,j})$,

recalling (2.4), we have

$$\begin{aligned} u(re^{i\theta}) &\geq \frac{1}{2\pi} \int_{J_{n,j} \setminus I_{n,j}} P(re^{i\theta}; e^{it}) dt \geq \alpha \int_{b_{n,j}}^{c_{n,j}} \frac{1-r}{(1-r)^2 + (t-\theta)^2} dt \\ &\geq \alpha \int_{b_{n,j}}^{c_{n,j}} \frac{1-r}{(1-r)^2 + (t-a_{n,j})^2} dt \\ &= \alpha \int_{\rho_n}^{2\rho_n} \frac{1-r}{(1-r)^2 + s^2} ds = \alpha \int_{\rho_n/(1-r)}^{2\rho_n/(1-r)} \frac{dx}{1+x^2}. \end{aligned}$$

The condition $re^{i\theta} \in G_{n,j}^\nu$ implies $1-r \leq 2^{-(1+p^{-1}+\epsilon)k(n)} \cdot 2^\nu = \rho_n \cdot 2^{-k(n)+\nu}$. In particular, $\rho_n/(1-r) \geq 1$. Hence $(1+x^2)^{-1} \geq (2x^2)^{-1}$ if $x \geq \rho_n/(1-r)$. The condition $re^{i\theta} \in G_{n,j}^\nu$ also requires $2^{-(1+p^{-1}+\epsilon)k(n)} \cdot 2^{(\nu-1)} < 1-r$, i.e., $2^{-k(n)+\nu-1} < (1-r)/\rho_n$. Thus

$$(2.6) \quad u(re^{i\theta}) \geq \frac{\alpha}{2} \int_{\rho_n/(1-r)}^{2\rho_n/(1-r)} \frac{1}{x^2} dx = \frac{\alpha}{4} \cdot \frac{1-r}{\rho_n} \geq \frac{\alpha}{4} 2^{-k(n)+\nu-1} = \frac{\alpha}{8} 2^{-k(n)+\nu}$$

if $re^{i\theta} \in G_{n,j}^\nu$ and $1 \leq \nu \leq k(n)$.

Now suppose that $re^{i\theta} \in G_{n,j}^0$, where $\theta \in I_{n,j}$. In this case we use the fact that $u(e^{it}) \geq 2^{-k(n)}$ for $t \in T_n$. Obviously, $[\theta - \rho_n, \theta + \rho_n] \subset T_n$. Therefore

$$\begin{aligned} (2.7) \quad u(re^{i\theta}) &\geq 2^{-k(n)} \frac{1}{2\pi} \int_{\theta-\rho_n}^{\theta+\rho_n} P(re^{i\theta}; e^{it}) dt \geq 2^{-k(n)} \alpha \int_{\theta-\rho_n}^{\theta+\rho_n} \frac{1-r}{(1-r)^2 + (t-\theta)^2} dt \\ &= 2^{-k(n)} \alpha \int_{-\rho_n}^{\rho_n} \frac{1-r}{(1-r)^2 + s^2} ds \geq 2^{-k(n)} \alpha \int_{-1}^1 \frac{dx}{1+x^2}, \end{aligned}$$

where the last \geq is due to the fact that $1-r \leq 2^{-(1+p^{-1}+\epsilon)k(n)} = 2^{-k(n)} \rho_n$.

Combining (2.7) with (2.6), we see that there is a constant $0 < c < 1$ which is independent of n, j such that $u(z) \geq c2^{-k(n)+\nu}$ if $z \in G_{n,j}^\nu$ and $0 \leq \nu \leq k(n)$. This implies that $1 - |\varphi(z)| = 1 - e^{-u(z)} \geq 1 - \exp(-c2^{-k(n)+\nu})$ if $z \in G_{n,j}^\nu$ and $0 \leq \nu \leq k(n)$. Thus, if we let $\delta = \inf_{0 < x \leq 1} x^{-1}(1 - e^{-x})$, then

$$(2.8) \quad \inf_{z \in G_{n,j}^\nu} (1 - |\varphi(z)|)^p \geq (\delta c)^p \cdot 2^{-pk(n)} \cdot 2^{p\nu}, \quad 0 \leq \nu \leq k(n).$$

