PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 131, Number 8, Pages 2329–2335
S 0002-9939(03)06829-1
Article electronically published on January 28, 2003

ON THE RINGS WHOSE INJECTIVE HULLS ARE FLAT

K. KHASHYARMANESH AND SH. SALARIAN

(Communicated by Wolmer V. Vasconcelos)

ABSTRACT. Let R be a commutative Noetherian ring with nonzero identity and let the injective envelope of R be flat. We characterize these kinds of rings and obtain some results about modules with nonzero injective cover over these rings.

1. Introduction

Let R be a commutative Noetherian ring with nonzero identity and let $E(R)$ be an injective envelope of R. It is well known that for the ring R the condition that $E(R)$ be flat generalizes the condition that R be Gorenstein (see for example [10, 5.1.2 (1), (4)]). In [2, Theorem 3], Cheathan and Enochs gave a characterization of these kinds of rings. In fact, they showed that $E(R)$ is flat if and only if for any flat R-module F, the injective envelope $E(F)$ is flat and these are equivalent to, for any injective module E, the flat cover $F(E)$ is injective. In this paper, we focus on the rings R for which $E(R)$ is flat. In section two, we extend the above equivalent conditions (see Theorem 2.2).

On the other hand, in [6], Golan and Teply showed that every module admits an injective (pre)cover. Of course, this may be zero. So the natural problem is when a nonzero module has nonzero injective cover. In [10, 2.4.8], it was shown that whenever every nonzero module over a ring R has a nonzero injective cover, the ring R must be Artinian. In section three we obtain some results about modules with nonzero injective cover over the rings with $E(R)$ flat. For example we show that if an R-module M has nonzero injective cover, then $\mathrm{Ass}R \cap \mathrm{Coass}M \neq \emptyset$. The converse is true if M is Artinian.

Throughout this paper, R is commutative Noetherian with nonzero identity and all modules are unitary. For an R-module X, $\mathrm{inj.dim}_R X$ stands for the injective dimension of X, $\mathrm{f.dim}_R X$ stands for flat dimension of X, $\mathrm{proj.dim}_R X$ stands for projective dimension of X, $E(X)$ stands for injective envelope of X and $F(X)$ stands for its flat cover. Finally we use $\mathbb{N}$ to denote the set of positive integers. All other notations are standard.

Received by the editors May 11, 2001 and, in revised form, March 26, 2002.
2000 *Mathematics Subject Classification.* Primary 13C11, 13H10.
Key words and phrases. Injective envelope, flat cover, injective cover, Gorenstein ring, Gorenstein injective module.

2. Characterizing the rings with $E(R)$ flat

We recall that a module C is called cotorsion if $\mathrm{Ext}^1_R(F,C)=0$ for any flat module F; it is strongly cotorsion if $\mathrm{Ext}^1_R(X,C)=0$ for any module X of finite flat dimension. Also C is called strongly torsion free if $\mathrm{Tor}^R_1(X,C)=0$ for all modules X of finite flat dimension. If F is flat and cotorsion, it was proved in [3, p.183] that F can be uniquely written in the form $F=\prod T_{\mathfrak{p}}$, where $T_{\mathfrak{p}}$ is a completion of a free $R_{\mathfrak{p}}$-module with respect to $\mathfrak{p}$-adic topology. Note that flat covers are known to exist over any rings (see [4, 7.4.4]).

A module M is Gorenstein injective if there is an exact sequence

$$\cdots \longrightarrow E^{-2} \longrightarrow E^{-1} \longrightarrow E^{0} \longrightarrow E^{1} \longrightarrow E^{2} \longrightarrow \cdots$$

of injective modules such that $M=\ker(E^0 \longrightarrow E^1)$ and such that $\mathrm{Hom}_R(E,-)$ leaves the sequence exact when E is injective. If M is Gorenstein injective, then $\mathrm{Ext}^i_R(L,M)=0$ for all $i\in\mathbb{N}$ and L such that $\mathrm{inj.dim}L<\infty$. Also an R-module N is called Gorenstein flat if there is an exact complex

$$\cdots \longrightarrow F^{-2} \longrightarrow F^{-1} \longrightarrow F^{0} \longrightarrow F^{1} \longrightarrow F^{2} \longrightarrow \cdots$$

with F^i flat and $N=\ker(F^0\longrightarrow F^1)$ such that $E\otimes_R -$ leaves it exact for every injective module E.

