GRAPHS THAT ARE NOT COMPLETE PLURIPOLAR

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Abstract. Let $D_1 \subset D_2$ be domains in $\mathbb{C}$. Under very mild conditions on $D_2$ we show that there exist holomorphic functions $f$, defined on $D_1$ with the property that $f$ is nowhere extendible across $\partial D_1$, while the graph of $f$ over $D_1$ is not complete pluripolar in $D_2 \times \mathbb{C}$. This refutes a conjecture of Levenberg, Martin and Poletsky (1992).

1. Introduction

Levenberg, Martin and Poletsky [6] have conjectured that if $f$ is a holomorphic function, which is defined on its maximal domain of existence $D \subset \mathbb{C}$, then the graph

$$\Gamma_f = \{(z, f(z)) : z \in D\}$$

of $f$ over $D$ is a complete pluripolar subset of $\mathbb{C}^2$. I.e., there exists a plurisubharmonic function on $\mathbb{C}^2$ such that it equals $-\infty$ precisely on $\Gamma_f$ (see e.g. [4]). They gave support for this conjecture in the sense that they could prove it for some lacunary series. More support was provided by Levenberg and Poletsky [7] and by the second author [9, 10, 11]. Nevertheless, in this paper we show that the conjecture is false.

In fact we have

Theorem 1.1. Let $D_1 \subset D_2$ be domains in $\mathbb{C}$. Assume that $D_2 \setminus D_1$ has a density point in $D_2$. Then there exists a holomorphic function $f$ with domain of existence $D_1$ such that the graph $\Gamma_f$ of $f$ over $D_1$ is not complete pluripolar in $D_2 \times \mathbb{C}$.

In case $D_2 \setminus D_1$ has no density point in $D_2$, it is known that $\Gamma_f$ is complete pluripolar in $D_2 \times \mathbb{C}$ (see [10]). If we take in Theorem 1.1 for $D_1$ the unit disc $\mathbb{D}$ and for $D_2$ the whole plane $\mathbb{C}$, we obtain the following corollary.

Corollary 1.2. There exists a holomorphic function $f$ defined on $\mathbb{D}$, which does not extend holomorphically across $\partial \mathbb{D}$, such that $\Gamma_f$ is not complete pluripolar in $\mathbb{C}^2$.

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Theorem then states that such a function can even be smooth up to the boundary of $D$.

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2. Graphs with non-trivial pluripolar hull

The pluripolar hull of a pluripolar set $K \subset \Omega$ is the set

$$K_\Omega^* = \{ z \in \Omega : u|_K = -\infty, u \in PSH(\Omega) \Rightarrow u(z) = -\infty \}.$$ 

By $\omega(z, E, D)$ we denote as usual the harmonic measure of a subset $E$ of the boundary of a domain $D$ in $\mathbb{C}$ at the point $z$ in $D$ (see e.g. [8]).

Let $A = \{ a_n \}_{n=1}^\infty$ be a countable dense subset of $\partial D$. Under the assumptions of Theorem 1.1, there exists an $a \in (\partial D) \cap D_2$ such that $a \notin A \{ a \}$. We may assume that $a \notin A$. Our function $f$ will be of the form

$$f(z) = \sum_{j=1}^{\infty} \frac{c_j}{z - a_j}.$$ 

We will choose $c_n$ very rapidly decreasing to 0. In particular,

$$\sum_{j=1}^{\infty} |c_j| < +\infty,$$

so the limit in (2.1) exists and is a holomorphic function on $D_1$.

Moreover, we will choose $c_n$ such that $\sum_{n=1}^{\infty} \frac{|c_j|}{|a - a_j|} < +\infty$. Hence, the series (2.1) will converge at $z = a$. We will denote its limit by $f(a)$. We will prove a version of Theorem 1.1 that elaborates on (2.1). However, no statement about extendibility is made at this point.