Because $p - 2 > 0$, it follows from the definition of $G_{n,j}^\nu$ that

$$(2.9) \quad \sup_{z \in G_{n,j}^\nu} (1 - |z|)^{p-2} \leq 2^{-(p-2)(1+p^{-1}+\epsilon)k(n)} \cdot 2^{(p-2)\nu}, \quad 0 \leq \nu \leq k(n).$$

From the definitions of $G_{n,j}^\nu$ and $I_{n,j}$ it obviously follows that

$$(2.10) \quad A(G_{n,j}^\nu) \leq 2^{-(p-1+\epsilon)k(n)} \cdot 2^{-(1+p^{-1}+\epsilon)k(n)} \cdot 2^\nu, \quad 0 \leq \nu \leq k(n).$$

Combining (2.8)-(2.10) and simplifying the exponents involved, we find that

$$(2.11) \quad \int_{G_{n,j}^\nu} \frac{(1 - |z|)^{p-2}}{(1 - |\varphi(z)|)^p} dA(z) \leq \frac{\sup_{z \in G_{n,j}^\nu} (1 - |z|)^{p-2}}{\inf_{z \in G_{n,j}^\nu} (1 - |\varphi(z)|)^p} A(G_{n,j}^\nu) \leq (c\delta)^{-p} \cdot 2^{-\nu} \cdot 2^{-\epsilon pk(n)}$$

for all $0 \leq \nu \leq k(n)$. Recall that $G_{n,j} = \bigcup_{\nu=0}^{k(n)} G_{n,j}^\nu$. Because of the factor $2^{-\nu}$ that appears on the right-hand side of (2.11), if we sum both sides of this inequality as ν ranges from 0 to $k(n)$, we see that the conclusion of the lemma holds for $C_4 = 2(c\delta)^{-p}$. \square

Lemma 5. For any $n \geq 1$ and $1 \leq j \leq 2^{2\epsilon k(n)}$, let $B_{n,j}$ be the middle third of $I_{n,j}$. That is, $B_{n,j} = (3^{-1}(b_{n,j} + 2a_{n,j}), 3^{-1}(2b_{n,j} + a_{n,j}))$, where $a_{n,j} < b_{n,j}$ are the end-points of $I_{n,j}$. Furthermore, for such n and j , define

$$(2.12) \quad E_{n,j} = \{re^{it} : t \in B_{n,j}, 0 < 1 - r \leq 2^{-(1+p^{-1}+\epsilon)k(n)}\}.$$

Then $\sup_{z \in E_{n,j}} u(z) \leq (1+6\beta)2^{-k(n)}$, where β is the constant that appears in (2.4).

Proof. Let $n \geq 1$ and $1 \leq j \leq 2^{2\epsilon k(n)}$ be given. Consider an arbitrary $z = re^{i\theta} \in E_{n,j}$, where $\theta \in B_{n,j}$. Since $B_{n,j}$ is the middle third of $I_{n,j}$, we have $|\theta - t| \geq \rho_n/3$ if $t \in \mathbf{R} \setminus I_{n,j}$, where $\rho_n = 2^{-(p^{-1}+\epsilon)k(n)}$ is the length of $I_{n,j}$. The condition $\theta \in B_{n,j}$ also implies $0 < \theta < 1/2$. Therefore $(-\pi, \pi] \setminus I_{n,j} \subset \{t \in \mathbf{R} : \rho_n/3 \leq |\theta - t| \leq \pi + (1/2)\}$. Now, because $u(e^{it}) = 2^{-k(n)}$ for $t \in I_{n,j}$ and $u(e^{it}) \leq 1$ for $t \in (-\pi, \pi] \setminus I_{n,j}$, we have

$$(2.13) \quad \begin{aligned} u(re^{i\theta}) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} P(re^{i\theta}; e^{it})u(e^{it})dt \leq 2^{-k(n)} + \frac{1}{2\pi} \int_{(-\pi, \pi] \setminus I_{n,j}} P(re^{i\theta}; e^{it})dt \\ &\leq 2^{-k(n)} + \beta \int_{\rho_n/3 \leq |\theta - t| \leq \pi + (1/2)} \frac{1 - r}{(1 - r)^2 + (\theta - t)^2} dt, \end{aligned}$$

where β is the constant that appears in (2.4). But

$$(2.14) \quad \begin{aligned} \int_{|\theta - t| \geq \rho_n/3} \frac{1 - r}{(1 - r)^2 + (\theta - t)^2} dt &= 2 \int_{\rho_n/3}^{\infty} \frac{1 - r}{(1 - r)^2 + s^2} ds \\ &= 2 \int_{\rho_n/3(1-r)}^{\infty} \frac{1}{1 + x^2} dx \\ &\leq 6(1 - r)/\rho_n. \end{aligned}$$