Lemma 2.1. *Let R be commutative Noetherian with finite Krull dimension d. If M is Gorenstein injective, then M is strongly cotorsion.*

Proof. Since M is Gorenstein injective, there exists a sequence

$$\cdots \longrightarrow E^{-2} \xrightarrow{\delta^{-2}} E^{-1} \xrightarrow{\delta^{-1}} E^{0} \xrightarrow{\delta^{0}} E^{1} \xrightarrow{\delta^{1}} E^{2} \longrightarrow \cdots$$

of injective modules such that $M=\ker\delta^0$ and such that $\mathrm{Hom}_R(E,-)$ leaves the sequence exact when E is injective. For each $i\in\mathbb{N}$, put $N_i:=\ker\delta^{-i}$ and break the above long exact sequence into short exact sequences. We have that

$$\mathrm{Ext}^1_R(F,M)\cong \mathrm{Ext}^2_R(F,N_1)\cong\cdots\cong \mathrm{Ext}^{d+1}_R(F,N_d).$$

Now, by Gruson and Jensen's theorem in [7], $\mathrm{proj.dim}F\le d$ and so $\mathrm{Ext}^{d+1}_R(F,N_d)=0$ and the proof is complete.

There have been attempts to dualize the theory of associated primes (see [12], [1] and [11]). However, these dualizing concepts are equivalent [11]. A prime ideal $\mathfrak{p}$ of R is said to be a coassociated prime of M if there exists an Artinian homomorphic image L of M with $\mathfrak{p}=0:_R L$. The set of coassociated prime ideals of M is denoted by $\mathrm{Coass}(M)$.

Theorem 2.2. *Let R be commutative Noetherian. The following conditions are equivalent:*

(1) *$E(R)$ is flat;*
(2) *$E(R)$ has finite flat dimension;*
(3) *The injective envelope $E(F)$ is flat for any flat module F;*
(4) *The flat cover $F(E)$ is injective for any injective module E;*
(5) *The flat cover $F(M)$ is injective for any strongly cotorsion module M;*
(6) *The injective envelope $E(M)$ is flat for any strongly torsion free module M;*
(7) *The injective envelope $E(M)$ is flat for any Gorenstein flat module M;*
(8) *If $\mathfrak{p}\in\mathrm{Coass}(E)$ for an injective R-module E, then $\hat{R}_{\mathfrak{p}}$ is injective.*

If moreover the Krull dimension of R is finite, then the above conditions are equivalent to:

(9) *The flat cover $F(M)$ is injective for any Gorenstein injective module M.*

Proof. The implications (1) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4) follow from [2, Theorem 3].

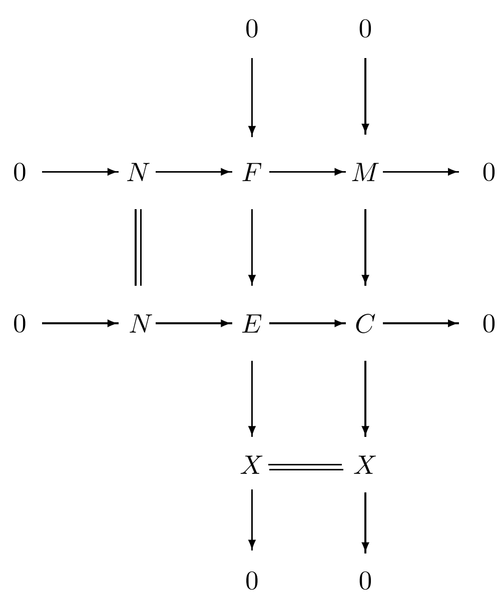
We will show that (2) $\Longrightarrow$ (1) $\Longrightarrow$ (7) $\Longrightarrow$ (2), (3) $\Longrightarrow$ (5) $\Longrightarrow$ (6) $\Longrightarrow$ (3), (1) $\Longrightarrow$ (8) $\Longrightarrow$ (4) and (5) $\Longleftrightarrow$ (9).

