PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 131, Number 8, Pages 2409–2414
S 0002-9939(03)06874-6
Article electronically published on January 15, 2003

ON A CLASS OF SUBLINEAR QUASILINEAR ELLIPTIC PROBLEMS

D. D. HAI

(Communicated by David S. Tartakoff)

ABSTRACT. We establish existence and multiplicity of positive solutions to the quasilinear boundary value problem

$$\begin{aligned} div(|\nabla u|^{p-2}\nabla u) &= -\lambda f(u) \text{ in } \Omega, \\ u &= 0 \quad \text{ on } \partial\Omega, \end{aligned}$$

where Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$, $f : [0,\infty) \to R$ is continuous and p-sublinear at ∞, and λ is a large parameter.

1. INTRODUCTION

Consider the quasilinear elliptic boundary value problem

$$\text{(I)} \qquad \begin{aligned} \Delta_p u = -\lambda f(u) \text{ in } \Omega, \\ u = 0 \text{ on } \quad \partial\Omega, \end{aligned}$$

where $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$, $p > 1$, Ω is a bounded domain in R^n with smooth boundary $\partial\Omega, f : [0,\infty) \to R$, and λ is a positive parameter.

Problem (I) has been studied extensively during recent years (see [2]–[11] and the references therein). In [5], Guo and Webb proved existence results for (I) for λ large when f is a smooth, nondecreasing positive function on $(0,\infty), \lim_{s\to\infty} s^{-\mu} f(s) = \beta > 0$ for some $\mu \in (0, p-1)$, and $\liminf_{x\to 0^+} \frac{f(s)}{s^{p-1}} > 0$, using Serrin's sweeping principle. They also considered uniqueness of positive solutions when $\frac{f(x)}{x^{p-1}}$ is decreasing on $(0,\infty)$. Multiplicity results were established in [4] when the above condition of f at 0 is replaced by $\lim_{s\to 0} \frac{f(s)}{s^{p-1}} = 0$. Related results for the radial case in a ball using ordinary differential equations techniques can be found in [6]. In this paper, we shall establish existence results to (I) for λ large when f is merely continuous and p-sublinear at ∞. In particular, we do not assume any conditions of f near 0, nor monotonicity and smoothness on f as in [4, 5]. We also establish a multiplicity result for (I) under additional assumptions. Our approach depends on $C^{1,\beta}$ regularity results in [7, 9] to create ordered sub- and supersolutions that provide a solution in between via the Schauder fixed point theorem. Note that our sub- and supersolutions are defined in the operator context and are not weak sub- and supersolutions in general.

Received by the editors March 7, 2002.
2000 *Mathematics Subject Classification.* Primary 35J25, 35J70.
Key words and phrases. Sub-supersolutions, quasilinear elliptic, positive solutions.

2. Existence results

We shall make the following assumptions:

(A.1) $f : [0, \infty) \to R$ is continuous and there exist positive numbers K, L_1 such that $f(x) > 2L_1$ for $x > K$.

(A.2) $\lim_{x\to\infty} \frac{f(x)}{x^{p-1}} = 0$.

Our main result is

Theorem 1. *Let* (A.1)-(A.2) *hold. Then there exists* $\lambda_0 > 0$ *such that* (I) *has a positive solution for* $\lambda > \lambda_0$. *If, in addition,* $f \geq 0$ *on* $(0,\infty)$ *and*

$$lim_{x\to 0^+} \frac{f(x)}{x^{p-1}} = 0,$$

then (I) *has at least two positive solutions for* $\lambda > \lambda_0$.

We first need some preliminary results. We shall denote by $||.||_p$ and $|.|_p$ the norms in $L^p(\Omega)$ and $C^p(\bar{\Omega})$ respectively.

The following lemma gives positivity of the p-Laplacian operator.