The condition $re^{i\theta} \in E_{n,j}$ requires $1 - r \leq 2^{-(1+p^{-1}+\epsilon)k(n)} = 2^{-k(n)}\rho_n$. Thus it follows from (2.13) and (2.14) that $u(z) \leq (1 + 6\beta)2^{-k(n)}$ for every $z \in E_{n,j}$. \square

Lemma 6. Let $U = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{2^{2\epsilon k(n)}} I_{n,j}$. Then for any $x \in \mathbf{R} \setminus U$ and $0 < a < \infty$, we have

$$(2.15) \quad m((x - a, x + a) \setminus U) \geq (2/3)a,$$

where m is the standard Lebesgue measure on \mathbf{R} .

Proof. The case where either $(x - a, x) \cap U = \emptyset$ or $(x, x + a) \cap U = \emptyset$ is trivial. Thus let us assume that $x \in \mathbf{R} \setminus U$ and $a > 0$ are such that $(x - a, x) \cap U \neq \emptyset$ and $(x, x + a) \cap U \neq \emptyset$. Because $x \notin U$, this means that there are at least two intervals in the family

$$\mathcal{F} = \{I_{n,j} : (x - a, x + a) \cap I_{n,j} \neq \emptyset, n \geq 1, 1 \leq j \leq 2^{2\epsilon k(n)}\}.$$

Let Y be the smallest open interval such that

$$Y \cap U = (x - a, x + a) \cap U.$$

Since $(x - a, x + a) \setminus U = \{Y \setminus U\} \cup \{(x - a, x + a) \setminus Y\}$, (2.15) will follow once we show that $m(Y \setminus U) \geq (1/3)m(Y)$.

From the arrangement $I_{n,j} \subset S_n \subset T_n = (2^{-(n+1)}, 2^{-n}]$ it is easy to see that we can re-enumerate the family \mathcal{F} as $\{(a_\nu, b_\nu) : 1 \leq \nu < N\}$, where $3 \leq N \leq \infty$, such that $b_{\nu+1} < a_\nu$ for all $1 \leq \nu < N - 1$. Thus $Y \cap U = \bigcup_{1 \leq \nu < N} I_\nu$, where $I_1 = (a_1, b'_1)$ with $a_1 < b'_1 \leq b_1$, $I_\nu = (a_\nu, b_\nu)$ if $2 \leq \nu < N - 1$, and, in the

event $N < \infty$, $I_{N-1} = (a'_{N-1}, b_{N-1})$ with $a_{N-1} \leq a'_{N-1} < b_{N-1}$. The definition of the $I_{n,j}$'s ensures that the length of the gap between $(a_{\nu+1}, b_{\nu+1})$ and (a_ν, b_ν) is greater than or equal to the length of (a_ν, b_ν) . That is, $a_\nu - b_{\nu+1} \geq b_\nu - a_\nu$, $1 \leq \nu < N - 1$. Also, $b_\nu - a_\nu \geq b_{\nu+1} - a_{\nu+1}$ by the definition of the $I_{n,j}$'s. Since $Y \setminus U = \bigcup_{1 \leq \nu < N-1} [b_{\nu+1}, a_\nu]$, we have

$$\begin{aligned} m(Y \setminus U) &= \sum_{1 \leq \nu < N-1} (a_\nu - b_{\nu+1}) \geq \sum_{1 \leq \nu < N-1} (b_\nu - a_\nu) \\ &\geq \frac{1}{2} \sum_{1 \leq \nu < N} (b_\nu - a_\nu) \geq \frac{1}{2} m(Y \cap U). \end{aligned}$$

Adding $(1/2)m(Y \setminus U)$ to both sides, we find that $(3/2)m(Y \setminus U) \geq (1/2)m(Y)$. Thus $m(Y \setminus U) \geq (1/3)m(Y)$ as promised. \square

Lemma 7. *There is a $c_7 > 0$ such that*

$$u(z) \geq c_7 \quad \text{for every } z \in D \setminus \left\{ \bigcup_{n=1}^\infty \bigcup_{j=1}^{2^{2\epsilon k(n)}} G_{n,j} \right\},$$

where $G_{n,j}$ is defined by (2.5).