(2) $\Longrightarrow$ (1) Let $\mathfrak{p} \in \mathrm{Ass}(R)$. Since $E(R/\mathfrak{p})$ is a summand of $E(R)$, it has finite flat dimension as an R-module and hence as an $R_{\mathfrak{p}}$-module. So, by [9, Proposition 2.1(3), (2)] $R_{\mathfrak{p}}$ is Gorenestein ring and $\mathrm{f.dim}_{R_{\mathfrak{p}}} E(R/\mathfrak{p}) = 0$. Therefore $\mathrm{f.dim}_R E(R/\mathfrak{p}) = 0$ which, in turn, implies that $E(R)$ is a flat R-module.

(1) $\Longrightarrow$ (7) Let $\mathfrak{p} \in \mathrm{Ass}(R)$. Since $E(R/\mathfrak{p})$ is a summand of $E(R)$, it is flat. Now, suppose that M is a Gorenestein flat R-module. Then M can be embedded to a flat R-module F. Hence $\mathrm{Ass}(M) \subseteq \mathrm{Ass}(F) \subseteq \mathrm{Ass}(R)$. Thus $E(M) = \bigoplus_{\mathfrak{p} \in \mathrm{Ass}(R)} \mu_0(\mathfrak{p}, M) E(R/\mathfrak{p})$ is flat.

(7) $\Longrightarrow$ (2) This is clear.

(3) $\Longrightarrow$ (5) Suppose that M is a strongly cotorsion R-module. Consider the full pushout diagram of $F \longrightarrow M$ and $F \longrightarrow E$ where F is a flat cover of M and E is an injective envelope of F:



Since E is flat, $\mathrm{f.dim} X \leq 2$. Hence the exact sequence

$$0 \longrightarrow M \longrightarrow C \longrightarrow X \longrightarrow 0$$

is split. Thus, by [10, 1.2.10], the flat cover of M is a summand of the flat cover of C. Now, since E is flat and, by Wakamutsu's Lemma (see for example [10, 2.1.1]), N is cotorsion, E is a flat precover of C. Hence, by [10, 1.2.7], the flat cover of C is injective and so the flat cover of M is injective.

(5) $\Longrightarrow$ (6) Let C be an injective cogenerator of R-modules. Set $D(-) := \mathrm{Hom}_R(-, C)$. Since M is strongly torsion free, it follows from the equality

$$D(\mathrm{Tor}_1^R(X, M)) = \mathrm{Ext}_R^1(X, D(M)),$$

for all R-modules X, that $D(M)$ is strongly cotorsion. Let $F \longrightarrow D(M) \longrightarrow 0$ be a flat cover of $D(M)$. Then F is injective. Hence we have an exact sequence $0 \longrightarrow D(D(M)) \longrightarrow D(F)$ in which $D(F)$ is flat. Thus the injective envelope $E(D(D(M)))$ is flat and so $E(M)$ is too.

(6) $\Longrightarrow$ (3) This is clear.

(1) $\Longrightarrow$ (8) Let E be an injective R-module. Let $\mathfrak{p} \in \mathrm{Coass}E$. Then, by [11, 1.7], there is a maximal ideal $\mathfrak{m}$ of R such that $\mathfrak{p} \in \mathrm{AssHom}_R(E, E(R/\mathfrak{m}))$. Also $\mathrm{Hom}_R(E, E(R/\mathfrak{m}))$ is flat. Hence $E(R/\mathfrak{p})$ is flat which implies that

$$\mathrm{Hom}_R(E(R/\mathfrak{p}), E(R/\mathfrak{p})) \cong \hat{R}_\mathfrak{p}$$

is injective.

