PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 131, Number 8, Pages 2617–2622
S 0002-9939(03)06933-8
Article electronically published on March 11, 2003

TWO COUNTABLY COMPACT TOPOLOGICAL GROUPS: ONE OF SIZE $\aleph_\omega$ AND THE OTHER OF WEIGHT $\aleph_\omega$ WITHOUT NON-TRIVIAL CONVERGENT SEQUENCES

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(Communicated by Alan Dow)

Abstract. E. K. van Douwen asked in 1980 whether the cardinality of a countably compact group must have uncountable cofinality in ZFC. He had shown that this was true under GCH. We answer his question in the negative. V. I. Malykhin and L. B. Shapiro showed in 1985 that under GCH the weight of a pseudocompact group without non-trivial convergent sequences cannot have countable cofinality and showed that there is a forcing model in which there exists a pseudocompact group without non-trivial convergent sequences whose weight is $\omega_1 < \mathfrak{c}$. We show that it is consistent that there exists a countably compact group without non-trivial convergent sequences whose weight is $\aleph_\omega$.

1. Introduction

It is well-known that there is a relation between the size of an infinite compact group G and its weight, $w(G)$, given by the equality $|G| = |G|^{w(G)}$. In particular, the cofinality of $|G|$ is uncountable. Since $\{0,1\}^\lambda$ is a compact group for each λ, there is a group of size κ, for κ infinite if and only if $\kappa = 2^\mu$ for some cardinal μ.

Given an infinite cardinal κ with $\kappa = \kappa^\omega$, it requires only standard closing off arguments to construct a subgroup of $\{0,1\}^\kappa$ of size κ which is countably compact. It is natural to ask if those are the only possible cardinalities for infinite countably compact groups. Its consistency was obtained by E. K. van Douwen [3] under GCH. Recall that, if κ has countable cofinality, then $\kappa^\omega > \kappa$, but under GCH both are equivalent.

This motivated van Douwen [3] to ask in ZFC:

If X is an infinite group (or homogeneous space) which is countably compact, is $|X|^\omega = |X|$? Is at least $\mathrm{cf}(|X|) \neq \omega$*?*

We answer his question in the negative:

Theorem 1.1. *The existence of countably compact groups whose size has countable cofinality is independent of* ZFC.

Received by the editors October 15, 2001 and, in revised form, January 15, 2002.
2000 *Mathematics Subject Classification.* Primary 54H11, 54A25, 54A35; Secondary 22A05.
Key words and phrases. Forcing, countably compact group, cofinality, weight.
This research was partially conducted while the author was visiting the Department of Mathematics of Universidade de Coimbra. This visit was supported by CCINT-USP and the local organizing committee of the Fourth Ibero American Congress of Topology and its Applications.

Note that $\aleph_n$ for $n \in \omega$ cannot be the size of a counterexample to van Douwen's question. Indeed, if there exists a countably compact group of size $\aleph_n$, then $\aleph_n \geq 2^\omega$ and therefore $(\aleph_n)^\omega = \aleph_n$.

We will show that $\aleph_\omega$ can be the size of a counterexample in a forcing model.

Sirota [8], answering a question of Arhangelskii, constructed a pseudocompact group of weight τ without non-trivial convergent sequences of weight τ, for any infinite τ satisfying $\tau = \tau^\omega$. Malykhin and Shapiro [7] showed that under GCH, those are the only possible weights for pseudocompact groups without non-trivial convergent sequences, and using forcing they showed that there exists a pseudocompact group of weight $\aleph_1 < (\aleph_1)^\omega$.

However, ω_1 has uncountable cofinality, thus it is natural to ask whether a pseudocompact group without non-trivial convergent sequences can have weight of countable cofinality. Also, it is natural to ask whether pseudocompactness can be replaced by countable compactness.

Theorem 1.2. *The existence of countably compact groups without non-trivial convergent sequences whose weight has countable cofinality is independent of* ZFC.

We show that it is consistent that there exists a countably compact group of size $\mathfrak{c}$ without non-trivial convergent sequences whose weight is $\aleph_\omega$.

2. A sketch of the counterexample to van Douwen's question

Motivated by [5], a countably compact group of size $\mathfrak{c}$ (with a non-productivity property) was constructed in [10] and [11], as the direct sum of a countable subgroup of $2^\mathfrak{c}$ and $\bigcup_{\alpha<\mathfrak{c}} 2^\alpha \times \{0\}^{\mathfrak{c}\setminus\alpha}$. We will obtain our example by replacing the countable group by a larger one.

