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ON THE FUNDAMENTAL GROUP OF MANIFOLDS WITH ALMOST NONNEGATIVE RICCI CURVATURE

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Abstract. Gromov conjectured that the fundamental group of a manifold with almost nonnegative Ricci curvature is almost nilpotent. This conjecture is proved under the additional assumption on the conjugate radius. We show that there exists a nilpotent subgroup of finite index depending on a lower bound of the conjugate radius.

1. Introduction

Gromov conjectured that if a manifold has almost nonnegative Ricci curvature, then the fundamental group is almost nilpotent. Fukaya and Yamaguchi proved this conjecture when the manifold has almost nonnegative sectional curvature [FY]. Recall that a group is *almost nilpotent* if and only if it has a nilpotent subgroup of finite index. In [FY], they conjectured that if M is an almost nonnegatively curved manifold, then there exists a nilpotent subgroup Γ of $\pi_1(M)$ such that $[\pi_1(M) : \Gamma] \leq \omega(n)$, where $\omega(n)$ is a constant depending only on n.

For a manifold with almost nonnegative Ricci curvature, if its conjugate radius is bounded below, we have weak coordinates which have a bounded weak $C^{0,\alpha}$-norm [PWY]. Using the smoothing procedure of Petersen, Wei and Ye [PWY], we can apply the fibration theorem of Fukaya [F]. Then the splitting theorem of Cheeger and Colding [CC] implies Gromov's conjecture immediately under the assumption on the conjugate radius.

We have the following result:

Theorem 1.1. *Let M be an n-dimensional compact Riemannian manifold with* $\operatorname{diam}(M) \leq 1$ *and* $\operatorname{conj}_M \geq c_0 > 0$. *Then there exist an $\epsilon > 0$ and an $\omega > 0$ both depending only on n and c_0 such that if* $\operatorname{Ric}_M \geq -(n-1)\epsilon^2$, *there exists a nilpotent subgroup Λ of $\pi_1(M)$ satisfying $[\pi_1(M) : \Lambda] \leq \omega$.*

We briefly review the definition of weak coordinate and weak norm. The (weak) norm of an n-dimensional Riemannian manifold (M, g) on scale $r > 0$ is defined in [PWY] as follows, where g is the metric of M.

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Definition 1.2. The $C^{k,\alpha}$-norm of an n-dimensional Riemannian manifold (M, g) on scale $r > 0$, denoted by $||(M, g)||_{C^{k,\alpha},r}$, is defined to be the infimum of positive numbers Q such that there exist imbeddings

$$\phi_\tau : B(0, r) \subset \mathbb{R}^n \to U_\tau \subset M$$

with images U_τ, $\tau \in I$ (an index set), with $\to$ having the following properties:
1) $e^{-2Q}\delta \le \phi_\tau^*(g) \le e^{2Q}\delta$, where δ is the Euclidean metric,
2) every metric ball $B(p, \frac{r}{10}e^{-Q})$ for $p \in M$ lies in some U_τ,
3) $r^{|l|+\alpha}||\partial^l g_{\tau,ij}||_{C^{0,\alpha}} \le Q$ for all multi-indices l with $0 \le |l| \le k$, where $g_\tau = \phi_\tau^* g$.

The weak norm $||(M, g)||^W_{C^{k,\alpha},r}$ is identically defined except that ϕ_τ is assumed to be a local diffeomorphism instead of a diffeomorphism. Then we may regard ϕ_τ as a *weak* coordinate chart. If we consider only the case that weak coordinate charts are harmonic, we call such a norm a *weak harmonic norm.* In [PWY], they generalized almost flat manifolds using the bounded weak $C^{0,\alpha}$-harmonic norm. We also have a generalized almost flat manifold using the bounded weak $C^{0,\alpha}$-norm (without harmonicity assumption) [Pa]. For a positive function $Q(r)$ satisfying $Q(r) \to 0$ as $r \to 0$ and $\alpha > 0$, we define the following class of n-dimensional complete Riemannian manifolds without any assumption on the harmonicity:

$$\mathcal{M}(n, \alpha, Q) = \{(M, g) \mid ||(M, g)||^W_{C^{0,\alpha},r} \le Q(r) \text{ for } 0 < r \le 1\}.$$

Theorem 1.3 ([Pa]). *There exists an $\epsilon(n, \alpha, Q) > 0$ depending on n, α and Q such that if $M \in \mathcal{M}(n, \alpha, Q)$ and $\operatorname{diam}(M) \le \epsilon(n, \alpha, Q)$, then a finite covering space of M is a nilmanifold.*

As an immediate corollary, we obtain that

Corollary 1.4. *Let M be an n-dimensional compact Riemannian manifold with $\operatorname{Ric}_M \ge -1$ and $\operatorname{conj}_M \ge c_0$. Then there exists an $\epsilon(n, c_0) > 0$ such that if $\operatorname{diam}(M) \le \epsilon$, then M is diffeomorphic to a nilmanifold up to finite cover.*

Using the smoothness of small balls as we see in Corollary 1.4, we obtain Theorem 1.1.

