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ESTIMATES ON THE MEAN GROWTH OF H^p FUNCTIONS IN CONVEX DOMAINS OF FINITE TYPE

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Abstract. Let D be a bounded convex domain of finite type in $\mathbb{C}^n$ with smooth boundary. In this paper, we prove the following inequality:

$$\left(\int_0^{\delta_0} \mathcal{M}_q^\lambda(f;t)\, t^{\lambda n(1/p-1/q)-1}dt\right)^{1/\lambda} \le C_{p,q}\|f\|_{p,0},$$

where $1 < p < q < \infty, f \in H^p(D)$, and $p \le \lambda < \infty$. This is a generalization of some classical result of Hardy-Littlewood for the case of the unit disc. Using this inequality, we can embed the H^p space into a weighted Bergman space in a convex domain of finite type.

1. Introduction and statement of results

Let D be a bounded domain in $\mathbb{C}^n$ with smooth boundary. For $z \in D$ let $\delta(z)$ denote the distance from z to ∂D. For $\alpha > 0$, we define a measure dV_α on D by $dV_\alpha = C_\alpha \delta^{\alpha-1} dV$ where dV is the volume element and C_α is chosen so that dV_α is a probability measure. As $\alpha \to 0^+$, the measures dV_α converge as measures on ∂D to the normalized surface measure on ∂D which we denote dV_0 (or sometimes $d\sigma$). We will denote the L^p space with respect to dV_α by L^p_α, and the associated norm by $\|\cdot\|_{p,\alpha}$. We will denote by $A^p_\alpha(D) = L^p_\alpha(D) \cap \mathcal{O}(D)$ the subspace of $L^p_\alpha(D)$ consisting of functions which are holomorphic on D. In particular, $A^p_0(D)$ is the Hardy class usually denoted by $H^p(D)$, which we identify in the usual way with a subspace of $L^p_0(D) = L^p(\partial D; d\sigma)$.

Let $\vec{N}$ be a real vector field in a neighborhood of ∂D which agree with the outward unit normal vector field on ∂D. For $z \in \partial D$ and $t > 0$ sufficiently small, say $0 < t < \delta_0$, the integral curve of $\vec{N}$ through z has a unique intersection point with the hypersurface $\{\delta = t\}$. We call this intersection point z_t. For any function f on D we define f_t on ∂D by $f_t(z) = f(z_t)$ for $z \in \partial D$, and we define means of f by

$$\mathcal{M}_p(f;t) = \left(\int_{\partial D} |f_t|^p d\sigma\right)^{1/p}.$$

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It follows from Fubini's theorem and elementary estimates for Jacobians that for $f \in \mathcal{O}(D)$ we have $f \in A^p_0(D)$ if and only if $\sup_{0<t<\delta_0} \mathcal{M}_p(f;t) < \infty$, and for $\alpha > 0$ we have $f \in A^p_\alpha(D)$ if and only if

$$\int_0^{\delta_0} \mathcal{M}_p^p(f;t)\ t^{\alpha-1}dt < \infty.$$

In this paper we get a sharp estimate on the mean growth of H^p functions on convex domains of finite type.

Theorem 1.1. *Let D be a bounded convex domain of finite type in $\mathbb{C}^n$ with smooth boundary. If $1 < p < q < \infty, f \in H^p(D)$, and $p \le \lambda < \infty$, then*

$$\left(\int_0^{\delta_0} \mathcal{M}_q^\lambda(f;t)\ t^{\lambda n(1/p-1/q)-1}dt\right)^{1/\lambda} \le C_{p,q}\|f\|_{p,0}.$$

In the present setting we do not know whether the estimate in Theorem 1.1 remains valid when $0 < p \le 1$. If we apply Theorem 1.1 with $\lambda = q$, then we get the following result.