**Theorem 2.1.** Let $D_1 \subset D_2$ be domains in $\mathbb{C}$, such that $D_2 \setminus D_1$ has a density point in $D_2$. There exists a sequence $\{ \rho_n \}_{n=1}^{\infty}$ of positive numbers such that for any sequence of complex numbers $\{ c_n \}_{n=1}^{\infty}$ with $|c_n| \leq R_n$ we have $(a, f(a)) \in (\Gamma_f)^*_D \times \mathbb{C}$, where $f$ is given by (2.1). Here $\Gamma_f$ is the graph of $f$ over $D_1$.

**Proof.** We may assume that $a = 0$. For $b \in \mathbb{C}$ and $r > 0$ we set $D(b, r) = \{ z \in \mathbb{C} : |z - b| < r \}$, $D_r = D(0, r)$ and $D = D_1$, the unit disc. Put

$$\pi_n^k(z) = e^{\frac{2\pi i k}{n}}, \quad z \in \mathbb{C}, \quad k, n \in \mathbb{N},$$

and

$$B = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} \pi_n^k(A).$$

Note that $0 \notin B$ and that $B$ is also countable (and therefore thin at 0). By Corollary 4.8.3 in [4], there exists an open set $U \supset B$ such that $U$ is thin at 0.

**Step 1.** We construct a sequence of radii $\{ \rho_n \}_{n=1}^{\infty}$ with special properties, the main one being that $\bigcup_{n=1}^{n} \bigcup_{k=1}^{n} \pi_n^k(D(a_n, \rho_n))$ is thin at 0.

It is a corollary of Wiener’s criterion (see [8], Theorem 5.4.2) that there exists a sequence $r_n \to 0$ such that

$$\partial D_{r_n} \cap U = \emptyset, \quad n \in \mathbb{N}.$$
Since $U$ is thin at 0, there exists a subharmonic function $u$ on $\mathbb{C}$ such that

$$\limsup_{U \ni z \to 0} u(z) = -\infty < u(0)$$

(see e.g. Proposition 4.8.2 in [4]). Moreover, by scaling and adding a constant, we can assume that $u(0) = -\frac{1}{2}$ and $u < 0$ on $\mathbb{D}$. By \((2.3)\) there exists a $\rho > 0$ such that $\overline{D}_\rho \subset D_2$, $\partial D_\rho \cap D_1 = \emptyset$, $\partial D_\rho \cap \emptyset = \emptyset$, and $u \leq -1$ on $U \cap D_\rho$ (take $\rho = r_n$ with sufficiently big $n$).

Let $J \subset \partial D_\rho \cap D_1$ be a closed arc. We can assume that

$$J = \{ e^{i\theta} r : \frac{2\pi k_0}{n_0} \leq \theta \leq \frac{2\pi (k_0 + 1)}{n_0} \}$$

for some $k_0, n_0 \in \mathbb{N}$.

Now we choose a sequence of positive numbers $\rho_n \in (0,1), n \in \mathbb{N}$, in the following way:

1. Let $0 < \rho_1 < 1$ be such that
   \begin{enumerate}
   \item[(a)] $\bigcup_{k=1}^{n_0} \pi^k_{n_0} (\mathbb{D}(a_1, \rho_1)) \subset U$;
   \item[(b)] $\mathbb{D}_\rho \setminus \bigcup_{k=1}^{n_0} \pi^k_{n_0} (\mathbb{D}(a_1, \rho_1))$ is connected.
   \end{enumerate}
2. Assume that $\rho_1, \ldots, \rho_{n-1}$ are chosen. Choose $0 < \rho_n < 1$ such that
   \begin{enumerate}
   \item[(a)] $\bigcup_{k=1}^{n_0} \pi^k_{n_0} (\mathbb{D}(a_n, \rho_n)) \subset U$;
   \item[(b)] $\mathbb{D}_\rho \setminus \bigcup_{k=1}^{n_0} \pi^k_{n_0} (\mathbb{D}(a_n, \rho_n))$ is connected.
   \end{enumerate}

Put $Y_n = \bigcup_{j=1}^{n_0} \pi^k_{n_0} (\mathbb{D}(a_j, \rho_j))$. So, $Y_n \subset U$ is a closed set such that $\mathbb{D}_\rho \setminus Y_n$ is a domain and $\partial \mathbb{D}_\rho \cap Y_n = \emptyset$ for any $n \in \mathbb{N}$.