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2. Splitting theorem and solvability of $\pi_1(M)$

In [FY], the authors proved the following generalized Bieberbach theorem, which is useful to find a nilpotent subgroup and a bound on the index of the nilpotent subgroup:

Theorem 2.1 ([FY]). *Let G be a closed subgroup of the group of isometries of $\mathbb{R}^n$. Then $\pi_0(G)$ contains a finite index, free Abelian subgroup whose rank is not greater than $\dim(\mathbb{R}^n/G)$.*

Corollary 2.2 ([FY]). *Suppose, in addition, that the quotient space $\mathbb{R}^n/G$ is compact. Then there exists a normal subgroup G' of G such that*
(1) *$[G : G'] < w_n$, where w_n is a number depending only on n,*
(2) *$\mathbb{R}^n/G'$ is isometric to a flat torus.*

Let M_i be a sequence of compact Riemannian manifolds with $\mathrm{Ric}_{M_i} \geq -(n-1)\epsilon^2 \to 0$, $\mathrm{conj}_M \geq c_0$ and $\mathrm{diam}(M) \leq 1$. We denote $\pi_1(M_i)$ by Π_i. From [CC], we have the following convergence with respect to the pointed Gromov-Hausdorff distance:

$$(\tilde{M}_i, \Pi_i) \to (\mathbb{R}^k \times X, \Pi)$$

for some compact length space X and an isometry group Π of X. Also $\gamma \in \Pi$ is of the form $(\gamma_1, \gamma_2)(x, y) = (\gamma_1(x), \gamma_2(y))$, where γ_1 and γ_2 are isometries on $\mathbb{R}^k$ and X, respectively [CG].

Now we review the construction of solvable subgroups with a bounded index. From Corollary 2.2, if we consider a finite ($\leq w(k)$) covering space of M_i, we may assume that M_i converges to a fiber bundle X_1 over T^{k_1}, where $k_1 \leq k$ and k_1 is the rank of the maximal discrete Abelian subgroup of Π. So we may assume that $M_i \to X_1$. Then γ_1 can be decomposed to $(\gamma_{11}, \gamma_{12})$, where γ_{11} and γ_{12} are isometries on $\mathbb{R}^{k_1}$ and $\mathbb{R}^{k-k_1}$, respectively. We may consider γ_{11} as an element of $\pi_1(T^{k_1})$. Let $p_1(\gamma)$ be γ_{11} and $p_2(\gamma)$ be (γ_{12}, γ_2). We denote $\mathrm{Ker}(p_1)$ by K. Then K is normal subgroup of Π and $\Pi/K \simeq \mathbb{Z}^{k_1}$. Using Theorem 3.10 in [FY], we can find a normal subgroup K_i of Π_i such that

$$(\tilde{M}_i, K_i) \to (\mathbb{R}^k \times X, K)$$

and

$$\Pi_i/K_i \simeq \Pi/K = \mathbb{Z}^{k_1}.$$

Using Theorem 1.3 or Corollary 1.4 from the previous section, we can find $\delta(n, c_0) > 0$ such that $\pi_1(\phi_i(B(0,\delta)))$ is almost nilpotent, so there exists a nilpotent subgroup $\tilde{H}_i$ such that

$$[\pi_1(\phi_i(B(0,\delta))) : \tilde{H}_i] \leq \kappa(n)$$

for some constant κ depending only on n, where ϕ_i is a weak coordinate of M_i (see [G] and [Pa]). Let $i : \phi_i(B(0,\delta)) \to M_i$ be the inclusion map. We denote $i_*(\tilde{H}_i)$ by H_i. Let

$$(\tilde{M}_i, H_i) \to (\mathbb{R}^k \times X, H)$$

for some isometry group H of $\mathbb{R}^k \times X$. If we take $\delta > 0$ sufficiently small, we may assume that $p_1(H) = \mathrm{Id}$, so $H \subset K$ and $H_i \subset i_*(\pi_1(\phi_i(B(0,\delta)))) \subset K_i$. From Definition 1.2, $\phi_i(B(0,\delta))$ contains an ϵ-ball for a constant $\epsilon(n, c_0) > 0$ depending on n and c_0.