Lemma 1 ([10])**.** *Let* $x = (x_1, ..., x_n), y = (y_1, ..., y_n)$ *be two points in* R^n, *and let* $|.|$ *and* $(.,.)$ *denote the Euclidean norm and the corresponding inner product in* R^n *respectively.*

Then

(i) *If* $p \geq 2$,

$$(|x|^{p-2}x - |y|^{p-2}y, x - y) \geq (\frac{1}{2})^{p-1}|x - y|^p.$$

(ii) *If* $1 < p \leq 2$,

$$(|x| + |y|)^{2-p}(|x|^{p-2}x - |y|^{p-2}y, x - y) \geq (p - 1)|x - y|^2.$$

Next, let $\phi \in C^1(\bar{\Omega})$ be the solution of

$$\begin{cases} \Delta_p\phi = -1 \text{ in } \Omega, \\ \phi = 0 \text{ on } \partial\Omega. \end{cases}$$

Then we have

Lemma 2. *Let* $g \in L^\infty(\Omega)$ *and let* $u \in W_0^{1,p}(\Omega)$ *satisfy*

$$\begin{cases} \Delta_p u = -g \textit{ in } \Omega, \\ u = 0 \textit{ on } \partial\Omega. \end{cases}$$

Then $u \in L^\infty(\Omega)$ *and*

$$||u||_\infty \leq M|\phi|_0,$$

where $M = ||g||_\infty^{\frac{1}{p-1}}$.

Proof. Since $\Delta_p(M\phi) = -M^{p-1} = -||g||_\infty$ and $\Delta_p u = -g$, it follows from the weak maximum principle [8] that $-M\phi \leq u \leq M\phi$, from which the lemma follows.

Lemma 3. *Let* $C, L_0, L_1,$ *and* L *be positive numbers with* $L_0 \leq L \leq L_1$. *Suppose that there exist* $h, h_{\lambda,L} \in L^\infty(\Omega)$ *such that*

$$\|h_{\lambda,L}\|_\infty \leq C \tag{2.1}$$

for every $\lambda > 0, L_0 \leq L \leq L_1$, *and*

$$\lim_{\lambda\to\infty} \|h_{\lambda,L} - h\|_2 = 0$$

uniformly for $L \in [L_0, L_1]$. *Let* $u_{\lambda,L}$ *and* u *be solutions of*

$$\Delta_p u_{\lambda,L} = -h_{\lambda,L} \text{ in } \Omega,\ u_{\lambda,L} = 0 \text{ on } \partial\Omega$$

and

$$\Delta_p u = -h \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega$$

respectively. Then

$$\lim_{\lambda\to\infty} |u_{\lambda,L} - u|_1 = 0 \tag{2.2}$$

uniformly for $L \in [L_0, L_1]$.

Proof. By (2.1) and Lemma 2, it follows that $u_{\lambda,L}$ are bounded in $L^\infty(\Omega)$ independent of λ, L. It then follows from regularity results in [7, 9] that there exist numbers $\beta \in (0,1)$ and $C_1 > 0$ such that $u_{\lambda,L},\ u \in C^{1,\beta}(\bar{\Omega})$ and

$$|u_{\lambda,L}|_1 < C_1 \tag{2.3}$$

for all $\lambda > 0, L \in [L_0, L_1]$.

Multiplying the equation

$$\Delta_p u_{\lambda,L} - \Delta_p u = -(h_{\lambda,L} - h)$$

by $u_{\lambda,L} - u$ and integrating gives

$$\begin{aligned} &\int_\Omega (|\nabla u_{\lambda,L}|^{p-2}\nabla u_{\lambda,L} - |\nabla u|^{p-2}\nabla u, \nabla u_{\lambda,L} - \nabla u)dx \\ &= \int_\Omega (h_{\lambda,L} - h)(u_{\lambda,L} - u)dx \\ &\le ||h_{\lambda,L} - h||_2\ ||u_{\lambda,L} - u||_2. \end{aligned} \tag{2.4}$$

On the other hand, Lemma 1 and (2.3) imply

$$\int_\Omega (|\nabla u_{\lambda,L}|^{p-2}\nabla u_{\lambda,L} - |\nabla u|^{p-2}\nabla u, \nabla u_{\lambda,L} - \nabla u)dx \ge C_2||\nabla(u_{\lambda,L} - u)||_2^q, \tag{2.5}$$

where $q = \max(p, 2)$, and C_2 is a positive constant depending only on p, C_1.