Step 2. We want to show that

\begin{equation}
(2.4) \quad \omega(0, \partial \mathbb{D}_\rho, \mathbb{D}_\rho \setminus Y_n) \geq \frac{1}{2}, \quad n \in \mathbb{N}.
\end{equation}

Fix $n \in \mathbb{N}$. Put $v_n(z) = -\omega(z, \partial \mathbb{D}_\rho, \mathbb{D}_\rho \setminus Y_n) + u(z)$. It suffices to show that

\begin{equation}
(2.5) \quad v_n \leq -1 \quad \text{on } \mathbb{D}_\rho \setminus Y_n.
\end{equation}

Observe that $-\omega(\cdot, \partial \mathbb{D}_\rho, \mathbb{D}_\rho \setminus Y_n) \leq 0$ and $u(\cdot) \leq 0$ on $\mathbb{D}_\rho \setminus Y_n$. Moreover, we have

$$\limsup_{z \to \partial \mathbb{D}_\rho, \mathbb{D}_\rho \setminus Y_n} -\omega(z, \partial \mathbb{D}_\rho, \mathbb{D}_\rho \setminus Y_n) \leq -1 \text{ and } \limsup_{z \to Y_n} u(z) \leq -1.$$  So, from the maximum principle for the subharmonic function $v_n$ we get \((2.5)\) and, therefore, \((2.4)\).

Step 3. Here we want show that

\begin{equation}
(2.6) \quad \omega(0, J, \mathbb{D}_\rho \setminus \bigcup_{j=1}^{n_0} \mathbb{D}(a_j, \frac{\rho_j}{2})) \geq \frac{1}{2n_0}, \quad n \in \mathbb{N}.
\end{equation}

Put

$$w_n(z) = \omega(z, \partial \mathbb{D}_\rho, \mathbb{D}_\rho \setminus Y_n) - \sum_{k=1}^{n_0} \omega(z, \pi^k_{n_0}(J), \mathbb{D}_\rho \setminus Y_n).$$

Note that $\bigcup_{k=1}^{n_0} \pi^k_{n_0}(J) = \partial \mathbb{D}_\rho$. Again from the maximum principle we obtain that $w_n \leq 0$ on $\mathbb{D}_\rho \setminus Y_n$, $n \in \mathbb{N}$.

Because $\pi^k_{n_0}(\mathbb{D}_\rho \setminus Y_n) = \mathbb{D}_\rho \setminus Y_n$, for any $k, n \in \mathbb{N}$, we find

$$\omega(0, \pi^k_{n_0}(J), \mathbb{D}_\rho \setminus Y_n) = \omega(0, J, \mathbb{D}_\rho \setminus Y_n), \quad k \in \mathbb{N}.$$
Hence,
\[ \omega(0, J, D_\rho \setminus \bigcup_{j=1}^{n} \overline{D}(a_j, \frac{\rho_j}{2})) \geq \omega(0, J, D_\rho \setminus Y_n) \geq \frac{1}{2n_0}, \quad n \in \mathbb{N}. \]

**Step 4.** Let \( \{ R_n \}_{n=1}^{\infty} \) be a sequence of positive numbers such that \( C_1 := \sum_{n=1}^{\infty} \frac{R_n}{n} < +\infty \) and, therefore, \( \sum_{n=1}^{\infty} R_n < C_1 \) (take e.g. \( R_n = \frac{1}{n^2} \)). Consider any sequence of complex numbers \( \{ c_n \}_{n=1}^{\infty} \) with \( |c_n| \leq R_n \) and let \( f \) be defined by (2.1).