Using the Bishop-Gromov comparison theorem, we obtain that for any $x \in \tilde{M}_i$,

$$\frac{\mathrm{vol}(B(x,20))}{\mathrm{vol}(B(x,\epsilon/100))} \leq C_1(n, c_0)$$

for some constant $C_1(n, c_0) > 0$ depending only on n and c_0. (In fact, C_1 depends on ϵ but ϵ is determined by c_0.) For $y \in B(x, 10)$, we have

$$\frac{\mathrm{vol}(B(x,10))}{\mathrm{vol}(B(y,\epsilon/100))} \leq \frac{\mathrm{vol}(B(y,20))}{\mathrm{vol}(B(y,\epsilon/100))} \leq C_1(n, c_0),$$

so the maximal number of ϵ-balls disjointly contained in a 10-ball is bounded by C_1. For convenience, if we assume that $\mathrm{diam}(X) \le 20$,

$$X = \bigcup_{i=1}^{N} B(y_i, \epsilon/10)$$

for $N \le C_1$. Then we obtain that

$$\begin{aligned}[K_i : H_i] &= [K_i : i_*(\pi_1(\phi_i(B(0,\delta))))][i_*(\pi_1(\phi_i(B(0,\delta)))) : H_i] \\ &\le \kappa(n) C_1(n, c_0).\end{aligned} \tag{2.1}$$

We choose $\{\alpha_1^{(i)}, \cdots, \alpha_{k_1}^{(i)}\} \subset \Pi_i$ such that $\alpha_j^{(i)} \to \alpha_j \in \Pi$ as $i \to \infty$ and $\{p_1(\alpha_j)\}$ is a basis of $\pi_1(T^{k_1})$, where T^{k_1} is mentioned above. Let $A(\alpha)$ be an automorphism on K_i defined by

$$A(\alpha)(\gamma) = \alpha\gamma\alpha^{-1}.$$

From (2.1), we can find $l \le (\kappa(n)C_1(n, c_0))^{k_1}$ such that

$$A(\alpha_j^{(i)})^l (H_i) = (\alpha_j^{(i)})^l H_i (\alpha_j^{(i)})^{-l} = H_i. \tag{2.2}$$

We denote $(\alpha_j^{(i)})^l$ by $\beta_j^{(i)}$. Let Γ_i be the subgroup generated by K_i and $\{\beta_j^{(i)} | j = 1, \cdots, k_1\}$. From (2.2), H_i is a normal subgroup of Γ_i and

$$\begin{aligned}[\Pi_i : \Gamma_i] &= [K_i : H_i][\langle \alpha_1^{(i)}, \cdots, \alpha_{k_1}^{(i)} \rangle : \langle \beta_1^{(i)}, \cdots, \beta_{k_1}^{(i)} \rangle] \\ &\le l^{k_1+1} \le (\kappa(n) C_1(n, c_0))^{2n^2}.\end{aligned}$$

We have

$$1 \to K_i \to \Gamma_i \to \mathbb{Z}^{k_1} \to 1$$

and

$$1 \to K_i/H_i \to \Gamma_i/H_i \to \mathbb{Z}^{k_1} \to 1.$$

Applying Lemma 4.4 in [FY], we obtain that

$$1 \to \mathbb{Z}^{k_1} \to \Gamma_i/H_i \to \Omega \to 1$$

for some finite group Ω such that $|\Omega| \le C_2(n)$ for some constant C_2 depending only on n. Hence we can find a solvable subgroup Γ_i' such that

$$\Gamma_i'/H_i \simeq \mathbb{Z}^{k_1}$$

and

$$[\Pi_i : \Gamma_i'] = [\Pi_i : \Gamma_i][\Gamma_i : \Gamma_i'] \le C_3(n) := (\kappa(n) C_1(n, c_0))^{2n^2} C_2(n) < \infty.$$

From this, the boundedness of indices of solvable subgroups of $\pi_1(M)$ is obtained.