Combining (2.4), (2.5), and Poincare's inequality, we obtain

$$\lim_{\lambda\to\infty} ||u_{\lambda,L} - u||_2 = 0 \tag{2.6}$$

uniformly for $L \in [L_0, L_1]$.

Suppose to the contrary that (2.2) does not hold. Then there exist a positive number ε and sequences $(L_n) \subset [L_0, L_1]$, $(\lambda_n) \to \infty$ such that

$$|u_{\lambda_n,L_n} - u|_1 \ge \varepsilon$$

for all n. Since (u_{λ_n,L_n}) is bounded in $C^{1,\beta}(\bar{\Omega})$, there exist $v \in C^1(\bar{\Omega})$ and a subsequence of (u_{λ_n,L_n}), which we still denote by (u_{λ_n,L_n}), such that

$$\lim_{n\to\infty} |u_{\lambda_n,L_n} - v|_1 = 0.$$

From this and (2.6), we deduce that $u = v$, a contradiction. This completes the proof of Lemma 3.

For each $\lambda > 0$, define the operator A_λ on $C(\bar{\Omega})$ by $A_\lambda v = u$ if

$$\Delta_p u = -\lambda f(v) \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega,$$

where we define $f(x) = f(0)$ if $x < 0$. Then $A_\lambda : C(\bar{\Omega}) \to C(\bar{\Omega})$ is completely continuous, and fixed points of A_λ are solutions of (I).

The next lemma produces subsolutions to (I).

Lemma 4. *Let* $0 < L_0 \leq L \leq L_1$*, where* L_1 *is given by* (A.1)*, and let* $\phi_{\lambda,L} = (\lambda L)^{\frac{1}{p-1}}\phi$*. Then there exists a positive number* λ_0 *independent of* L *such that for* $\lambda \geq \lambda_0$*,*

$$v \geq \phi_{\lambda,L} \Rightarrow A_\lambda v \geq \phi_{\lambda,L}.$$

Proof. Let $v \in C(\bar{\Omega})$ with $v \geq \phi_{\lambda,L}$, and let $u_\lambda = A_\lambda v$, $\tilde{u}_{\lambda,L} = \frac{u_\lambda}{(2\lambda L)^{\frac{1}{p-1}}}$. Then

$$v(x) \geq (\lambda L_0)^{\frac{1}{p-1}}\phi(x) > K \text{ if } \phi(x) > \frac{K}{(\lambda L_0)^{\frac{1}{p-1}}},$$

where K is given by (A.1). Define

$$h_{\lambda,L}(x) = \begin{cases} 1 \quad \text{if} \quad \phi(x) > \frac{K}{(\lambda L_0)^{\frac{1}{p-1}}}, \\ \dfrac{M_0}{2L} \text{ if } \phi(x) \leq \frac{K}{(\lambda L_0)^{\frac{1}{p-1}}}, \end{cases}$$

where M_0 is a number such that $f(x) \geq M_0$ for all x.

By (A.1),

$$\Delta_p(\tilde{u}_{\lambda,L}) = -\frac{f(v)}{2L} < -h_{\lambda,L} \text{ in } \Omega,$$

and the maximum principle implies that $\tilde{u}_{\lambda,L} \geq \bar{u}_{\lambda,L}$, where $\bar{u}_{\lambda,L}$ is the solution of

$$\Delta_p(\bar{u}_{\lambda,L}) = -h_{\lambda,L} \text{ in } \Omega,\ \bar{u}_{\lambda,L} = 0 \text{ on } \partial\Omega.$$

Note that

$$||h_{\lambda,L} - 1||_2 \leq (\frac{|M_0|}{2L} + 1)(\int_{\left\{x\in\Omega:\phi(x)\leq\frac{K}{(\lambda L_0)^{\frac{1}{p-1}}}\right\}} dx)^{\frac{1}{2}} \to 0$$

as $\lambda \to \infty$ uniformly for $L \in [L_0, L_1]$. Since $\Delta_p\phi = -1$ in Ω, it follows from Lemma 3 that

$$\lim_{\lambda\to\infty} |\bar{u}_{\lambda,L} - \phi|_1 = 0$$

uniformly for $L \in [L_0, L_1]$. Let $\gamma > 0$. Then by the mean value theorem