(8) $\Longrightarrow$ (4) Let E be an injective R-module and let $T_\mathfrak{p} \neq 0$ appear in $F(E)$ for some prime ideal $\mathfrak{p}$ of R. We will show that $T_\mathfrak{p}$ is injective and so $F(E)$ is injective. Since, by [5, 2.2],

$$0 \neq k(\mathfrak{p}) \otimes_{R_\mathfrak{p}} \mathrm{Hom}_R(R_\mathfrak{p}, E) \cong \mathrm{Hom}_R(\mathrm{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), R_\mathfrak{p}), E),$$

we have that $\mathfrak{p} \in \mathrm{Ass}R$. Now, let $\mathfrak{m}$ be a maximal ideal of R such that $\mathfrak{p} \subseteq \mathfrak{m}$. Since $E(R/\mathfrak{m})$ is Artinian, by [8, 2.1] and [11],

$$\{\mathfrak{q} \in \mathrm{Ass}R : \mathfrak{q} \subseteq \mathfrak{m}\} = \mathrm{Att}E(R/\mathfrak{m}) = \mathrm{Coass}E(R/\mathfrak{m}).$$

Hence $\mathfrak{p} \in \mathrm{Coass}E(R/\mathfrak{m})$ and so, by our assumption, $\hat{R}_\mathfrak{p}$ is injective. Now, it follows from the fact that $T_\mathfrak{p}$ is a summand of a product of copies of $\hat{R}_\mathfrak{p}$ (see for example [10, 4.1.10]) that $T_\mathfrak{p}$ is injective.

(5) $\Longrightarrow$ (9) This follows from Lemma 2.1.

(9) $\Longrightarrow$ (5) This is clear.

3. Results about modules over the rings with $E(R)$ flat

Theorem 3.1. *Let R be commutative Noetherian. If the injective envelope $E(R)$ is flat, then for any R-module M, the following conditions are equivalent:*

(1) *$F(M)$ is injective;*

(2) *every injective precover $E \xrightarrow{\varphi} M$ of M is surjective and* $\ker\varphi$ *is cotorsion;*

(3) *there is an injective precover $E \xrightarrow{\varphi} M$ of M such that φ is surjective and* $\ker\varphi$ *is cotorsion.*

Proof. (1) $\Longrightarrow$ (2) Since $F(M)$ is injective, every injective precover $E \xrightarrow{\varphi} M$ is surjective. Set $K := \ker\varphi$. We must show that K is cotorsion. To do this, consider the full pullback diagram of $F(M) \longrightarrow M$ and $E \longrightarrow M$:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & H & = & H & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & C & \longrightarrow & F(M) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since $\varphi : E \longrightarrow M$ is an injective precover, $\mathrm{Ext}^1_R(X, K) = 0$ for every injective module X. In particular, $\mathrm{Ext}^1_R(F(M), K) = 0$. So the upper exact row is split. Hence K is a summand of C. On the other hand, since H is cotorsion and E is injective, C is cotorsion. Hence K is cotorsion.

(2) $\Longrightarrow$ (3) This is clear.

(3) $\Longrightarrow$ (1) Let M be an R-module and let

$$0 \longrightarrow K \longrightarrow E \longrightarrow M \longrightarrow 0$$

be an exact sequence such that E is an injective precover of M and K is cotorsion. Then, it is easy to see that $F(E)$ is a flat precover of M. Thus $F(M)$ is a summand of $F(E)$ and so $F(M)$ is injective because $F(E)$ is injective.

The following example, which was suggested by the referee, provides a noninjective module whose flat cover is injective.

Example 3.2. Let k be a field. Let $R = k[[x^3, x^4, x^5]]$. Then R is not Gorenestein since 3, 4 and 5 do not generate a symmetric submonoid of $\mathbb{N}$. Note that R is a cotorsion (in fact pure injective) R-module. On the other hand, $k((x))$ (the field of fractions of R) is a flat and injective R-module. Set $M := k((x))/R$. So $k(x) \longrightarrow M$ is a flat (pre)cover of M. Therefore $F(M)$ is injective but M is not injective, since R is not Gorenestein.

Recall that whenever R is Gorenstein, $E(R)$ is flat (see for example [10, 5.1.2]).

Theorem 3.3. *Let R be a Gorenstein ring. Then M is strongly cotorsion if and only if M is cotorsion and Gorenstein injective.*

Proof. ($\Longrightarrow$) It follows from [9, Lemma 4.1].