3. Proof of Theorem 1.1

We will find nilpotent subgroups $\Lambda_i \subset \Gamma_i'$ such that $[\Pi_i : \Lambda_i] < \infty$. Since $\tilde{M}_i \to \mathbb{R}^k \times X$, we have a pointed Hausdorff approximation

$$f_i : B(o_i, 1/\epsilon_i) \subset \tilde{M}_i \to \mathbb{R}^k \times X,$$

as $\epsilon_i \to 0$. For the precise definition of Hausdorff approximation, see [FY]. Assume that $f_i(o_i)$ converges to $o = (o_1, o_2) \in \mathbb{R}^k \times X$. Let N be $\{(x, o_2) | x \in \mathbb{R}^k\} \subset \mathbb{R}^k \times X$ and N_i be $f_i^{-1}(N)$.

As in the previous section, we have a quotient map $q_i : \Gamma_i' \to \mathbb{Z}^{k_1}$. We take a subset $B_i = \{\gamma_1^{(i)}, \cdots, \gamma_{k_1}^{(i)}\}$ of Γ_i' such that $q_i(B_i)$ is a basis of $\mathbb{Z}^{k_1}$. Since $\gamma_j^{(i)}$

converges to an isometry $\gamma_j = (p_1(\gamma_j), p_2(\gamma_j))$ of the $\mathbb{R}^k \times X$, there exist $l_j \leq C_1$ such that

$$d(\gamma_j^{l_j}(o), N) \leq \epsilon,$$

so

$$d((\gamma_j^{(i)})^{l_j}(o_i), N_i) \leq 2\epsilon$$

for C_1 and ϵ in section 2. We denote $\gamma_j^{l_j}$ and $(\gamma_j^{(i)})^{l_j}$ by γ'_j and ${\gamma'}_j^{(i)}$, respectively. Let x_j be $p_1(\gamma'_j(o))$. We take a point $x_j^{(i)}$ in $f_i^{-1}(x_j)$. (It is possible that $f_i^{-1}(x_j) = \emptyset$. But we can find a sequence $x_{j,h} \to x_j$ as $h \to \infty$ in $\mathbb{R}^k \times X$ such that $x_{j,h} \in \mathrm{Im}(f_i)$. So we may assume its existence.)

In Definition 1.2, a weak coordinate for an r-ball is used. Now we use a weak coordinate-like map for the tubular neighborhood of geodesic segment from o_i to $x_j^{(i)}$. We consider a weak coordinate-like map by the same method as in Definition 1.2: We denote $d(\gamma_j(o), o)$ by s_j. Let $c_j^{(i)}(t)$ be the shortest geodesic segment from o_i to $x_j^{(i)}$. We may assume that $c_j^{(i)}(t)$ is a unit speed curve. On $T_\delta = (-10\delta, s_j + 10\delta) \times B(0, 10\delta)$, we define ψ_i as follows:

$$\begin{aligned} \psi_i : T_\delta &\to \tilde{M}_i, \\ (t, y) &\mapsto \exp_{c_j^{(i)}(t)} y, \end{aligned}$$

where $B(0, R)$ is the R-ball centered at 0 in $\mathbb{R}^{n-1}$ and δ is the positive constant in section 2. Then ψ_i is a nonsingular map. We consider a metric $\psi_i^* g_i$ on T_δ, where g_i is the metric of M_i. From [AC], $\{(T_\delta, \psi_i^* g_i)\}$ is $C^{0,\alpha}$-compact. So T_δ has a $C^{0,\alpha}$-bounded norm. Then there exists a weak coordinate-like map

$$\phi_i : T_\delta \to \tilde{M}_i$$

satisfying the conditions in Definition 1.2. Using the compactness result of [AC], we know that $(T_\delta, \phi_i^* g_i)$ converges to a $C^{0,\alpha}$-Riemannian manifold. By the fibration theorem in [F] and [W], there exists an almost Riemannian submersion

$$\Psi_i : T_\delta \to I,$$

where $I = (-10\delta, s_j + 10\delta)$. Then we have the following convergence:

$$(T_\delta, \phi_i^* g_i) \to (T, g_0) \subset (I \times Y, g_0),$$

where $Y \subset \mathbb{R}^{n-1}$ has a $C^{0,\alpha}$-Riemannian metric and g_0 is an almost product metric of the metric on $I \subset \mathbb{R}$ and the metric on Y. There exists a diffeomorphism