Theorem 1.2. *Let D be as in Theorem 1.1 and assume that $\alpha \ge 0, 1 < p \le q < \infty$, and $n/p = (n+\alpha)/q$. Then $H^p(D) \subset A^q_\alpha(D)$ and the inclusion is continuous.*

These results were first proved by Hardy-Littlewood for the case of the unit disc ([6], p. 87). When D is the unit ball and the strictly pseudoconvex domain, the results were proved by Beatrous-Burbea [2] and Beatrous [1]. The key point is the reproducing kernel with right estimate matching quasimetric on ∂D. For the case of convex domains of finite type we use the holomorphic support function with best possible non-isotropic estimates constructed by Diederich-Fornaess [4].

2. Reproducing kernels

Throughout, $D = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ is a smoothly bounded, convex domain of finite type m defined by a real-valued function ρ with convex infralevel sets. The defining function ρ can be chosen in such a way that there exists a neighborhood U of ∂D such that $|\partial\rho(z)| > 1$ for all $z \in U$. Diederich-Fornaess [4] constructed a good C^∞-family $S(z,\zeta)$ of support functions on D, holomorphic in $z \in \bar{D}$ and C^∞ in ζ chosen in a suitable neighborhood U of ∂D with the following estimates. For $\zeta \in U$ let $\vec{n}_\zeta$ denote the outer unit vector normal to the level set $\{\rho = \rho(\zeta)\}$ at ζ and let $\vec{v}$ be any unit vector complex tangential to this level set at ζ. Define

$$a_{\alpha\beta}(\zeta,\vec{v}) := \frac{\partial^{\alpha+\beta}}{\partial\lambda^\alpha\partial\bar{\lambda}^\beta}\rho(\zeta+\lambda\vec{v})|_{\lambda=0}.$$

Then there are constants $K, c, d > 0$, such that one has for all points z written as $z = \zeta + \mu\vec{n}_\zeta + \lambda\vec{v}$ with $\mu, \lambda \in \mathbb{C}$ the estimate

$$\begin{aligned} 2\ \mathrm{Re}\ S(z,\zeta) \le &- |\mathrm{Re}\ \mu| - K(\mathrm{Im}\ \mu)^2 \\ &- c\sum_{j=2}^m \sum_{\alpha+\beta=j} |a_{\alpha\beta}(\zeta,\vec{v})||\lambda|^j + d\sup\{0, \rho(z)-\rho(\zeta)\}. \end{aligned} \tag{2.1}$$

In (5) and (6) of [3] C^∞ functions $Q_j(z,\zeta), j = 1,\dots,n$, holomorphic in z, were defined, such that

$$S(z,\zeta) = \sum_{j=1}^n Q_j(z,\zeta)(z_j - \zeta_j).$$

Henkin [7] proved the integral representation

$$f(z) = c\int_{\zeta\in\partial D} f(\zeta)\frac{Q\wedge(\bar\partial^T Q)^{n-1}}{S(z,\zeta)^n}, \quad z\in D,$$

for $f\in L^1(\partial D)\cap\mathcal{O}(D)$, where $\bar\partial^T Q$ means the tangential components of $\bar\partial Q$. We define the reproducing kernel

$$K(z,\zeta) = c\frac{Q(z,\zeta)\wedge(\bar\partial^T Q(z,\zeta))^{n-1}}{S(z,\zeta)^n}, \quad z\in D,\ \zeta\in\partial D.$$

3. Integral estimates for the reproducing kernels

For $z\in U$ and $0<\epsilon<\epsilon_0$ we define some sort of boundary distances by

$$\tau(z,\vec v,\epsilon) := \sup\{r>0 : |\rho(z+\lambda\vec v)-\rho(z)|<\epsilon, |\lambda|\le r, \lambda\in\mathbb{C}\}.$$

