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RECONSTRUCTION OF FUNCTIONS IN SPLINE SUBSPACES FROM LOCAL AVERAGES

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Abstract. In this paper, we study the reconstruction of functions in spline subspaces from local averages. We present an average sampling theorem for shift invariant subspaces generated by cardinal B-splines and give the optimal upper bound for the support length of averaging functions. Our result generalizes an earlier result by Aldroubi and Gröchenig.

1. Introduction and the main result

The sampling theorem is one of the most powerful results in signal analysis. It states that if a signal (function) $f(x)$ satisfies certain conditions, then it is uniquely determined by its sampled values at some discrete points $\{x_k : k \in \mathbb{Z}\}$ and

$$f(x) = \sum_{k\in\mathbb{Z}} f(x_k) S_k(x)$$

for some sampling functions $\{S_k(x)\}$. For example, every $f \in B_\Omega := \{f : \operatorname{supp} \hat{f} \subset [-\Omega, \Omega]\}$ can be reconstructed by the formula $f(x) = \sum_{k\in\mathbb{Z}} f(\frac{k\pi}{\Omega}) \frac{\sin(\Omega x - k\pi)}{\Omega x - k\pi}$. This is the classical Shannon sampling theorem. Although the assumption that a signal is band-limited is eminently useful, it is not always realistic since a band-limited signal is of infinite duration. Thus, it is natural to investigate other signal classes for which a sampling theorem holds. A simple model is to consider shift-invariant subspaces, e.g., wavelet subspaces, which generalize the space of band-limited functions and are of the form

$$V = \overline{\operatorname{span}}\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$$

for some kernel function $\varphi(x)$. In fact, there have been many results concerning the sampling in shift-invariant subspaces for both regular and irregular sampling. For details, see [1]–[7], [10]–[19], [22]–[25] and [27].

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In particular, for the spline subspace

$$V_m = \{\sum_{k\in\mathbb{Z}} c_k\varphi_m(\cdot - k) : \{c_k\} \in \ell^2\}$$

generated by the cardinal B-spline

$$\varphi_m = \chi_{[0,1]} * \cdots * \chi_{[0,1]} \quad (\ m+1 \ \text{terms}\ ),\ \ m \geq 1,$$

Liu [16] proved that every $f \in V_m$ is uniquely determined and can be reconstructed by an iterative algorithm from its samples $f(x_k)$ if $x_{k+1} - x_k$ is small enough. Furthermore, Aldroubi and Gröchenig [1] proved that if $\alpha \leq x_{k+1} - x_k \leq \beta$ for some constants $0 < \alpha < \beta < 1$, then every $f \in V_m$ is uniquely determined by $f(x_k)$.

For physical reasons, e.g., the inertia of the measurement apparatus, measured sampled values obtained in practice may not be values of a function f precisely at times x_k, but only local averages of f near x_k. Specifically, measured sampled values are

$$\langle f, u_k\rangle = \int f(x)u_k(x)dx$$

for some collection of averaging functions $u_k(x), k \in \mathbb{Z}$, which satisfy the following properties:

$$\text{supp } u_k \subset [x_k - \frac{\delta}{2}, x_k + \frac{\delta}{2}], \quad u_k(x) \geq 0, \quad \text{and} \quad \int u(x)dx = 1.$$

Observe that the averaging procedure is allowed to vary from point to point.

It is clear that from local averages one should obtain at least a good approximation of the original signal if δ is small enough. Wiley, Butzer and Lei [5, 6, 24] studied the approximation error when local averages are used as sampled values. Furthermore, Gröchenig [14] proved that if sampling points $\{x_k : k \in \mathbb{Z}\}$ satisfy $0 < x_{k+1} - x_k \leq \delta < \frac{1}{\sqrt{2}\Omega}$, then every $f \in B_\Omega$ is uniquely determined by local averages $\langle f, u_k\rangle$ around x_k and can be reconstructed by an iteration scheme. In [13], Feichtinger and Gröchenig proved that if $\delta := \sup_{k\in\mathbb{Z}}(x_{k+1} - x_k) < \frac{\pi}{\Omega}$, then every $f \in B_\Omega$ is uniquely determined by $\frac{1}{y_k - y_{k-1}} \int_{y_{k-1}}^{y_k} f(x)dx$ with $y_k = \frac{x_k + x_{k+1}}{2}$.