Lemma 2.1. *Let H be a subgroup of $2^\mathfrak{c}$ such that for each injective sequence $\{h_n : n \in \omega\} \subseteq H$ there exists $\alpha < \mathfrak{c}$ such that $\{h_n|_{[\alpha,\mathfrak{c})} : n \in \omega\}$ has $0 \in 2^{[\alpha,\mathfrak{c})}$ as an accumulation point. Then the group G generated by H and $\{x \in 2^\mathfrak{c} : (\exists \alpha < \mathfrak{c})(\{\beta \in \mathfrak{c} : x(\beta) \neq 0\} \subseteq \alpha)\}$ is a countably compact group of size $|H| + 2^{<\mathfrak{c}}$.*

Proof. Let $\{y_n : n \in \omega\}$ be any sequence in the group generated. Then there exists $\{h_n : n \in \omega\} \subseteq H$ and $\alpha < \mathfrak{c}$ such that $y_n - h_n|_{[\alpha,\mathfrak{c})} = 0 \in 2^{[\alpha,\mathfrak{c})}$. Without loss of generality, we can assume that $\{h_n : n \in \omega\}$ is either constant h or injective.

In the first case, the sequence $(y_n - h)|_\alpha$ has an accumulation point $x \in 2^\alpha$. Then $x \cup h|_{[\alpha,\mathfrak{c})}$ is an accumulation point of $\{y_n : n \in \omega\}$. In the second case, by hypothesis, there exists $\beta > \alpha$ such that $\{h_n|_{[\beta,\mathfrak{c})} : n \in \omega\}$ has $0 \in 2^{[\beta,\mathfrak{c})}$ as an accumulation point. Since 2^β is compact, there exists $y \in 2^\beta$ such that $(y, 0)$ is an accumulation point of the sequence $\{(y_n|_\beta, h_n|_{[\beta,\mathfrak{c})}) : n \in \omega\} = \{(y_n|_\beta, y_n|_{[\beta,\mathfrak{c})}) : n \in \omega\}$. Therefore, $y \cup 0|_{[\beta,\mathfrak{c})}$ is an accumulation point of $\{y_n : n \in \omega\}$.

Clearly the size of the second group is $2^{<\mathfrak{c}}$, therefore the size of the group generated is $|H| + 2^{<\mathfrak{c}}$. □

We observe that the construction of a group H of size $\kappa \leq \mathfrak{c}$ can be done using Martin's Axiom for partial orders of size κ, but for one of size $\aleph_\omega > \mathfrak{c}$, it is not enough. Indeed, GCH implies MA and van Douwen showed that this implies the non-existence of $\aleph_\omega$-sized countably compact groups. The forcing model in which a group H of size $\aleph_\omega$ exists will be described in the following section:

Theorem 2.2. *Given a model of CH and a regular cardinal $\kappa > \mathfrak{c}$, there exists a countably closed, ω_2-cc forcing such that, in the extension, for every cardinal $\lambda \in [\mathfrak{c}, \kappa]$ there exists a countably compact group of size λ.*

In particular, we obtain the example in the title.

Example 2.3. It is consistent that $\mathfrak{c} = \aleph_1 < \aleph_\omega$ and there exists a group of size $\aleph_\omega$ that is countably compact.

Start with a model of GCH and apply Theorem 2.2 for the cardinal $(\aleph_\omega)^+$ in the ground model. Let $\lambda = \aleph_\omega$. Since CH holds and the forcing is countably closed, CH holds in the extension. Furthermore, the forcing is cardinal preserving, therefore, $\aleph_\omega$ in the extension is λ. Thus, there is a countably compact group of size $\aleph_\omega$ in the extension.

3. The forcing

Assume CH and let κ be a regular cardinal greater than ω_1. Let $\{f_\xi : \xi \in \kappa\}$ be an enumeration of all $1-1$ sequences in $[\kappa]^{<\omega}$.

Throughout this construction, we shall denote $z_F = \sum_{\mu \in F} z_\mu$ whenever F is a finite set of indexes and $\{z_\mu : \mu \in F\}$ is a family of functions from a fixed ordinal into 2.