The quantity τ measures the size of the largest complex disc centered at z lying on the line spanned by $\vec v$ that fits in the domain $\{\zeta : \rho(\zeta)<\rho(z)+\epsilon\}$. Next we define the ϵ-extremal basis $(\vec v_1,\dots,\vec v_n)$ centered at z of McNeal [8]. The first vector $\vec v_1$ is the unit vector in the direction of $\partial\rho(z)$; chosen $\vec v_1,\dots,\vec v_{i-1}$, $\vec v_i$ is a unit vector orthogonal to $\vec v_1,\dots,\vec v_{i-1}$. In this way we obtain a basis $(\vec v_1,\dots,\vec v_n)$ depending on both z and $\epsilon>0$. We denote the i-th component of the coordinates with respect to this basis by w_i. We call the coordinates ϵ-extremal coordinates at z. We write $\tau_i(z,\epsilon) := \tau(z,\vec v_i,\epsilon)$. Since D is finite type m, we have (see [8])

$$\tau_1(z,\epsilon)\sim\epsilon \quad\text{and}\quad \epsilon^{1/2}\lesssim\tau_i(z,\epsilon)\lesssim\epsilon^{1/m} \quad\text{for}\quad 2\le i\le n.$$

We can now define the non-isotropic polydiscs at z with radius ϵ by

$$P_\epsilon(z) := \{\zeta = z+\sum_{i=1}^n w_i\vec v_i : |w_i|\le\tau_i(z,\epsilon), i=1,\dots,n\}.$$

We transform the forms $Q(z,\zeta)$ by pulling back to the ζ-variable, obtaining

$$Q^*(z,w) := \sum_{i=1}^n Q_i(z,\zeta(w))d\zeta_i(w).$$

We write

$$Q^*(z,w) = \sum_{i=1}^n Q_i^*(z,w)dw_i.$$

Then $|Q\wedge(\bar\partial^T Q)^{n-1}|$ is bounded from above by the sum over all terms of the form

$$|Q^*_{i_1}(z,w)|\prod_{l=2}^n\left|\frac{\partial Q^*_{i_l}(z,w)}{\partial\bar w_{j_l}}\right| \tag{3.1}$$

where $(i_1,\dots,i_n)$ is a multi-index with values from $\{1,\dots,n\}$ and $i_k\ne i_l$ for $k\ne l$ and $(j_2,\dots,j_n)$ is a multi-index with values in $\{2,\dots,n\}$ with $j_k\ne j_l$ for $k\ne l$.

According to Lemma 5.1 of [3] we have

$$|Q^*_{i_l}(z,w)|\lesssim\frac{\epsilon}{\tau_{i_l}(z,\epsilon)}, \quad \zeta(w)\in P_\epsilon(z) \tag{3.2}$$

and

$$\left|\frac{\partial Q^*_{i_l}(z,w)}{\partial\bar w_{j_l}}\right|\lesssim\frac{\epsilon}{\tau_{i_l}(z,\epsilon)\tau_{j_l}(z,\epsilon)}, \quad \zeta(w)\in P_\epsilon(z). \tag{3.3}$$

By (3.2) and (3.3) each term in (3.1) is bounded by

$$\frac{\epsilon}{\tau_{i_1}(z,\epsilon)}\prod_{l=2}^{n}\frac{\epsilon}{\tau_{i_l}(z,\epsilon)\tau_{j_l}(z,\epsilon)}.$$

If we put this together into $|Q\wedge(\bar\partial^T Q)^{n-1}|$ we get

$$|Q\wedge(\bar\partial^T Q)^{n-1}| \lesssim \frac{\epsilon^{n-1}}{\prod_{j=2}^{n}\tau_j(z,\epsilon)^2}. \tag{3.4}$$

For integral estimates we define a family of polyannuli based on non-isotropic polydiscs. We choose a constant $C_1>0$ such that $C_1P_{\epsilon/2}(z)\supset\frac{1}{2}P_\epsilon(z)$ for $\epsilon>0$ and put for integer i

$$P_\epsilon^i(z) := C_1P_{2^i\epsilon}(z)\setminus\frac{1}{2}P_{2^i\epsilon}(z).$$

Then we see that

$$\bigcup_{i=0}^{\infty}P_\epsilon^{-i}(z)\supset P_\epsilon(z)\setminus\{z\}$$

and for $\epsilon<\epsilon_0$

$$\bigcup_{i=0}^{\infty}P_\epsilon^{i}(z)\supset P_{\epsilon_0}(z)\setminus P_\epsilon(z).$$

Lemma 3.1 ([5]). *For integer i we have*

$$|S(z,\zeta)|\gtrsim 2^i\epsilon \tag{3.5}$$

uniformly in $z\in D\cap U,\ \zeta\in P_\epsilon^i(z)\cap\partial D$ (or uniformly in $\zeta\in\partial D,\ z\in P_\epsilon^i(\zeta)\cap D$).