Put
\[ f_n(z) = \sum_{j=1}^{n} \frac{c_j}{z - a_j} - \sum_{j=n+1}^{\infty} \frac{c_j}{a_j}, \quad n \in \mathbb{N}. \]

Then \( |f_n(z)| \leq 2C_1 \) for every \( z \in \mathbb{D}_\rho \setminus \bigcup_{j=1}^{n} \overline{D}(a_j, \frac{\rho_j}{2}) \) and all \( n \).

Let \( h \in \text{PSH}(D_2 \times \mathbb{C}) \) have the property that \( h(z, f(z)) = -\infty, \quad z \in D_1. \)

The function \( s_n \) defined on \( D_2 \setminus \{ a_1, \ldots, a_n \} \) by \( s_n(z) := h(z, f_n(z)) \) is subharmonic. Let \( A_n := \sup_{z \in \mathbb{H}} s_n(z) \) and let \( C_2 := \sup_{z \in \mathbb{D}_\rho, |w| \leq 2C_1} h(z, w). \) Then \( A_n \to \sup_{z \in \mathbb{D}_\rho} h(z, f(z)) = -\infty \) as \( n \to \infty. \)

From the two-constant theorem (see e.g. [8], Theorem 4.3.7) we infer
\[ \frac{C_2 - s_n(0)}{C_2 - A_n} \geq \omega(0, J, D_\rho \setminus \bigcup_{j=1}^{n} \overline{D}(a_j, \frac{\rho_j}{2})) \geq \frac{1}{2n_0}, \quad n \in \mathbb{N}. \]

Letting \( n \to \infty \), we conclude that \( h(0, f(0)) = s_n(0) = -\infty \) and therefore \( (0, f(0)) \in (\Gamma f)_D \times \mathbb{C}. \)

For the proof of Theorem 1.1 we need to know that the function defined by (2.1) is not extendible across the boundary of \( D_1 \). This will be done in the next section.

### 3. Non-extendible sums

Without additional conditions on \( a_n \) and \( c_n \), a function defined by (2.1) may well extend holomorphically beyond the boundary of \( D_1 \)—think of the Lambert-type series \( \sum_{n=1}^{\infty} c_n \frac{z^n}{n^2}; \) cf. [8]. It may even yield \( 0 \) on \( D_1 \); cf. [1]. We will see that a suitable choice of \( a_n \) and \( c_n \) prevents this from happening. We are grateful to Marek Jarnicki and Peter Pflug who suggested the idea of the proof of the next lemma.

**Lemma 3.1.** Let \( D \) be a domain in \( \mathbb{C} \). Then there exist a dense subset \( A = \{ a_n \}_{n=1}^{\infty} \) of \( \partial D \) and a sequence \( \{ R_n \}_{n=1}^{\infty} \) of positive numbers such that for any sequence of complex numbers \( \{ c_n \}_{n=1}^{\infty} \) with \( 0 < |c_n| \leq R_n \) the holomorphic function \( f \) given by (2.1) is not holomorphically extendible across \( \partial D \).

**Proof.** Let \( B = \{ b_n \}_{n=1}^{\infty} \) be a dense subset of \( D \) (take e.g. \( U = D \cap \mathbb{Q}^2 \)). For any \( b_n \in B \) there exists a point \( a \in \partial D \) such that \( \text{dist}(b_n, \partial D) = |b_n - a| \). We denote by \( a_n \) one of them. Set \( A = \{ a_n \}_{n=1}^{\infty} \). Note that \( A \) is a dense subset of \( \partial D \). Taking subsequence of \( \{ a_n \}_{n=1}^{\infty} \) we may assume that \( a_i \neq a_j, \ i \neq j \).

Fix \( n \in \mathbb{N} \). Let \( B_n = \{ z \in D : \text{dist}(z, \partial D) = |z - a_n| \} \subset D. \) Note that \( B_n \cup \{ a_n \} \) is a closed set on the plane and \( B = \bigcup_{n=1}^{\infty} B_n \) is dense in \( D \) (because \( B \supset B_n \)). Moreover, if \( z_0 \in B_n \), then the open segment with the ends at the points \( z_0 \) and \( a_n \) is contained in \( B_n \).
For any \( j \in \mathbb{N} \) we put \( \epsilon_{nj} = \text{dist}(a_j, B_n) \). Since \( a_j \notin B_n \) for \( j \neq n \), we see that \( \epsilon_{nj} > 0 \) for \( j \neq n \).