Definition 3.1. We say that $p = (\alpha_p, \{x^p_\eta : \eta \in E_p\}, \{A^p_\delta : \delta \in D_p\})$ is an element of $\mathbb{P}$ if $\alpha_p \in \omega_1$, $D_p, E_p \in [\kappa]^\omega$, $E_p = D_p \cup \bigcup_{n\in\omega \wedge \delta \in D_p} f_\delta(n)$, $x^p_\eta \in 2^{\alpha_p}$ for each $\eta \in E_p$ and $A^p_\delta \in [\omega]^\omega$ for each $\delta \in D_p$. Given $p, q \in \mathbb{P}$, we say that $p \leq q$ if $\alpha_p \geq \alpha_q$, $D_p \supseteq D_q$, $E_p \supseteq E_q$, $\forall \eta \in E_q (x^p_\eta|_{\alpha_q} = x^q_\eta)$, $\forall \delta \in D_q (A^p_\delta \subseteq^* A^q_\delta)$ and the sequence $\{x^p_{f_\delta(n)}|_{[\alpha_q, \alpha_p)} : n \in A^p_\delta\}$ converges to $0|_{[\alpha_q,\alpha_p)} \in 2^{[\alpha_q,\alpha_p)}$ for every $\delta \in D_p$.

Lemma 3.2. *The set $\mathbb{P}$ endowed with the partial ordering above is countably closed and ω_2-cc.*

Proof. Given a decreasing sequence $\{p_n : n \in \omega\}$ with $p_n = (\alpha_n, \{x^n_\eta : \eta \in E_n\}, \{A^n_\delta : \delta \in D_n\})$, let $p_\omega = (\alpha_\omega, \{x^\omega_\eta : \eta \in E_\omega\}, \{A^\omega_\delta : \delta \in D_\omega\})$, where $\alpha_\omega = \bigcup_{n\in\omega} \alpha_n$, $D_\omega = \bigcup_{n\in\omega} D_n$, $E_\omega = \bigcup_{n\in\omega} E_n$, $x^\omega_\eta = \bigcup_{n\in\omega\wedge\eta\in E_n} x^n_\eta$, and A^ω_δ is an infinite subset of ω such that for each $\delta \in D_\omega$, $A^\omega_\delta \subseteq^* A^n_\delta$ for each $n \in \omega$ such that $\delta \in D_n$. Clearly $p_\omega \in \mathbb{P}$ and $p_\omega \leq p_n$ for each $n \in \omega$. Hence, $\mathbb{P}$ is countably closed.

Let $\{p_\mu : \mu < \omega_2\}$ be a subset of $\mathbb{P}$. Using the Δ-system lemma, there exists $E \in [\kappa]^{\leq\omega}$ and $I \in [\omega_2]^{\omega_2}$ such that $E_\mu \cap E_\beta = E$ for any pair $\{\mu, \beta\} \in [I]^2$. We can also assume that there exists $\alpha \in \omega_1$ such that $\alpha_\mu = \alpha$ for every $\mu \in I$. Furthermore, there are only $\mathfrak{c} = \omega_1$ functions from E to 2^α and there are only $\mathfrak{c} = \omega_1$ functions from E to $[\omega]^\omega$, thus, we can assume that for every pair $\{\mu, \beta\} \in [I]^2$, $x^\beta_\eta = x^\mu_\eta$ for every $\eta \in E$ and $A^\beta_\delta = A^\mu_\delta$ for every $\delta \in D_\mu \cap D_\beta \subseteq E$.

Fix $\mu, \beta \in I$ distinct and let $p = (\alpha, \{x^\mu_\eta : \eta \in E_\mu\} \cup \{x^\beta_\eta : \eta \in E_\beta \setminus E\}, \{A^\mu_\delta : \delta \in D_\mu\} \cup \{A^\beta_\delta : \delta \in D_\beta \setminus D_\mu\})$. Then $p \leq p_\beta$ and $p \leq p_\mu$. Therefore, $\mathbb{P}$ has the ω_2-cc property. □

We define now some dense subsets of $\mathbb{P}$.

Definition 3.3. Let $\phi : F \longrightarrow 2$, where $F \in [\kappa]^{<\omega}$ and $\alpha \in \omega_1$. Define $\mathcal{D}_{\phi,\alpha} = \{p \in \mathbb{P} : \operatorname{dom}\phi \subseteq D_p \wedge (\exists\gamma \in [\alpha, \alpha_p)\forall\xi \in \operatorname{dom}\phi)(\phi(\xi) = x^p_\xi(\gamma))\}$.