In [20], we gave an average sampling theorem for shift invariant subspaces with equally spaced sampling points and arbitrary averaging functions. For the case of band-limited signals, we gave an analogy of Kadec's 1/4-theorem. In [21], we studied average sampling theorems for spline subspaces with standard averaging functions $u_k = \chi_{[x_k - 1/2, x_k + 1/2]}$.

In this paper, we study the reconstruction of functions in spline subspaces from local averages with arbitrary averaging functions and irregular sampling points. We present an average sampling theorem and give the optimal upper bound for the support length of averaging functions. Our result generalizes Aldroubi and Gröchenig's irregular sampling theorem. In fact, we prove the following.

Theorem 1.1. *Let $\{x_k : k \in \mathbb{Z}\}$ be a real sequence such that $\lim_{k\to\pm\infty} x_k = \pm\infty$ and*

$$0 < x_{k+1} - x_k \leq \beta < 1, \quad k \in \mathbb{Z},$$

for some constant β. Then for any $0 < \delta < 1 - \beta$ and averaging functions $u_k(x)$ with supp $u_k \subset [x_k - \frac{\delta}{2}, x_k + \frac{\delta}{2}]$*, there is a frame $\{S_k(x) : k \in \mathbb{Z}\}$ for V_m such that* *for any* $f \in V_m$,

$$f(x) = \sum_{k\in\mathbb{Z}} (\frac{x_{k+1} - x_{k-1}}{2})^{1/2} \langle f, u_k \rangle S_k(x), \tag{1.1}$$

where the convergence is both in $L^2(\mathbb{R})$ *and uniform on* $\mathbb{R}$.

Furthermore, the conclusion fails if $\delta \geq 1 - \beta$.

Notation and definition. The Fourier transform and the Zak transform of $f \in L^2(\mathbb{R})$ is defined by $\hat{f}(\omega) = \int_{\mathbb{R}} f(x)e^{-ix\omega}dx$ and $Zf(x,\omega) = \sum_{k\in\mathbb{Z}} f(x+k)e^{-ik\omega}$ respectively.

We call $u(x)$ an averaging function if $u(x) \geq 0, u(x) \in L^1(\mathbb{R})$ and $\int_{\mathbb{R}} u(x)dx = 1$.

2. Some basic facts in frame theory

In this section, we introduce some basic facts in frame theory.

A family of functions $\{\varphi_k : k \in \mathbb{Z}\}$ belonging to a separable Hilbert space $\mathcal{H}$ is said to be a frame if there exist positive constants A and B such that $A\|f\|^2 \leq \sum_{k\in\mathbb{Z}} |\langle f, \varphi_k\rangle|^2 \leq B\|f\|^2$ for every $f \in \mathcal{H}$. The constants A and B are called the lower and upper frame bounds respectively.

A frame that ceases to be a frame when any one of its elements is removed is said to be an exact frame. It is well known that exact frames and Riesz bases are identical (see [26]).

Let $\{\varphi_k : k \in \mathbb{Z}\}$ be a frame for some Hilbert space $\mathcal{H}$ with bounds A and B. Define the frame operator T as follows: $Tf = \sum_{n\in\mathbb{Z}}\langle f, \varphi_n\rangle\varphi_n, \forall f \in \mathcal{H}$. Then T is bounded and invertible and $\{T^{-1}\varphi_k : k \in \mathbb{Z}\}$ is also a frame for $\mathcal{H}$ with bounds $\frac{1}{B}$ and $\frac{1}{A}$, called the dual frame of $\{\varphi_k : k \in \mathbb{Z}\}$. It can be shown that for any $f \in \mathcal{H}$,

$$f = \sum_{n\in\mathbb{Z}} \langle f, T^{-1}\varphi_n\rangle\varphi_n = \sum_{n\in\mathbb{Z}} \langle f, \varphi_n\rangle T^{-1}\varphi_n.$$

For the general theory on frames and Riesz bases, see [9, 26].

For frames in shift invariant subspaces, we have the following.

Lemma 2.1. *Let* $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ *be a frame for some closed subspace* V *of* $L^2(\mathbb{R})$. *Suppose that* φ *is continuous and* $\sum_{k\in\mathbb{Z}} |\varphi(x-k)|^2 \leq L < +\infty$. *Then for any frame* $\{S_k : k \in \mathbb{Z}\}$ *of* V_0, $\sum_{k\in\mathbb{Z}} |S_k(x)|^2$ *is bounded on* $\mathbb{R}$.