Put

\[
R_j = \frac{\min\{\epsilon_{1j}, \ldots, \epsilon_{(j-1)j}\}}{j^2}, \quad j \in \mathbb{N}.
\]

For any \( n \in \mathbb{N} \) and any \( j > n \) we have \( R_j \leq \frac{\epsilon_{nj}}{j} \) and therefore

\[
\sum_{j \neq n} R_j \frac{|z - a_j|}{|z - a_n|} \leq \sum_{j \neq n} \frac{R_j}{\epsilon_{nj}} < +\infty, \quad z \in B_n.
\]

Take a sequence of complex numbers \( \{c_n\}_{n=1}^\infty \) with \( 0 < |c_n| \leq R_n \). Then for a fixed \( n \in \mathbb{N} \) we have

\[
(3.1) \quad \liminf_{B_n, \exists z \rightarrow a_n} |(z - a_n) f(z)| \geq |c_n| - \lim_{B_n, \exists z \rightarrow a_n} |z - a_n| \cdot \limsup_{B_n, \exists z \rightarrow a_n} \sum_{j \neq n} \frac{|c_j|}{|z - a_j|} = |c_n| > 0.
\]

Observe that for any \( n \in \mathbb{N} \) the Taylor series at any point of \( z_0 \in B_n \) has a radius of convergence equal to \( \text{dist}(z_0, \partial D) \) (because of (3.1) and \( |z_0 - a_n| = \text{dist}(z_0, \partial D) \)).

Hence, by Lemma 1.7.5 from [3] we see that \( D \) is the domain of existence of \( f \).

**Proof of Theorem 3.1.** If a set \( E \subset \Omega \) is complete pluripolar in a domain \( \Omega \), then \( E^*_\Omega = E \). By Lemma 3.1 and Theorem 2.1 there exists a holomorphic function \( f \) on \( D_1 \) for which \( D_1 \) is a domain of existence and \( (\Gamma f)^{1,2}_E \neq \Gamma f \). Hence, \( \Gamma f \) is not complete pluripolar in \( D_2 \times \mathbb{C} \).

**Theorem 3.2.** There exists a sequence \( \{a_n\}_{n=1}^\infty \subset \mathbb{C} \setminus \overline{D} \) and a sequence \( \{c_n\}_{n=1}^\infty \) such that the function \( f \) defined by (2.1) is \( C^\infty \) on \( \overline{D} \) and is nowhere extendible over the boundary of \( D \), while \( \Gamma f \) is not complete pluripolar in \( \mathbb{C}^2 \).

**Proof.** Let \( r_j = 1 + 1/(j + 1) \). The sequence \( a_n \) is formed by

\[
a_{2^j+k} = r_j e^{2\pi i k/2^j}, \quad k = 0, \ldots, 2^j - 1, \quad j = 0, 1, \ldots.
\]

The proof of Theorem 2.1 provides us with a sequence \( \{R_n\} \) such that for every sequence \( \{c_n\} \) with \( |c_n| < R_n \) the series (2.1) represents a function on \( D \), the graph of which is not complete pluripolar. Assembling all \( a_n \in C(0, r_j) \) we find that there exists a sequence \( \{R_j^*\} \) such that for every choice of \( 0 < \epsilon < R_j^* \) the function \( f_\epsilon \) on \( \overline{D} \) defined by

\[
(3.2) \quad f_\epsilon(z) = \sum_{j=0}^\infty \frac{\epsilon_j}{r_j^* - z^{2^j}}
\]

has a graph that is not complete pluripolar.