Lemma 3.4. *The sets defined in Definition 3.3 are dense in $\mathbb{P}$.*

The proof of the density of the sets above will be given later in this section. We will first prove the main theorem.

Proof of Theorem 2.2. Assume CH, let $\kappa > \mathfrak{c}$ be a regular cardinal and let $\mathbb{P}$ be the partial order defined in Definition 3.1. From Lemma 3.2, this partial ordering is countably closed and ω_2-cc. Let $\mathcal{G}$ be a generic filter for the partial order $\mathbb{P}$ which interesects each dense set in Definition 3.3. For each $\xi \in \kappa$, let $x_\xi = \bigcup_{p \in \mathcal{G} \wedge \xi \in D_p} x^p_\xi$.

Let ϕ be the function of domain $\{\xi\}$ such that $\phi(\xi) = 0$. For every $\alpha < \omega_1$ there exists $p \in \mathcal{G} \cap \mathcal{D}_{\phi,\alpha}$ such that $\alpha_p > \alpha$ and $\xi \in D_p$. Therefore, $x_\xi \in 2^{\omega_1}$.

We claim that $\{x_\xi : \xi < \kappa\}$ is linearly independent. Indeed, if F is a non-empty finite subset of κ, there exist $\phi : F \longrightarrow 2$ such that $\sum_{\mu \in F} \phi(\mu) \neq 0$ and $p \in \mathcal{G} \cap \mathcal{D}_{\phi,\omega}$. Then, there exists $\gamma < \alpha_p$ such that $x_F(\gamma) = x^p_F(\gamma) = \sum_{\mu\in F} \phi(\mu) \neq 0 \in 2$. Thus, $x_F \neq 0 \in 2^{\omega_1}$.

Let I be a subset of κ. We claim that the group H_I generated by $\{x_\xi : \xi \in I\}$ satisfies the conditions of Lemma 2.1. Indeed, let $\{y_n : n \in \omega\}$ be an injective sequence in H_I. There exists a function $f : \omega \longrightarrow [I]^{<\omega}$ such that $y_n = x_{f(n)}$ for each $n \in \omega$. Since the forcing does not add new countable subsets of the ground model, there exists $\zeta < \kappa$ such that $f = f_\zeta$. Let $p \in \mathcal{G}$ such that $\zeta \in D_p$. If F is a finite subset of $[\alpha_p, \omega_1)$, then there exists $q \in \mathcal{G}$ such that $\alpha_q > \max F$ and $q \leq p$. Thus, $\{n \in \omega : y_n|_F = 0|_F\} \supseteq^* A^q_\zeta$. Therefore, $\{y_n|_{[\alpha_p,\omega_1)} : n \in \omega\}$ has $0 \in 2^{[\alpha_p,\omega_1)}$ as an accumulation point and H_I satisfies the conditions of Lemma 2.1.

As H_I is a group with a basis of size I, we conclude that H_I and I have the same size if I is infinite. Thus applying Lemma 2.1, we obtain countably compact groups of any size between $\mathfrak{c}$ and κ. □

We will be done by proving the density of the sets in Definition 3.3.

Proof of Lemma 3.4. Let q be an arbitrary element of $\mathbb{P}$ and fix $\phi : F \longrightarrow 2$, with $F \in [\kappa]^{<\omega}$ and $\alpha < \omega_1$. Define $\alpha_r = \max\{\alpha, \alpha_q\}$, $D_r = D_q \cup \operatorname{dom}\phi$, $E_r = D_r \cup \bigcup_{n\in\omega \wedge \delta \in D_r} f_\delta(n)$.

For each $\eta \in E_r \setminus E_q$ define $x^r_\eta = 0 \in 2^{\alpha_r}$; for each $\eta \in E_q$ define $x^r_\eta = x^q_\eta \cup 0|_{[\alpha_q,\alpha_r)}$.