Proof. Let $\{\tilde{\varphi}(\cdot - k) : k \in \mathbb{Z}\}$ be the dual frame of $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$. Then for any $f \in V, f(x) = \sum_{k\in\mathbb{Z}}\langle f, \tilde{\varphi}(\cdot - k)\rangle\varphi(x-k)$. Therefore,

$$\|f\|_\infty^2 \leq \sup_x \sum_{k\in\mathbb{Z}} |\langle f, \tilde{\varphi}(\cdot - k)\rangle|^2 \sum_{k\in\mathbb{Z}} |\varphi(x-k)|^2 \leq \frac{L}{A}\|f\|_2^2.$$

It follows that for any $x \in \mathbb{R}$,

$$\sum_{k\in\mathbb{Z}} |S_k(x)|^2 = \sup_{\|c\|_2=1} \left|\sum_{k\in\mathbb{Z}} c_k S_k(x)\right|^2 \leq \sup_{\|c\|_2=1} \frac{L}{A} \left\|\sum_{k\in\mathbb{Z}} c_k S_k\right\|_2^2 \leq \frac{LM}{A},$$

where M is the upper frame bound of $\{S_k(x) : k \in \mathbb{Z}\}$. □

Lemma 2.2. *Suppose that $\{u_k : k \in \mathbb{Z}\}$ is a sequence of compactly supported averaging functions and there exist positive constants A and B such that*

$$A\|f\|^2 \leq \sum_{k\in\mathbb{Z}} |\langle f, u_k\rangle|^2 \leq B\|f\|^2, \quad \forall f \in V_m. \tag{2.1}$$

Then there is a frame $\{S_k : k \in \mathbb{Z}\}$ for V_m such that $f(x) = \sum_{k\in\mathbb{Z}}\langle f, u_k\rangle S_k(x), \forall f \in V_m$, where the convergence is both in $L^2(\mathbb{R})$ and uniform on $\mathbb{R}$.

Proof. By [8, Theorem 4.5], $\{\varphi_m(\cdot - n) : n \in \mathbb{Z}\}$ is a Riesz basis for V_m. Let $\{\tilde{\varphi}_m(\cdot - n) : n \in \mathbb{Z}\}$ be the dual Riesz basis. For any $k \in \mathbb{Z}$, let

$$h_k = \sum_{n\in\mathbb{Z}} \langle u_k, \varphi_m(\cdot - n)\rangle \tilde{\varphi}_m(\cdot - n).$$

Since both φ_m and u_k are compactly supported, the above series contains only finite non-zero terms. Hence $h_k \in V_m$. For any $f \in V_m$, we have

$$\begin{aligned}
\langle f, h_k\rangle &= \sum_{n\in\mathbb{Z}} \langle \varphi_m(\cdot - n), u_k\rangle \cdot \langle f, \tilde{\varphi}_m(\cdot - n)\rangle \\
&= \Big\langle \sum_{n\in\mathbb{Z}} \langle f, \tilde{\varphi}_m(\cdot - n)\rangle \varphi_m(\cdot - n), u_k \Big\rangle \\
&= \langle f, u_k\rangle.
\end{aligned}$$

Now (2.1) shows that $\{h_k : k \in \mathbb{Z}\}$ is a frame for V_m. Let $\{S_k : k \in \mathbb{Z}\}$ be the dual frame. Then for any $f \in V_m$, $f(x) = \sum_{k\in\mathbb{Z}}\langle f, h_k\rangle S_k(x) = \sum_{k\in\mathbb{Z}}\langle f, u_k\rangle S_k(x)$. The uniform convergence follows by Lemma 2.1. □

3. Proof of Theorem 1.1

First, we introduce a result by Aldroubi and Gröchenig. The following proposition is a special case of [1, Theorem 1].

Proposition 3.1. *Let $\{x_k : k \in \mathbb{Z}\}$ be a real sequence such that*

$$0 < \alpha < x_{k+1} - x_k < \beta < 1, \quad \forall k \in \mathbb{Z},$$

for some constants α and β. Then there exist positive constants C_1 and C_2 depending only on $m, \alpha,$ and β such that

$$C_1\|f\|_2^2 \leq \sum_{k\in\mathbb{Z}} |f(x_k)|^2 \leq C_2\|f\|_2^2, \quad \forall f \in V_m.$$

Lemma 3.2. $\inf_{f\in V_m} \frac{1}{\|f\|_2^2} \sum_{k\in\mathbb{Z}} |f(\frac{m}{2} + k)|^2 = 0.$

Proof. Since $\varphi_m(x)$ is symmetric with respect to $x = \frac{m+1}{2}$ and $\sum_{k\in\mathbb{Z}} \varphi_m(x-k) = 1$ for any $x \in \mathbb{R}$, it is easy to check that $Z\varphi_m(\frac{m}{2}, \pi) = 0$.