Proof. Let $W = \{re^{it} : 3/4 < r < 1, t \in (-1/4, 3/4)\}$. Since $u(e^{it}) = 1$ when $t \in (-\pi, \pi] \setminus (0, 1/2]$, we have $\lim_{r \uparrow 1} u(re^{it}) = 1$ uniformly for $t \in (-\pi, \pi] \setminus (-1/8, 5/8)$. Hence it suffices to find a $c_7 > 0$ such that

$$u(z) \geq c_7 \quad \text{for all } z \in W \setminus \left\{ \bigcup_{n=1}^\infty \bigcup_{j=1}^{2^{2\epsilon k(n)}} G_{n,j} \right\}.$$

For any $0 \leq r < 1$ and $\theta \in \mathbf{R}$, define $I(\theta, r) = (\theta - 3(1 - r), \theta + 3(1 - r))$. Then

$$(2.16) \quad \frac{1 - r}{(1 - r)^2 + (\theta - t)^2} = \frac{1}{1 - r} \cdot \left\{ 1 + \left(\frac{\theta - t}{1 - r} \right)^2 \right\}^{-1} \geq \frac{10^{-1}}{1 - r} \chi_{I(\theta, r)}(t).$$

Let $\theta \in (-1/4, 3/4)$ and $3/4 < r < 1$. Then $I(\theta, r) \subset (-\pi, \pi]$ and $u(e^{it}) = 1$ for $t \in I(\theta, r) \setminus U$, where $U = \bigcup_{n=1}^\infty \bigcup_{j=1}^{2^{2\epsilon k(n)}} I_{n,j}$ as in Lemma 6. By (2.4) and (2.16),

$$\begin{aligned} (2.17) \quad u(re^{i\theta}) &= \frac{1}{2\pi} \int_{-\pi}^\pi P(re^{i\theta}; e^{it}) u(e^{it}) dt \geq \frac{\alpha}{10} \cdot \frac{1}{1 - r} \int_{I(\theta, r)} u(e^{it}) dt \\ &\geq \frac{\alpha}{10} \cdot \frac{m(I(\theta, r) \setminus U)}{1 - r}. \end{aligned}$$

Let us further assume $re^{i\theta} \in W \setminus \{ \bigcup_{n=1}^\infty \bigcup_{j=1}^{2^{2\epsilon k(n)}} G_{n,j} \}$. We consider the following two cases:

(i) If $\theta \in (-1/4, 3/4) \setminus U$, then we apply Lemma 6 to the case where $x = \theta$ and $a = 3(1 - r)$ to obtain

$$\begin{aligned} m(I(\theta, r) \setminus U) &= m((\theta - 3(1 - r), \theta + 3(1 - r)) \setminus U) \\ &\geq (2/3) \cdot 3(1 - r) = 2(1 - r). \end{aligned}$$

By (2.17), we have $u(re^{i\theta}) \geq \alpha/5$ in this case.

(ii) If $\theta \in U$, then there exist an $n \geq 1$ and a $1 \leq j \leq k(n)$ such that $\theta \in I_{n,j}$. Now, because $re^{i\theta} \notin G_{n,j}$, (2.5) tells us $1 - r > 2^{-(p^{-1} + \epsilon)k(n)} = \rho_n$,

the length of $I_{n,j}$. Since the distance between θ and $J_{n,j} \setminus I_{n,j}$ is less than ρ_n , we can pick a $\theta' \in J_{n,j} \setminus I_{n,j}$ such that $|\theta - \theta'| < \rho_n < 1 - r$. Thus $I(\theta, r) = (\theta - 3(1 - r), \theta + 3(1 - r)) \supset (\theta' - 2(1 - r), \theta' + 2(1 - r))$. Since $J_{n,j} \setminus I_{n,j} \subset \mathbf{R} \setminus U$, we now apply Lemma 6 to the case where $x = \theta'$ and $a = 2(1 - r)$ to obtain $m(I(\theta, r) \setminus U) \geq m((\theta' - 2(1 - r), \theta' + 2(1 - r)) \setminus U) \geq (2/3) \cdot 2(1 - r)$. By (2.17), we have $u(re^{i\theta}) \geq 2\alpha/15$ in this case. This completes the proof. \square