For each $\delta \in D_r \setminus D_q$ define $A^r_\delta = \omega$ and for each $\delta \in D_q$ define $A^q_\delta = A^r_\delta$. Clearly $r = (\alpha_r, \{x^r_\eta : \eta \in E_r\}, \{A^r_\delta : \delta \in D_r\}) \leq q$. We shall extend r to some $p \in \mathcal{D}_{\phi,\alpha}$ with $\alpha_p = \alpha_r + 1$, $D_p = D_r$, $E_p = E_r$. We will define a function $\psi : E_p \longrightarrow 2$ which will be used to define each $x^p_\eta(\alpha_r)$. Enumerate D_p as $\{\beta_n : n \in \omega\}$ such that each element of D_p appears infinitely often in the enumeration. Set $F_0 = \operatorname{dom}\phi$ and let $\psi|_{F_0} = \phi$. We shall define $\psi|_{F_n}$ and k_n by induction such that $F_n \cup f_{\beta_n}(k_n) \subseteq F_{n+1}$, $k_n \in A^r_{\beta_n}$, $k_n < k_{n+1}$, and $\sum_{\mu \in f_{\beta_n}(k_n)} \psi(\mu) = 0$.

Suppose that F_n, $\psi|_{F_n}$ and k_{n-1} are defined for each $n \leq m$ satisfying the inductive hypothesis. Let $k_m > k_{m-1}$ such that $k_m \in A^r_{\beta_m}$ and $f_{\beta_m}(k_m) \setminus F_m \neq \emptyset$. Set $F_{m+1} = F_m \cup f_{\beta_m}(k_m)$ and define ψ on $F_{m+1} \setminus F_m$ so that $\sum_{\mu \in f_{\beta_m}(k_m)} \psi(\mu) = 0$. At stage ω, ψ is defined on $F = \bigcup_{n\in\omega} F_n$. Define $\psi(\mu) = 0$ for each $\mu \in E_p \setminus F$. Set $x^p_\eta = x^r_\eta \cup \{\langle \alpha_r, \psi(\eta)\rangle\}$ for each $\eta \in E_p$ and $A^p_\delta = \{k_n : \beta_n = \delta\} \subseteq A^r_\delta$ for each $\delta \in D_p$. It is easy to see that $p \leq r$ and $p \in \mathcal{D}_{\phi,\alpha}$. Thus, $\mathcal{D}_{\phi,\alpha}$ is dense in $\mathbb{P}$. □

4. Countably compact groups without non-trivial convergent sequences whose weight is $\aleph_\omega$

The next theorem shows how to increase the weight of countably compact groups without non-trivial convergent sequences of size $\mathfrak{c}$.

Theorem 4.1. *If there exists an infinite countably compact group topology without non-trivial convergent sequences on a free Abelian group G, then there exists a countably compact group topology without non-trivial convergent sequences of size $\mathfrak{c}$ whose weight is κ, for any $\kappa \in [\mathfrak{c}, 2^{\mathfrak{c}}]$.*

As a corollary to the proof of Theorem 4.1, we have the following:

Theorem 4.2. *If there exists an infinite countably compact group topology without non-trivial convergent sequences on an Abelian group G, then there exists a countably compact group topology without non-trivial convergent sequences of size $\mathfrak{c}$ and weight κ, for any $\kappa \in [\mathfrak{c}, 2^{\mathfrak{c}}]$.*

Allowing convergent sequences, van Douwen [3] and Comfort and Remus [1] have obtained countably compact groups whose weight has countable cofinality in ZFC. It is still open if there are countably compact groups without non-trivial convergent sequences in ZFC (this has been asked by van Douwen in [2]).

Example 4.3. If $\mathrm{MA}_{\mathrm{countable}}$ holds and $\mathfrak{c} \leq \aleph_\omega \leq 2^{\mathfrak{c}}$, then there exists a countably compact group topology without non-trivial convergent sequences of weight $\aleph_\omega$ on the free Abelian group of size $\mathfrak{c}$.

From [6], there exists, under $\mathrm{MA}_{\mathrm{countable}}$, a countably compact group topology without non-trivial convergent sequences of weight $\mathfrak{c}$ on the free Abelian group of size $\mathfrak{c}$. Applying Theorem 4.1, there exists one of size $\aleph_\omega$, as $\mathfrak{c} < \aleph_\omega < 2^{\mathfrak{c}}$.