For any $\varepsilon > 0$, by the continuity of $Z\varphi_m(x, \omega)$, there exists some $0 < \delta < \pi$ such that

$$|Z\varphi_m(\frac{m}{2}, \omega)| \leq \varepsilon, \qquad |\omega - \pi| \leq \delta.$$

Let $C(\omega) = \sum_{k\in\mathbb{Z}} c_k e^{-ik\omega}$ be a 2π-periodic function such that $C(\omega) = 1$ for $\omega \in [\pi - \delta, \pi + \delta]$ and $C(\omega) = 0$ for $\omega \in [0, \pi - \delta] \cup [\pi + \delta, 2\pi]$. Define $\hat{f}(\omega) = C(\omega)\hat{\varphi}_m(\omega)$.

Then we have

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}}|f(\frac{m}{2}+k)|^2 = \frac{1}{2\pi}\int_0^{2\pi}|Zf(\frac{m}{2},\omega)|^2 d\omega \\
= \quad & \frac{1}{2\pi}\int_0^{2\pi}|C(\omega)Z\varphi_m(\frac{m}{2},\omega)|^2 d\omega \le \frac{\varepsilon^2}{2\pi}\int_{\pi-\delta}^{\pi+\delta}|C(\omega)|^2 d\omega = \varepsilon^2\sum_{k\in\mathbb{Z}}|c_k|^2.
\end{aligned}$$

Since $\{\varphi_m(\cdot - n) : n \in \mathbb{Z}\}$ is a Riesz basis for V_m with lower bound $A_m > 0$, the above inequalities show that $\sum_{k\in\mathbb{Z}}|f(\frac{m}{2}+k)|^2 < \frac{\varepsilon^2}{A_m}$. Since ε is arbitrary, the conclusion follows. □

On the other hand, since $\varphi_m(x)$ is compactly supported and absolutely continuous for any $m \ge 1$, we also have

Lemma 3.3. *For any $f \in V_m, f' \in L^2(\mathbb{R})$.*

We are now ready to give the proof of the main result.

Proof of Theorem 1.1. For any $f \in V_m$, let

$$f_1(x) = \frac{f(x) + \overline{f(x)}}{2}, \qquad f_2(x) = \frac{f(x) - \overline{f(x)}}{2i}.$$

Then $f_1, f_2 \in V_m$ and $f(x) = f_1(x) + if_2(x)$.

Since $f_p(x)$ is continuous on $\mathbb{R}$ and $u_k(x) \ge 0$, by the first mean value theorem, there are some $x_{p,k} \in [x_k - \frac{\delta}{2}, x_k + \frac{\delta}{2}]$ such that

$$\langle f_p, u_k\rangle = f_p(x_{p,k}), \quad p = 1, 2.$$

Put $0 < \alpha < \frac{1}{3}(1-\beta-\delta)$. Let

$$\begin{aligned}
n_0 &= 0,\\
n_k &= \begin{cases} \min\{n : n > n_{k-1}, x_n - x_{n_{k-1}} \ge \alpha\}, & k > 0,\\ \max\{n : n < n_{k+1}, x_{n_{k+1}} - x_n \ge \alpha\}, & k < 0.\end{cases}
\end{aligned}$$

Then

$$\alpha \le x_{n_k} - x_{n_{k-1}} \le \beta + \alpha, \quad k \in \mathbb{Z}.$$

For any $k \in \mathbb{Z}$, let j_k and j'_k be the minimum and maximum of $\{j : \frac{x_{n_{k-1}}+x_{n_k}}{2} < x_j \le \frac{x_{n_k}+x_{n_{k+1}}}{2}\}$ respectively. Then $\{j : j_k \le j \le j'_k\}_{k\in\mathbb{Z}}$ is a partition of $\mathbb{Z}$.