$$|\bar{u}_{\lambda,L}(x) - \phi(x)| \leq |\bar{u}_{\lambda,L} - \phi|_1 d(x, \partial\Omega) < \gamma c\phi(x),\ x \in \Omega,$$

for λ large, where $c > 0$ is such that $d(x, \partial\Omega) \leq c\phi(x)$ for all $x \in \Omega$. Hence for such λ,

$$\bar{u}_{\lambda,L}(x) \geq (1 - \gamma c)\phi(x),\ x \in \Omega,$$

from which it follows that

$$u_\lambda(x) \geq (2\lambda L)^{\frac{1}{p-1}}\bar{u}_{\lambda,L}(x) \geq (\lambda L)^{\frac{1}{p-1}}\phi(x) = \phi_{\lambda,L}(x),\ x \in \Omega,$$

if γ is sufficiently small. This completes the proof of Lemma 4.

We are now in a position to give the

Proof of Theorem 1. Let $\lambda > \lambda_0$, where λ_0 is given by Lemma 4.

Let $\bar{\phi}_\lambda = M_\lambda\phi$ where $M_\lambda >> 1$. Let $v \in C(\bar{\Omega})$ satisfy $v \leq \bar{\phi}_\lambda$. We claim that $\tilde{u}_\lambda \equiv A_\lambda v \leq \bar{\phi}_\lambda$. Indeed, let $\tilde{f}(x) = \sup_{z\leq x} f(z)$. Then $\tilde{f}$ is nondecreasing and

$$\Delta_p\tilde{u}_\lambda = -\lambda f(v) \geq -\lambda\, \tilde{f}(M_\lambda|\phi|_0) \geq -M_\lambda^{p-1} \text{ in } \Omega,$$

where we use the fact that $\lim_{x\to\infty}\frac{\tilde{f}(x)}{x^{p-1}} = 0$. Since $\Delta_p(\bar{\phi}_\lambda) = -M_\lambda^{p-1}$, the maximum principle implies $\tilde{u}_\lambda \le \bar{\phi}_\lambda$, as claimed.

Thus $A_\lambda : [\phi_{\lambda,L}, \bar{\phi}_\lambda] \to [\phi_{\lambda,L}, \bar{\phi}_\lambda]$, where $\phi_{\lambda,L}$ is defined in Lemma 4. The Schauder fixed point theorem then gives a fixed point u of A_λ in $[\phi_{\lambda,L}, \bar{\phi}_\lambda]$, which is a positive solution of (I). This completes the proof of the first part of Theorem 1. Suppose next that $f \ge 0$ and $\lim_{x\to 0^+}\frac{f(x)}{x^{p-1}} = 0$. Let $\phi_\varepsilon = \varepsilon\phi$. We claim that for $\varepsilon > 0$ small,

$$v \le \phi_\varepsilon \Rightarrow u_\lambda \equiv A_\lambda v \le \phi_\varepsilon.$$

Indeed, we have

$$\Delta_p u_\lambda = -\lambda f(v) \text{ in } \Omega,$$

and for $\varepsilon > 0$ small,

$$\lambda f(v) \le \lambda\, \tilde{f}(v) \le \lambda\, \tilde{f}(\varepsilon|\phi|_0) \le \varepsilon^{p-1} = -\Delta_p\phi_\varepsilon \text{ in } \Omega,$$

where $\tilde{f}(x) = \sup_{z\le x} f(z)$. Here we have used the fact that $\lim_{x\to\infty}\frac{\tilde{f}(x)}{x^{p-1}} = 0$. By the maximum principle, $u_\lambda \le \phi_\varepsilon$, as claimed. By decreasing ε if necessary, we assume $\phi_\varepsilon < \phi_{\lambda,L_0}$ in Ω. Define

$$X = C_0^1(\bar{\Omega}),\ X_1 = [0,\phi_\varepsilon], X_2 = [\phi_{\lambda,L_0}, \bar{\phi}_\lambda],\ \text{and } X_3 = [0,\bar{\phi}_\lambda],$$