Proof of Theorem 4.1. By standard closing off arguments and from the fact that subgroups of free Abelian groups are free Abelian, we can assume that G has cardinality $\mathfrak{c}$. Since G is Abelian and countably compact, we can assume that it is a subgroup of $\mathbb{T}^\lambda$ for some cardinal λ. There are only $\mathfrak{c}$ many sequences and finite sums of elements of G; thus, we can project G injectively into $\mathbb{T}^\theta$, with $\theta \leq \mathfrak{c}$ so that its projection is free Abelian and does not contain non-trivial convergent sequences. Therefore, we can assume that the weight of G is at most $\mathfrak{c}$.

Let $\{y_\xi : \xi < \mathfrak{c}\}$ be an independent set of generators for G.

Let I be the set of all even ordinals in $\mathfrak{c}$. Fix an enumeration $\{f_\xi : \xi \in I\}$ of all injective functions f such that $\operatorname{dom} f = \omega$, $\operatorname{dom} f(n) \in [\mathfrak{c}]^\omega \setminus \{\emptyset\}$ and $\operatorname{rng} f(n) \subseteq \mathbb{Z} \setminus \{0\}$ for every $n \in \omega$. Furthermore, let $\bigcup_{n\in\omega} \operatorname{dom} f_\xi(n) \subseteq \xi$ for every $\xi \in I$.

In this construction, we denote $x_{f(n)} = \sum_{\mu \in \operatorname{dom} f(n)} f(n)(\mu) x_\mu$.

By induction, suppose that for $\mu < \xi < \mathfrak{c}$, we have chosen $x_\mu \in G$ satisfying the following:

(i) $\{x_\beta : \beta < \mu\}$ is independent for every $\mu < \xi$ and

(ii) if $\mu \in I$, then x_μ is an accumulation point of the sequence $\{x_{f_\mu(n)} : n \in \omega\}$.

We will show that x_ξ can be chosen so that the inductive hypothesis is satisfied. Indeed, let H be the subgroup generated by $\{x_\beta : \beta < \xi\}$. For each $h \in H$ let J_h be a finite subset of $\mathfrak{c}$ such that $h \in \langle\{y_\beta : \beta \in J_h\}\rangle$. Let $J = \bigcup_{h\in H} J_h$ and choose $x_\xi \in G \setminus \langle\{y_\beta : \beta \in J\}\rangle$ if ξ is odd or $x_\xi \in \overline{\{x_{f_\mu(n)} : n \in \omega\}} \setminus \langle\{y_\beta : \beta \in J\}\rangle$ if ξ is even. Note that the last difference is non-empty since the first set has size $\mathfrak{c}$ and the second set has size less than $\mathfrak{c}$. Clearly x_ξ satisfies the inductive hypothesis.

Let p_ξ be an ultrafilter on ω such that x_ξ is the p_ξ-limit of the sequence $\{x_{f_\xi(n)} : n \in \omega\}$.

Fix $\kappa \in [\theta, 2^{\mathfrak{c}}]$. Let $\{z_\xi : \xi \in \mathfrak{c} \setminus I\}$ be a dense subset of $\mathbb{T}^\kappa$. By induction on $\mu \in I$, let z_μ be the p_μ-limit of $\{z_{f_\mu(n)} : n \in \omega\}$. The group generated by $\{(x_\xi, z_\xi) : \xi < \mathfrak{c}\} \subseteq G \times \mathbb{T}^\kappa$ is as required. □

Using an argument similar to the one in Theorem 4.1 on the second example in [6], we can obtain the following example:

Example 4.4. It is consistent with CH that there exists a free Abelian group of size $2^{\mathfrak{c}}$ and weight κ for each κ in $[\mathfrak{c}, 2^{2^{\mathfrak{c}}}]$ and $2^{\mathfrak{c}}$ can be arbitrarily large. In particular, it is consistent that there exists a countably compact free Abelian group of weight $\lambda > 2^{\mathfrak{c}}$, with λ of countable cofinality.

Note. The author and I. Castro Pereira modified [6] to obtain a group in Theorem 2.2 that is free Abelian and without non-trivial convergent sequences.

Acknowledgment

It is a pleasure to thank P. Koszmider for some comments concerning this paper and Professor Comfort for information concerning the references. I also thank the Topology Seminar at Morelia to whom I presented a draft of the paper.

The author is grateful to the referee for a careful reading of the paper. All his suggestions make the notation more consistent and the reading of the same more pleasant. Future papers of the author should benefit from the referee's TEX advice.

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