We observe that independent of the choice of \( \epsilon_j \),

\[
\sum_{j=1}^n \frac{\epsilon_j}{r_j^* - z^{2^j}} = \sum_{k=1}^\infty d_{n,k} z^k
\]

is holomorphic on \( D_{r_n} \) with singularities on the boundary. (The \( d_{n,k} \) are defined by the equality.) Therefore we have \( \limsup_{k \rightarrow \infty} |d_{n,k}^{1/k}| = 1/r_n \). Hence there is a \( k_n \) such that

\[
(3.3) \quad |d_{n,k_n}| > r_n^{-k_n}.
\]
We will now make an appropriate choice for the $\varepsilon_j$ to insure that $f$ cannot be extended over the boundary of $\mathbb{D}$. Along the way we will determine constants $C_j$ that are needed for smoothness at the boundary.

Choose $\varepsilon_0 = R'_0$. Then

$$f_{\varepsilon_0}(z) = \frac{1}{r_0 - z} = \sum_{k=1}^{\infty} d_{0,k} z^k$$

with $\limsup_{k \to \infty} |d_{0,k}^{1/k}| = 1/r_0$; in particular there is a $C_0$ such that $|d_{0,k}| < C_0$.

Suppose $\varepsilon_0, \ldots, \varepsilon_{n-1}$ and $C_0, \ldots, C_{n-1}$ have been chosen in such a way that we have found $k_0, \ldots, k_{n-1}$ with

$$|d_{l,k_j}| > r_j^{-k_j} \quad \text{for } j = 1, \ldots, n-1, \quad l = j, \ldots, n-1,$$

and

$$|d_{j,k}| < \frac{C_l}{k^l} \quad \text{for } l = 0, \ldots, n-1, \quad j = 0, \ldots, n-1, \quad \text{and all } k.$$

Then choose

$$C_n > \sup_k |d_{j,k}| k^n \quad j = 0, \ldots, n-1.$$

This is finite because of $(3.4)$. Next choose $\varepsilon_n < R'_n$ so small that

(1) The inequality $(3.5)$ holds for $l = 0, \ldots, n$ and $j = 0, \ldots, n$. This is possible because of $(3.4)$.

(2)

$$|d_{n,k_j}| > r_j^{-k_j} \quad \text{for } j = 1, \ldots, n-1,$$

which is again possible because of $(3.4)$.

Having chosen $\varepsilon_n$, we can by $(3.3)$ choose $k_n$ so large that $|d_{n,k_n}| > r_n^{-k_n}$. Observe that the coefficients $d_{n,k}$ converge to the coefficients $d_k$ of the power series expansion of $f_\varepsilon$ as $n \to \infty$. From $(3.3)$ we see that $|d_{k_j}| > r_j^{-k_j}$ so that the radius of convergence of the power series of $f_\varepsilon$ is at most 1, and since $f_\varepsilon$ is holomorphic on $\mathbb{D}$, it equals 1. So $f_\varepsilon$ has a singular point $b$ on $C(0,1)$. We split $f_\varepsilon$ as

$$f_\varepsilon = f_1 + f_2 = \left( \sum_{j=0}^{n-1} + \sum_{j=n}^{\infty} \right) \frac{\varepsilon_j}{r_j^{2j} - z^{2j}}.$$

Then $f_1$ is holomorphic in a neighborhood of the closed unit disc and $f_2$ has at least one singular point on $C(0,1)$, but $f_2$ is invariant under rotation over a $2^n$-th root of unity, which implies that there is a singularity in each arc of length $> 2\pi/2^n$. Therefore, $f$ can nowhere be extended analytically over $C(0,1)$.

Next we show that $f$ is smooth up to the boundary of $\mathbb{D}$. We have to show that there exist constants $C_l > 0$ such that for every $l$

$$|d_k| \leq \frac{C_l}{k^l}, \quad \text{for all } k,$$

but this follows from $(3.5)$. □
Remark 3.3. Let $D$ be a domain in $\mathbb{C}$ and let $A$ be a closed polar subset of $D$. Using methods presented in this paper and in [10], the authors give in [2] a complete characterization of the holomorphic functions $f$ on $D \setminus A$ such that $\Gamma_f = \{(z, f(z)) : z \in D \setminus A\}$ is complete pluripolar in $D \times \mathbb{C}$.

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