where $[z,w] = \{u \in X : z \le u \le w\}$. Then $A_\lambda : X_k \to X_k$ and A_λ has a fixed point $\tilde{u}_2$ in X_2. By the maximum principle [11], every nontrivial fixed point of A_λ in X_3 is a positive solution of (I). Since 0 is an interior point of X_1 with respect to X_3, if A_λ has a fixed point u_1 on the boundary of X_1 in X_3, then $0 < u_1 < \tilde{u}_2$ in Ω and thus (I) has two distinct positive solutions. Since $A_\lambda : [\phi_{\lambda,L_1}, \bar{\phi}_\lambda] \to [\phi_{\lambda,L_1}, \bar{\phi}_\lambda]$, A_λ has a fixed point $\tilde{u}$ in $[\phi_{\lambda,L_1}, \bar{\phi}_\lambda]$. Note that every point in $[\phi_{\lambda,L_1}, \bar{\phi}_\lambda]$ is an interior point of X_2 in X_3. If A_λ has a fixed point u_2 on the boundary of X_2 in X_3, then $u_2 \ne \tilde{u}$ and we have two solutions. Finally, if A_λ does not have any fixed points on the boundaries of X_1 and X_2 in X_3, then by a result of Amann [1], there exists a fixed point $\bar{u}$ of A_λ such that $\bar{u} \in X_3\backslash(X_1 \cup X_2)$. In particular, $\bar{u}$ is a second positive solution of (I). This completes the proof of Theorem 1.

Remark 1. (i) Conditions $\lim_{x\to\infty}\frac{f(x)}{x^{p-1}} = 0$ and $\lim_{x\to 0^+}\frac{f(x)}{x^{p-1}} = 0$ in Theorem 1 can be replaced by $\liminf_{x\to\infty}\frac{f(x)}{x^{p-1}} = 0$ and $\liminf_{x\to 0^+}\frac{f(x)}{x^{p-1}} = 0$ respectively, provided that f is nondecreasing.

(ii) The multiplicity result in Theorem 1 was established in [4] under additional assumptions that $f \in C^1(R^+)$ is strictly increasing on R^+, and there exist $\alpha_1, \alpha_2 > 0$ such that $f(s) \le \alpha_1 + \alpha_2 s^\mu$ for some $\mu \in (0, p-1)$.

References

[1] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *Siam Rev.* **18** (1976), 620-709. MR **54:**3519; errata MR **57:**7269

[2] P. Drabek and J. Hernandez, Existence and uniqueness of positive solutions for some quasilinear elliptic problems, *Nonlinear Anal.* **44** (2001), 189-204. MR **2001m:**35104

[3] P. Drabek, A. Kufner, and F. Nicolosi, Quasilinear Elliptic Equations with Degenerations and Singularities, de Gruyter Series in Nonlinear Analysis and Applications, Vol. **5**, Berlin, New York, 1997. MR **98k:**35068

[4] Z. M. Guo, Some existence and multiplicity results for a class of quasilinear eigenvalue problems, *Nonlinear Anal.* **18** (1992), 957-971. MR **93c:**35045

[5] Z. M. Guo and J. R. L. Webb, Uniqueness of positive solutions for quasilinear elliptic equations when a parameter is large, *Proc. Roy. Soc. Edinburgh* **124 A** (1994), 189-198. MR **95a:**35046

[6] D. D. Hai and R. Shivaji, Existence and uniqueness for a class of quasilinear elliptic boundary value problems (submitted)

[7] G. M. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, *Nonlinear Anal.* **12** (1988), 1203-1219. MR **90a:**35098

[8] S. Sakaguchi, Concavity properties of solutions to some degenerate quasilinear elliptic Dirichlet problems, *Ann. Scula Norm. Sup. Pisa Cl. Sci.* **14** (1987), 403-421. MR **89h:**35133

[9] P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, *J. Diff. Eqns.* **51** (1984), 126-150. MR **85g:**35047

[10] T. Oden, Qualitative Methods in Nonlinear Mechanics, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1986.

[11] J. L. Vazquez, A strong maximum principle for some quasilinear elliptic equations, *Appl. Math. Optim.* **12** (1984), 191-202. MR **86m:**35018

Department of Mathematics, Mississippi State University, Mississippi State, Mississippi 39762

E-mail address: `dang@ra.msstate.edu`