For any $n_k < j \le j'_k$, since $j'_k < n_{k+1}$, by the definition of n_k,

$$\min\{x_j - x_{n_k}, x_{n_{k+1}} - x_j\} < \alpha.$$

But $x_{n_{k+1}} - x_j \ge \frac{x_{n_{k+1}}-x_{n_k}}{2} \ge x_j - x_{n_k} > 0$, so $x_j - x_{n_k} < \alpha$. A similar argument shows that $x_{n_k} - x_j < \alpha$ for $j_k \le j < n_k$. Hence for any $j_k \le j \le j'_k$, $|x_j - x_{n_k}| < \alpha$. Therefore,

$$x_{p,j} \in [x_{n_k} - \alpha - \frac{\delta}{2}, x_{n_k} + \alpha + \frac{\delta}{2}], \quad j_k \le j \le j'_k.$$

It follows from the continuity of $f_p(x)$ and $\sum_{j_k\le j\le j'_k}(x_{j+1}-x_{j-1}) = x_{j'_k+1}-x_{j_k-1}+x_{j'_k}-x_{j_k}$ that there is some $y_{p,k}\in[x_{n_k}-\alpha-\frac{\delta}{2}, x_{n_k}+\alpha+\frac{\delta}{2}]$ such that

$$
\begin{aligned}
(3.1)\qquad & \sum_{j_k\le j\le j'_k}\frac{x_{j+1}-x_{j-1}}{2}|f_p(x_{p,j})|^2 \\
&= \frac{x_{j'_k+1}-x_{j_k-1}+x_{j'_k}-x_{j_k}}{2}\sum_{j_k\le j\le j'_k}\frac{x_{j+1}-x_{j-1}}{x_{j'_k+1}-x_{j_k-1}+x_{j'_k}-x_{j_k}}|f_p(x_{p,j})|^2 \\
&= \frac{x_{j'_k+1}-x_{j_k-1}+x_{j'_k}-x_{j_k}}{2}|f_p(y_{p,k})|^2.
\end{aligned}
$$

Let $\{z_{p,k} : k\in\mathbb{Z}\}$ be a rearrangement of $\{y_{p,k} : k\in\mathbb{Z}\}$ such that $z_{p,k}\le z_{p,k+1}$. Since $0<x_{n_{k+1}}-x_{n_k}\le\beta+\alpha$ and $|y_{p,k}-x_{n_k}|\le\alpha+\frac{\delta}{2}$, it is easy to check that every interval with length $\beta+\delta+3\alpha$ must contain at least one point of $\{y_{p,k} : k\in\mathbb{Z}\}$. Hence

$$0\le z_{p,k+1}-z_{p,k}\le\beta+\delta+3\alpha.$$

Since $\beta+\delta+3\alpha<1$, similar to the choice of x_{n_k} we can prove that there is a subsequence $\{z_{p,i_k} : k\in\mathbb{Z}\}$ of $\{z_{p,k} : k\in\mathbb{Z}\}$ such that

$$\varepsilon\le z_{p,i_{k+1}}-z_{p,i_k}\le\beta+\delta+3\alpha+\varepsilon<1$$

for some $\varepsilon>0$. By Proposition 3.1, there is a positive constant C_1 depending only on m,ε and $\beta+\delta+3\alpha+\varepsilon$ such that

$$\sum_{k\in\mathbb{Z}}|f(z_{p,i_k})|^2\ge C_1\|f\|_2^2,\quad \forall f\in V_m.$$

Hence

$$(3.2)\qquad \sum_{k\in\mathbb{Z}}|f_p(y_{p,k})|^2=\sum_{k\in\mathbb{Z}}|f_p(z_{p,k})|^2\ge C_1\|f_p\|_2^2.$$

Noting that

$$\frac{x_{j'_k+1}-x_{j_k-1}+x_{j'_k}-x_{j_k}}{2}\ge\frac{x_{j'_k+1}-x_{j_k-1}}{2}\ge\frac{x_{n_{k+1}}-x_{n_{k-1}}}{4}\ge\frac{\alpha}{2},$$

we derive from (3.1) and (3.2) that

$$
\begin{aligned}
(3.3)\ \sum_{k\in\mathbb{Z}}\frac{x_{k+1}-x_{k-1}}{2}|\langle f_p,u_k\rangle|^2 &= \sum_{k\in\mathbb{Z}}\frac{x_{k+1}-x_{k-1}}{2}|f_p(x_{p,k})|^2 \\
&= \sum_{k\in\mathbb{Z}}\sum_{j_k\le j\le j'_k}\frac{x_{j+1}-x_{j-1}}{2}|f_p(x_{p,j})|^2 \\
&= \sum_{k\in\mathbb{Z}}\frac{x_{j'_k+1}-x_{j_k-1}+x_{j'_k}-x_{j_k}}{2}|f_p(y_{p,k})|^2 \\
&\ge \frac{\alpha C_1}{2}\|f_p\|_2^2.
\end{aligned}
$$