($\Longleftarrow$) First of all we show that $F(M)$ is injective. To do this, we only need to show that whenever $T_{\mathfrak{p}} \neq 0$ appear in the flat cover of M, it is injective. Using the same notations as we used in the proof of Lemma 2.1, since $\mathrm{inj.dim} R_{\mathfrak{p}} < \infty$, it is easy to see that $\mathrm{Ext}^1_R(R_{\mathfrak{p}}, N_i) = 0$ for all i. Hence, for all i, $\mathrm{Hom}_R(R_{\mathfrak{p}}, N_i)$ and $\mathrm{Hom}_R(R_{\mathfrak{p}}, M)$ are Gorenstein injective $R_{\mathfrak{p}}$-modules. So, in view of Theorem

3.1 and Lemma 2.1, $F(\mathrm{Hom}_R(R_{\mathfrak{p}}, M))$ is injective. On the other hand, by [5, 2.2], $T_{\mathfrak{p}}$ appears in the flat cover of $\mathrm{Hom}_R(R_{\mathfrak{p}}, M)$. Therefore $T_{\mathfrak{p}}$ is injective and $F(M)$ is too. Since E^{-1} is an injective precover of M, by Lemma 2.1, N_1 is cotorsion. Similarly, for each i, N_i is cotorsion. Let L be an R-module with finite flat dimension n. Then, by using the induction on n, it is easy to see that $\mathrm{Ext}^i_R(L, N) = 0$ for all $i > n$ and all cotorsion modules N and so $\mathrm{Ext}^{n+1}_R(L, N_n) = 0$. Hence $\mathrm{Ext}^1_R(L, M) \cong \mathrm{Ext}^2_R(L, N_1) \cong \cdots \cong \mathrm{Ext}^{n+1}_R(L, N_n) = 0$. Therefore M is strongly cotorsion.

Theorem 3.4. *Suppose that $E(R)$ is flat. If the R-module M has a nonzero injective cover, then $\mathrm{Ass}R \cap \mathrm{Coass}M \neq \emptyset$. The converse is true if M is Artinian.*

Proof. Let M be an R-module with nonzero injective cover $E \xrightarrow{\varphi} M$. Since $\mathrm{Coass}\varphi(E) \neq \emptyset$, it is enough to show that $\mathrm{Coass}\varphi(E) \subseteq \mathrm{Ass}R \cap \mathrm{Coass}M$. Let $\mathfrak{p} \in \mathrm{Coass}\varphi(E)$. Then, by [11, 1.14], $\mathfrak{p} \in \mathrm{Att}E \subseteq \mathrm{Ass}R$. Thus, by [2, Theorem 3], $\mathrm{ht}\mathfrak{p} = 0$. Therefore, by [11, 2.6], $\mathfrak{p} \in \mathrm{Coass}M$.

For the proof of the last statement let M be an Artinian module and let $\mathfrak{p} \in \mathrm{Ass}R \cap \mathrm{Att}M \neq \emptyset$. In view of [11, 1.7], there exists a maximal ideal $\mathfrak{m}$ of R such that $\mathfrak{p} \subseteq \mathfrak{m}$ and $\mathrm{Hom}_R(R/\mathfrak{p}, E(R/\mathfrak{m}))$ is a homomorphic image of M. Now, let F be a flat cover of $\mathrm{Hom}_R(R/\mathfrak{p}, E(R/\mathfrak{m}))$ and let $\mathfrak{q}$ be a prime ideal of R. By using the isomorphisms

$$\begin{aligned} k(\mathfrak{q}) \otimes_{R_{\mathfrak{p}}} \mathrm{Hom}_R(R_{\mathfrak{q}}, \mathrm{Hom}_R(R/\mathfrak{p}, E(R/\mathfrak{m}))) &\cong k(\mathfrak{q}) \otimes_{R_{\mathfrak{p}}} \mathrm{Hom}_R(R_{\mathfrak{q}} \otimes_R R/\mathfrak{p}, E(R/\mathfrak{m})) \\ &\cong \mathrm{Hom}_R(\mathrm{Hom}_R(k(\mathfrak{q}), R_{\mathfrak{q}} \otimes_R R/\mathfrak{p}), E(R/\mathfrak{m})) \end{aligned}$$