3. PROOF OF THEOREM 3

We must show that the analytic function $\varphi : D \rightarrow D$ defined by (2.3) satisfies (1.3) and has the property that $C_\varphi \notin \mathcal{C}_p$. Let us first verify (1.3).

Let $G = \bigcup_{n=1}^\infty \bigcup_{j=1}^{2^{2\epsilon k(n)}} G_{n,j}$, where $G_{n,j}$ is given by (2.5). By Lemma 7, $|\varphi(z)| = e^{-\operatorname{Re}\{h(z)\}} = e^{-u(z)} \leq e^{-c\tau}$ when $z \in D \setminus G$. Therefore

$$(3.1) \quad \int_{D \setminus G} \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^p d\lambda(z) = \int_{D \setminus G} \frac{(1 - |z|^2)^{p-2}}{(1 - |\varphi(z)|^2)^p} dA(z) < \infty.$$

Invoking Lemma 4, we have

$$(3.2) \quad \begin{aligned} \int_G \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^p d\lambda(z) &\leq \sum_{n=1}^\infty \sum_{j=1}^{2^{2\epsilon k(n)}} \int_{G_{n,j}} \frac{2^{p-2}(1 - |z|)^{p-2}}{(1 - |\varphi(z)|)^p} dA(z) \\ &\leq 2^{p-2} C_4 \sum_{n=1}^\infty 2^{-\epsilon p k(n)} \cdot 2^{2\epsilon k(n)} = 2^{p-2} C_4 \sum_{n=1}^\infty 2^{-\epsilon(p-2)k(n)}. \end{aligned}$$

Because $p > 2$ and $\epsilon > 0$, the above is finite and (1.3) follows from (3.1) and (3.2).

To show that $C_\varphi \notin \mathcal{C}_p$ or, what amounts to the same, $\operatorname{tr}((C_\varphi^* C_\varphi)^{p/2}) = \infty$, we need the following inequality: For any $1 < \rho < \infty$ and $0 < x < 1$, we have

$$(3.3) \quad \sum_{\ell=0}^\infty (\ell + 1)^\rho x^\ell \geq \frac{1}{(1 - x)^{\rho+1}}.$$

The proof of (3.3) is elementary. We begin with the identity

$$(1 - x)^{-2} = \sum_{\ell=0}^\infty (\ell + 1)x^\ell = \sum_{\ell=0}^\infty \{(\ell + 1)^\rho x^\ell\}^{1/\rho} \cdot \{x^\ell\}^{(\rho-1)/\rho}$$

and apply Hölder's inequality with conjugate exponents ρ and $\rho/(\rho - 1)$. This gives us $(1 - x)^{-2} \leq (\sum_{\ell=0}^\infty (\ell + 1)^\rho x^\ell)^{1/\rho} (1 - x)^{-(\rho-1)/\rho}$. Multiplying both sides by $(1 - x)^{(\rho-1)/\rho}$, we find that $(1 - x)^{-(\rho+1)/\rho} \leq (\sum_{\ell=0}^\infty (\ell + 1)^\rho x^\ell)^{1/\rho}$. Clearly, (3.3) follows from this.

Let $e_\ell(z) = (\ell + 1)^{1/2} z^\ell$, $\ell = 0, 1, 2, \dots$. Recall that $\{e_\ell : \ell \geq 0\}$ is the standard orthonormal basis for the Bergman space $L^2_a(D, dA)$. Because $p/2 > 1$ and $\|e_\ell\| = 1$, it follows from the spectral decomposition of $C_\varphi^* C_\varphi$ and Hölder's inequality that $\langle (C_\varphi^* C_\varphi)^{p/2} e_\ell, e_\ell \rangle \geq \{ \langle C_\varphi^* C_\varphi e_\ell, e_\ell \rangle \}^{p/2} = \|C_\varphi e_\ell\|^p$ [7, Proposition 6.3.3]. But $C_\varphi e_\ell = (\ell + 1)^{1/2} \varphi^\ell$. Therefore