On the other hand, it can be shown (see Section 4) that there is some constant C_2 such that

$$(3.4)\qquad \sum_{k\in\mathbb{Z}}\frac{x_{k+1}-x_{k-1}}{2}|\langle f_p,u_k\rangle|^2=\sum_{k\in\mathbb{Z}}\frac{x_{k+1}-x_{k-1}}{2}|f_p(x_{p,k})|^2\le C_2\|f_p\|_2^2.$$

Since $f(x) = f_1(x) + if_2(x)$ and f_1, f_2, u_k are real functions, we have $\|f\|_2^2 = \|f_1\|_2^2 + \|f_2\|_2^2$ and $|\langle f, u_k\rangle|^2 = |\langle f_1, u_k\rangle|^2 + |\langle f_2, u_k\rangle|^2$. Therefore, (3.3) and (3.4) hold if we substitute f for f_p. Now the first part of this theorem follows by Lemma 2.2.

Next we are to prove the second part. We only need to show that the conclusion fails if $\delta = 1 - \beta$.

First, we consider the case of $\frac{1}{2} \le \beta < 1$.

Let $x_{2k} = k + \frac{m-1}{2} - \frac{\beta}{2}$ and $x_{2k+1} = k + \frac{m-1}{2} + \frac{\beta}{2}$. Then $1-\beta \le x_{k+1} - x_k \le \beta$. Put $\varepsilon_k = \frac{1-\beta}{2+|k|}$. Choose some averaging functions $u_k(x)$ such that

$$\begin{aligned} \text{supp } u_{2k} &\subset [k-1+\frac{m}{2}, k-1+\frac{m}{2}+\varepsilon_k], \\ \text{supp } u_{2k+1} &\subset [k+\frac{m}{2}-\varepsilon_k, k+\frac{m}{2}]. \end{aligned}$$

Then supp $u_k \subset [x_k - \frac{\delta}{2}, x_k + \frac{\delta}{2}]$.

For any $\varepsilon > 0$, there exists some $f \in V_m$ such that

$$\sum_{k\in\mathbb{Z}} |f(\frac{m}{2}+k)|^2 \le \varepsilon \|f\|_2^2, \tag{3.5}$$

thanks to Lemma 3.2. Since $\varphi_m(x)$ is real-valued, we can assume that $f(x)$ is also real-valued. Let $f_n(x) = f(x-n), n \ge 1$. Then $f_n \in V_m$. By the first mean value theorem, there are some $x_{n,2k} \in [k-1+\frac{m}{2}, k-1+\frac{m}{2}+\varepsilon_k]$ and $x_{n,2k+1} \in [k+\frac{m}{2}-\varepsilon_k, k+\frac{m}{2}]$ such that

$$\langle f_n, u_k\rangle = f_n(x_{n,k}).$$

For any $p > 0$, we have

$$\begin{aligned} &\sum_{|k|\ge p} |f_n(x_{n,2k}) - f_n(k-1+\frac{m}{2})|^2 \\ &\quad = \sum_{|k|\ge p} \left| \int_0^{x_{n,2k}-k+1-\frac{m}{2}} f_n'(k-1+\frac{m}{2}+t)dt \right|^2 \\ &\quad \le \sum_{|k|\ge p} \varepsilon_k \int_0^{\varepsilon_k} \left| f_n'(k-1+\frac{m}{2}+t)\right|^2 dt \\ &\quad \le \varepsilon_p \int_0^{\varepsilon_p} \sum_{k\in\mathbb{Z}} \left| f_n'(k-1+\frac{m}{2}+t)\right|^2 dt \\ &\quad \le \varepsilon_p \|f_n'\|_2^2 = \varepsilon_p \|f'\|_2^2. \end{aligned}$$

It follows from (3.5) that

$$\sum_{|k|\ge p} |f_n(x_{n,2k})|^2 \le (\varepsilon_p^{1/2}\|f'\|_2 + \varepsilon^{1/2}\|f\|_2)^2. \tag{3.6}$$