$$\Phi_i : T \to T_\delta$$

such that

$$||\Phi_i^*(\phi_i^* g_i) - g_0|| \to 0$$

in $C^{0,\alpha}$-topology. From [CH], we know that isometries converge in $C^{1,\alpha}$-topology. Isometries from a small ball $B(o_i, \delta) \subset \tilde{M}_i$ to $\mathrm{Im}(\phi_i) \subset \tilde{M}_i$ can be lifted to isometries of a δ-ball in T_δ. We denote the lifting of $\alpha_i \in \mathrm{Isom}(M_i)$ by $\tilde{\alpha}_i \in \mathrm{Isom}(T_\delta)$. Then any elements $\gamma^{(i)} \in \pi_1(\phi_i(B(0, \delta)))$ can be lifted to an isometry $\tilde{\gamma}^{(i)}$ on a δ-ball in T_δ, where $\phi_i(0) = o_i$.

If $\gamma^{(i)} \in \tilde{H}_i$, then an almost isometric map $\Phi_i^{-1}\gamma^{(i)}\Phi_i : T \subset \mathbb{R}^n \to T \subset \mathbb{R}^n$ can be written in the following form: We denote by $\tau(x)$ a map f such that $f \in C^1$ and $f(0) = Df(0) = 0$. Then we have

$$G(x) = \Phi_i^{-1}\gamma^{(i)}\Phi_i(x) = b + Bx + \tau(d(x, \Psi_i(x))),$$

where $b = G(0)$ and $||B - \mathrm{I}|| \leq \epsilon_1 ||b||$ for some small $\epsilon_1 > 0$ since it is an isometry on $B(o_i, \delta)$ which is almost flat by [Pa] and [PWY].

We define F_j as follows:

$$F_j = \Phi_i^{-1}{\gamma'}_j^{(i)}\Phi_i$$

and write F_j as follows:

$$F_j(x) = a_j + A_j x + \tau(x),$$

where $a_j = F_j(0)$. Similarly as section 2, we denote $F_j G F_j^{-1}$ by $A(F_j)(G)$. Then we also obtain that for $||x|| \leq 10b$,

$$A(F_j)(G)(0) = (\mathrm{I} - A_j B A_j^{-1})a_j + A_j b + \tau(b).$$

Since A_j is an almost isometry and $||B - I|| \leq \epsilon_1 ||b||$, we have $d(A(F_j)(G)(o), A_j b) \leq 10\epsilon_1 ||a_j|| ||b||$. The cardinality of $\{\gamma \in H_i | b - \delta \leq d(\gamma(o), o) \leq b + \delta\}$ is bounded by 2^{k_1} by the choice of a nilpotent group H_i. Considering the orbit $\{A(F_j^l)(G)(0) \mid l = 1, 2, \cdots\}$, we have $||A_j^s - \mathrm{I}|| \leq \epsilon_1 ||a_j||$ for $s \leq 2^{k_1} \leq 2^n$. So we obtain that

$$||A(F_j^s)(G)(0) - G(0)|| \leq 10\epsilon_1 s ||a|| ||b||$$

and

$$d((\gamma_j')^s \gamma (\gamma_j')^{-s}(o), \gamma(o)) \leq 10\epsilon_1 ||\gamma^s|| ||\gamma||,$$

where $||\alpha|| = d(\alpha(o), o)$ for $\alpha \in \Pi$. From this, $\Lambda_i = \langle \{H_i, ({\gamma'}_j^{(i)})^s \mid j = 1, 2, \cdots, k_1\}\rangle$ is a nilpotent group since

$$||[\alpha, \beta]|| = d([\alpha, \beta](o_i), o_i) \leq 10\epsilon_1 ||\alpha|| ||\beta||$$

for $\alpha, \beta \in \Lambda_i$ [Pa]. Then $[\Pi_i : \Lambda_i] \leq (\kappa(n) C_1(n, c_0))^{2n^2} C_2(n) C_1(n)^{2n^2} < \infty$. Until now, we proved that for any sequence of manifolds $\{M_i\}$ which satisfies the condition of Theorem 1.1, there exists a subsequence $\{M_{i_j}\}$ such that $\pi_1(M_{i_j})$'s have nilpotent subgroups whose indices are bounded.

If we assume that Theorem 1.1 is false, then we can find a sequence of manifolds $\{M_i\}$ such that $\lim_{i\to\infty}[\pi_1(M_i) : \Lambda_i] = \infty$ for any nilpotent subgroups Λ_i, which is a contradiction to the fact we have proved. This completes the proof.

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