Let $K_t(z,\zeta)$ be the kernel on $\partial D\times\partial D$ defined by $K_t(z,\zeta)=K(z_t,\zeta)$. Then we get the following estimates.

Proposition 3.2. *Let $s>1$. Then we have*

$$\int_{\zeta\in\partial D}|K_t(z,\zeta)|^s d\sigma(\zeta)\lesssim t^{-n(s-1)}\quad \textit{uniformly in}\quad z\in\partial D, \tag{3.6}$$

$$\int_{z\in\partial D}|K_t(z,\zeta)|^s d\sigma(z)\lesssim t^{-n(s-1)}\quad \textit{uniformly in}\quad \zeta\in\partial D. \tag{3.7}$$

Proof. We treat only the case (3.6), since the other case (3.7) is similar. We denote

$$I(X)=\int_{\partial D\cap X}|K_t(z,\zeta)|^s d\sigma(\zeta).$$

For $z\in\partial D$ we write $z_t=\zeta+\mu\vec n_{z_t}+\lambda\vec v$. Then we have from (2.1) that

$$-2\ \mathrm{Re}\ S(z_t,\zeta)\geq|\mathrm{Re}\ \mu|+K(\mathrm{Im}\ \mu)^2+c\sum_{j=2}^{m}\sum_{\alpha+\beta=j}|a_{\alpha\beta}(\zeta,\vec v)||\lambda|^j.$$

Since D is finite type m, it follows that

$$\sum_{j=2}^{m}\sum_{\alpha+\beta=j}|a_{\alpha\beta}(\zeta,\vec v)||\lambda|^j\gtrsim|\lambda|^m.$$

Since $m\geq 2$, we have $|z_t-\zeta|^m=|\mu\vec n_{z_t}+\lambda\vec v|^m\lesssim|\mathrm{Re}\ \mu|+K(\mathrm{Im}\ \mu)^2+|\lambda|^m$. Thus it follows that

$$-2\ \mathrm{Re}\ S(z_t,\zeta)\gtrsim|z_t-\zeta|^m$$

and hence the only singularity of $K_t(z,\zeta)$ occurs for $\zeta = z_t$. Thus $I(\partial D) \le C$ if $t \ge \delta_0$ and $I(\partial D \setminus V) \le C$ for some small neighborhood V of z_t. Hence it is enough to prove that for fixed $\epsilon_0 > 0$ and small $t > 0$

$$I(P_{\epsilon_0}(z_t)) \lesssim t^{-n(s-1)}. \tag{3.8}$$

For fixed $z \in \partial D$ we define $\rho = |\rho(z_t)| \sim t$ and then split the polydisc $P_{\epsilon_0}(z_t)$ into the two parts $P_\rho(z_t)$ and $P_{\epsilon_0}(z_t) \setminus P_\rho(z_t)$. Remember that $P_\rho(z_t) \setminus \{z_t\}$ can be covered by $\bigcup_{i=0}^\infty P_\rho^{-i}(z_t)$. By (3.4), it follows that

$$\begin{aligned} I(P_\rho^{-i}(z_t)) &= \int_{\partial D \cap P_\rho^{-i}(z_t)} |K_t(z,\zeta)|^s d\sigma(\zeta) \\ &\lesssim \int_{\partial D \cap P_\rho^{-i}(z_t)} \frac{1}{|S(z_t,\zeta)|^{ns}} \frac{(2^{-i}\rho)^{(n-1)s}}{\prod_{j=2}^n \tau_j(z_t, 2^{-i}\rho)^{2s}} d\sigma(\zeta). \end{aligned}$$