On the other hand, since

$$\begin{aligned}\sum_{|k|\le n/2}|f_n(x_{n,2k})-f_n(k-1+\frac{m}{2})|^2 &\le \sum_{|k|\le n/2}\varepsilon_k\int_0^{\varepsilon_k}|f_n'(k-1+\frac{m}{2}+t)|^2dt\\ &\le \int_0^1\sum_{|k|\le n/2}|f_n'(k-1+\frac{m}{2}+t)|^2dt\\ &\le \int_0^1\sum_{|l|\ge n/2}|f'(l-1+\frac{m}{2}+t)|^2dt\\ &\le \int_{|t|\ge\frac{n-m}{2}}|f'(t)|^2dt\\ &:= \|T_{\frac{n-m}{2}}f'\|_2^2,\end{aligned}$$

we have

$$\text{(3.7)}\quad \begin{aligned}\left(\sum_{|k|\le n/2}|f_n(x_{n,2k})|^2\right)^{1/2} &\le \left(\sum_{k\in\mathbb{Z}}|f_n(k-1+\frac{m}{2})|^2\right)^{1/2}+\|T_{\frac{n-m}{2}}f'\|_2\\ &\le \varepsilon^{1/2}\|f\|_2+\|T_{\frac{n-m}{2}}f'\|_2.\end{aligned}$$

Putting (3.6) and (3.7) together, we have

$$\text{(3.8)}\quad \begin{aligned}\sum_{k\in\mathbb{Z}}|f_n(x_{n,2k})|^2 &= \sum_{|k|>n/2}|f_n(x_{n,2k})|^2+\sum_{|k|\le n/2}|f_n(x_{n,2k})|^2\\ &\le (\varepsilon_{n/2}^{1/2}\|f'\|_2+\varepsilon^{1/2}\|f\|_2)^2+(\varepsilon^{1/2}\|f\|_2+\|T_{\frac{n-m}{2}}f'\|_2)^2.\end{aligned}$$

A similar argument shows that (3.8) holds if we substitute $f_n(x_{n,2k+1})$ for $f_n(x_{n,2k})$. Therefore,

$$\sum_{k\in\mathbb{Z}}|\langle f_n,u_k\rangle|^2\le 2(\varepsilon_{n/2}^{1/2}\|f'\|_2+\varepsilon^{1/2}\|f\|_2)^2+2(\varepsilon^{1/2}\|f\|_2+\|T_{\frac{n-m}{2}}f'\|_2)^2.$$

Noting that $\|f_n\|_2=\|f\|_2$ for any $n\ge 1$, we have by the above inequality that

$$\limsup_{n\to\infty}\frac{1}{\|f_n\|_2^2}\sum_{k\in\mathbb{Z}}|\langle f_n,u_k\rangle|^2\le 4\varepsilon.$$

Since ε is arbitrary, we have $\inf_{f\in V_m}\frac{1}{\|f\|_2^2}\sum_{k\in\mathbb{Z}}|\langle f,u_k\rangle|^2=0$ and hence

$$\inf_{f\in V_m}\frac{1}{\|f\|_2^2}\sum_{k\in\mathbb{Z}}\frac{x_{k+1}-x_{k-1}}{2}|\langle f,u_k\rangle|^2=0.$$

For the general case, suppose that $\frac{1}{r+1}\le\beta<\frac{1}{r}$ for some integer $r\ge 1$. Let $x_{(r+1)k+l}=k+\frac{m-1}{2}+l\beta-\frac{\beta}{2},\quad 0\le l\le r$. Then $1-r\beta\le x_{k+1}-x_k\le\beta$. Define $\varepsilon_k=\frac{1-\beta}{2+|k|}$. Choose averaging functions $u_k(x)$ such that

$$\begin{aligned}\text{supp } u_{(r+1)k+l} &\subset [k+\frac{m}{2}-\varepsilon_k,k+\frac{m}{2}],\quad 1\le l\le\frac{r+1}{2},\\ \text{supp } u_{(r+1)k+l} &\subset [k+\frac{m}{2},k+\frac{m}{2}+\varepsilon_k],\quad \frac{r+1}{2}<l\le r+1.\end{aligned}$$