$$\langle (C_\varphi^* C_\varphi)^{p/2} e_\ell, e_\ell \rangle \geq (\ell + 1)^{p/2} \|\varphi^\ell\|^p \geq (\ell + 1)^{p/2} \left\{ \sum_{n=1}^\infty \int_{E_n} |\varphi|^{2\ell} dA \right\}^{p/2},$$

where $E_n = \bigcup_{j=1}^{2^{2\epsilon k(n)}} E_{n,j}$, $n \in \mathbf{N}$, and $E_{n,j}$ is defined by (2.12). Define

$$f_n = \inf_{z \in E_n} |\varphi(z)|, \quad n \in \mathbf{N}.$$

It is elementary that $\{\sum_n a_n\}^\rho \geq \sum_n a_n^\rho$ if $\rho > 1$ and $a_n \geq 0$. Hence

$$\begin{aligned} \langle (C_\varphi^* C_\varphi)^{p/2} e_\ell, e_\ell \rangle &\geq (\ell + 1)^{p/2} \left\{ \sum_{n=1}^\infty f_n^{2\ell} A(E_n) \right\}^{p/2} \\ &\geq (\ell + 1)^{p/2} \sum_{n=1}^\infty (f_n^{2\ell} A(E_n))^{p/2}. \end{aligned}$$

Therefore

$$\begin{aligned} \text{tr}((C_\varphi^* C_\varphi)^{p/2}) &= \sum_{\ell=0}^\infty \langle (C_\varphi^* C_\varphi)^{p/2} e_\ell, e_\ell \rangle \geq \sum_{\ell=0}^\infty (\ell + 1)^{p/2} \sum_{n=1}^\infty (f_n^{2\ell} A(E_n))^{p/2} \\ (3.4) \quad &= \sum_{n=1}^\infty (A(E_n))^{p/2} \sum_{\ell=0}^\infty (\ell + 1)^{p/2} (f_n^p)^\ell \geq \sum_{n=1}^\infty \frac{(A(E_n))^{p/2}}{(1 - f_n^p)^{1+(p/2)}}, \end{aligned}$$

where the second \geq follows from an application of (3.3) with $\rho = p/2$. Thus the conclusion $C_\varphi \notin \mathcal{C}_p$ will follow once we show that there is a $d > 0$ such that

$$(3.5) \quad \frac{(A(E_n))^{p/2}}{(1 - f_n^p)^{1+(p/2)}} \geq d \quad \text{for every } n \geq 1.$$

To prove (3.5), we estimate $A(E_n)$ and $1 - f_n^p$. It follows from (2.12) that $A(E_{n,j}) \geq (2\pi)^{-1} 2^{-(1+p^{-1}+\epsilon)k(n)} m(B_{n,j}) = (6\pi)^{-1} 2^{-(1+(2/p)+2\epsilon)k(n)}$. Hence

$$(3.6) \quad A(E_n) = \sum_{j=1}^{2^{2\epsilon k(n)}} A(E_{n,j}) \geq \frac{1}{6\pi} 2^{-(1+(2/p)+2\epsilon)k(n)} \cdot 2^{2\epsilon k(n)} = \frac{1}{6\pi} 2^{-(1+(2/p))k(n)}.$$

Since $|\varphi(z)| = e^{-u(z)}$, it follows from Lemma 5 that $|\varphi(z)| \geq \exp(-M2^{-k(n)})$ for every $z \in E_n$, where $M = 1 + 6\beta$. That is, $f_n \geq \exp(-M2^{-k(n)})$. Therefore

$$(3.7) \quad 1 - f_n^p \leq 1 - \exp(-pM2^{-k(n)}) \leq pM2^{-k(n)},$$

where the second \leq follows from the inequality $x^{-1}(1 - e^{-x}) \leq 1$ for $0 < x < \infty$, which holds because the function $x \mapsto x^{-1}(1 - e^{-x})$ is decreasing on $(0, \infty)$. Combining (3.6) and (3.7), we see that (3.5) holds for $d = (pM)^{-1-(p/2)} (6\pi)^{-p/2}$. This completes the proof of Theorem 3.

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