Since $\rho = |\rho(z_t)| = |\rho(z_t) - \rho(\zeta)| \lesssim |\mathrm{Re}\ \mu|$, it follows that $|S(z_t,\zeta)| \gtrsim \rho$ and hence we have

$$\begin{aligned} I(P_\rho^{-i}(z_t)) &\lesssim \frac{(2^{-i})^{(n-1)s}}{\rho^s} \int_{|v_1| < \tau_1(z_t, 2^{-i}\rho)} dv_1 \prod_{j=2}^n \int_{|w_j| < \tau_j(z_t, 2^{-i}\rho)} \frac{du_j dv_j}{\tau_j(z_t, 2^{-i}\rho)^{2s}} \\ &\lesssim \frac{(2^{-i})^{(n-1)s}}{\rho^s} \tau_1(z_t, 2^{-i}\rho) \prod_{j=2}^n \tau_j(z_t, 2^{-i}\rho)^{-2(s-1)} \\ &\lesssim \frac{(2^{-i})^{(n-1)s}}{\rho^s} (2^{-i}\rho)(2^{-i}\rho)^{-(s-1)(n-1)} = \rho^{-n(s-1)}(2^{-i})^n. \end{aligned}$$

Thus we get

$$\begin{aligned} I(P_\rho(z_t)) &\le \sum_{i=0}^\infty I(P_\rho^{-i}(z_t)) \\ &\lesssim \sum_{i=0}^\infty \rho^{-n(s-1)}(2^{-i})^n \lesssim \rho^{-n(s-1)} \sim t^{-n(s-1)}. \end{aligned} \tag{3.9}$$

To estimate the integral over $P_{\epsilon_0}(z_t) \setminus P_\rho(z_t)$ we use the covering by $\bigcup_{i=0}^\infty P_\rho^i(z_t)$. By (3.4) and (3.5), we have

$$\begin{aligned} I(P_\rho^i(z_t)) &\lesssim \int_{\partial D \cap P_\rho^i(z_t)} \frac{1}{|S(z_t,\zeta)|^{ns}} \frac{(2^i\rho)^{(n-1)s}}{\prod_{j=2}^n \tau_j(z_t, 2^i\rho)^{2s}} d\sigma(\zeta) \\ &\lesssim \frac{1}{(2^i\rho)^s} \tau_1(z_t, 2^i\rho) \prod_{j=2}^n \tau_j(z_t, 2^i\rho)^{-2(s-1)} \\ &\lesssim \rho^{-n(s-1)}(2^i)^{-n(s-1)}. \end{aligned}$$

Thus we get

$$\begin{aligned} I(P_{\epsilon_0}(z_t) \setminus P_\rho(z_t)) &\le \sum_{i=0}^\infty I(P_\rho^i(z_t)) \\ &\lesssim \sum_{i=0}^\infty \rho^{-n(s-1)}(2^i)^{-n(s-1)} \lesssim \rho^{-n(s-1)} \sim t^{-n(s-1)}. \end{aligned} \tag{3.10}$$

By (3.9) and (3.10), we proved the required estimate (3.8). □

4. Proofs of main results

For $f \in L^1(\partial D; d\sigma)$ we define

$$\mathcal{K}f(z) = \int_{\zeta\in\partial D} f(\zeta)K(z,\zeta), \quad z \in D.$$

By using Proposition 3.2, we can prove as in ([1], Lemma 2.8) that

$$\left(\int_0^{\delta_0} \mathcal{M}_q^\lambda(\mathcal{K}f;t)\, t^{\lambda n(1/p-1/q)-1}dt\right)^{1/\lambda} \le C_{p,q}\|f\|_{p,0}, \tag{4.1}$$

where $1 < p < q < \infty$ and $q \le \lambda < \infty$. Since $K(z,\zeta)$ is the reproducing kernel for holomorphic functions, we get Theorem 1.1 from (4.1). Moreover, for $\alpha > 0$, $1 < p \le q < \infty$, and $n/p = (n+\alpha)/q$, if we apply (4.1) with $\lambda = q$ we have

$$\int_D |\mathcal{K}f|^q dV_\alpha \lesssim \|f\|_{p,0}^q.$$

Thus the integral operator $\mathcal{K} : L^p(\partial D; d\sigma) \to A^q_\alpha(D)$ is bounded and so we get Theorem 1.2.

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