in conjunction with [5, 2.2], it is easy to see that $F = T_{\mathfrak{p}}$. Since $\mathfrak{p} \in \mathrm{Ass}R$, we have that $E(R/\mathfrak{p})$ is flat. Thus $\mathrm{Hom}_R(E(R/\mathfrak{p}), E(R/\mathfrak{p})) \cong \hat{R}_{\mathfrak{p}}$ is injective and so $T_{\mathfrak{p}} = F$ is injective. Therefore it is enough to show that $\mathrm{Hom}_R(T_{\mathfrak{p}}, M) \neq 0$. To do this consider the exact sequence

$$0 \longrightarrow K \longrightarrow M \longrightarrow \mathrm{Hom}_R(R/\mathfrak{p}, E(R/\mathfrak{m})) \longrightarrow 0.$$

Since K is Artinian, it is thus cotorsion, and so the sequence

$$\mathrm{Hom}_R(T_{\mathfrak{p}}, M) \longrightarrow \mathrm{Hom}_R(T_{\mathfrak{p}}, \mathrm{Hom}_R(R/\mathfrak{p}, E(R/\mathfrak{m}))) \longrightarrow 0$$

is exact. Now, since $T_{\mathfrak{p}}$ is a flat cover of $\mathrm{Hom}_R(R/\mathfrak{p}, E(R/\mathfrak{m}))$, $\mathrm{Hom}_R(T_{\mathfrak{p}}, M) \neq 0$ and the proof is complete.

Acknowledgments

The authors are deeply grateful to the referee for his careful reading of the manuscript and for suggesting Example 3.2 and part (2) of Theorem 2.2. We would like to thank the Institute for Studies in Theoretical Physics and Mathematics (IPM) for their financial support.

References

1. L. Chambless, *Coprimary decomposition, N-dimension and divisibility, Application to Artinian modules*, Comm. Algebra **9** (1981), 1131-1149. MR **82k:**16032
2. T. Cheatham and E. Enochs, *Injective hulls of flat modules*, Comm. Algebra **8** (1980), 1989-1998. MR **81m:**16026
3. E. Enochs, *Flat covers and flat cotorsion modules*, Proc. Amer. Math. Soc. **92** (1984), 179-184. MR **85j:**13016
4. E. Enochs and O. M. Jenda, Relative homological algebra, Degruyter expositions in Mathematics 30, 2000. MR **2001h:**16013

5. E. Enochs and J. Xu, *On invariants dual to the Bass numbers*, Proc. Amer. Math. Soc. **125** (1997) 951-960. MR **97f:**13015
6. J. S. Golan and M. L. Teply, *Torsion-free covers*, Israeal J. Math. **15** (1973), 237-256. MR **48:**4034
7. L. Gruson and C.U. Jensen, *Dimensions cohomologiques reliées aux foncteurs* $\varprojlim^{(i)}$, in Lecture Notes in Math. Vol. 867, 234-294, Springer, Berlin-New York, 1981. MR **83d:**16026
8. A. H. Toroghy and R. Y. Sharp, *Asymptotic behaviour of ideals relative injective modules over a commutative Noetherian rings*, Proc. Edinburgh Math. Soc. **34** (1991), 155-160. MR **92b:**13005
9. J. Xu, *Minimal injective and flat resolutions of modules over Gorenstein rings*, J. of Algebra **175** (1995), 451-477. MR **96h:**13025
10. J. Xu, *Flat covers of modules*, Lecture Notes in Mathematics 1634 (Springer, Berlin 1996). MR **98b:**16003
11. S. Yassemi, *Coassociated primes,* Comm. in Algebra **23** (1995), 1473-1498. MR **96e:**13003
12. H. Zöschinger, *Linear-kompakte moduln über notherschen Ringen*, Arch. Math. **41** (1983), 121-130. MR **85d:**13024

Institute for Studies in Theoretical Physics and Mathematics, P.O. Box 19395-5746, Tehran, Iran – and – Department of Mathematics, Damghan University, P.O. Box 36715-364, Damghan, Iran

E-mail address: khashyar@mail.ipm.ir

Institute for Studies in Theoretical Physics and Mathematics, P.O. Box 19395-5746, Tehran, Iran – and – Department of Mathematics, Damghan University, P.O. Box 36715-364, Damghan, Iran