It is easy to check that supp $u_k \subset [x_k - \frac{\delta}{2}, x_k + \frac{\delta}{2}]$. Similar to the case $\frac{1}{2} \le \beta < 1$ we can prove that

$$\inf_{f \in V_m} \frac{1}{\|f\|_2^2} \sum_{k\in\mathbb{Z}} \frac{x_{k+1} - x_{k-1}}{2} |\langle f, u_k\rangle|^2 = 0. \tag{3.9}$$

If there is a frame $\{S_k : k \in \mathbb{Z}\}$ for V_m such that $f = \sum_{k\in\mathbb{Z}} (\frac{x_{k+1}-x_{k-1}}{2})^{1/2} \langle f, u_k\rangle S_k$ holds for any $f \in V_m$, by [9, Proposition 3.2.4], we have

$$\sum_{k\in\mathbb{Z}} \frac{x_{k+1} - x_{k-1}}{2} |\langle f, u_k\rangle|^2 \ge \sum_{k\in\mathbb{Z}} |\langle f, \tilde{S}_k\rangle|^2 \ge \tilde{A}\|f\|_2^2,$$

where $\{\tilde{S}_k(x) : k \in \mathbb{Z}\}$ is the dual frame of $\{S_k(x) : k \in \mathbb{Z}\}$ and $\tilde{A}$ is the lower frame bound. This is a contradiction to (3.9), which completes the proof. □

4. Iterative reconstruction algorithm

By Theorem 1.1, we can reconstruct any $f \in V_m$ from local averages by (1.1). However, it may be difficult to find the frame $\{S_k(x)\}$ in practice. In this case, the reconstruction of $f \in V_m$ from local averages can be performed by applying the following Feichtinger-Gröchenig algorithm.

Proposition 4.1 (see [13])**.** *Suppose that $\{\varphi_k : k \in \mathbb{Z}\}$ is a frame with bounds A and B for some Hilbert space $\mathcal{H}$. Let λ be a constant such that $0 < \lambda < \frac{2}{B}$. For any $f \in \mathcal{H}$, define*

$$\begin{aligned} Sf &= \lambda \sum_{k\in\mathbb{Z}} \langle f, \varphi_k\rangle \varphi_k, \\ f_0 &= Sf, \\ f_{k+1} &= f_k + S(f - f_k), \quad k \ge 0. \end{aligned}$$

Then $\lim_{k\to\infty} f_k = f$ *and* $\|f - f_k\| \le \gamma^{k+1}\|f\|$, *where* $\gamma = \max\{|1 - \lambda A|, |1 - \lambda B|\} < 1$.

For Theorem 1.1, the operator S is of a slightly different form:

$$Sf := \lambda \sum_{k\in\mathbb{Z}} \frac{x_{k+1} - x_{k-1}}{2} \langle f, u_k\rangle h_k,$$

where h_k is defined as in Lemma 2.2.

In the above iterative reconstruction algorithm, the relaxation parameter λ plays an important role. If the exact value of the frame bounds are known, then $\lambda = \frac{2}{A+B}$ will be the best one since γ will be minimized in this case. Unfortunately, we have only some estimates for the frame bounds in most cases. Whereas it is not difficult to get an estimate for the upper bound of the frame $\{(\frac{x_{k+1}-x_{k-1}}{2})^{1/2} h_k : k \in \mathbb{Z}\}$ (see below), for the lower bound only existence is known. To ensure the convergence of the iterative reconstruction algorithm, we have to choose a small λ, which in turn may be responsible for slow convergence.

To apply Proposition 4.1, an upper frame bound must be found. Let $\{x_k\}, \delta, \beta$, and $\{u_k\}$ be defined as in Theorem 1.1. Define $y_k = \frac{x_k + x_{k+1}}{2}$. We have

$$\begin{aligned}
&\left\| f - \sum_{k\in\mathbb{Z}} \langle f, u_k\rangle \chi_{[y_{k-1}, y_k]} \right\|_2^2 = \sum_{k\in\mathbb{Z}} \int_{y_{k-1}}^{y_k} |f(x) - \langle f, u_k\rangle|^2 dx \\
= &\sum_{k\in\mathbb{Z}} \int_{y_{k-1}}^{y_k} \left| \int_{x_k - \frac{\delta}{2}}^{x_k + \frac{\delta}{2}} [f(x) - f(t)] u_k(t) dt \right|^2 dx \\
\le &\sum_{k\in\mathbb{Z}} \int_{y_{k-1}}^{y_k} \int_{x_k - \frac{\delta}{2}}^{x_k + \frac{\delta}{2}} |f(x) - f(t)|^2 u_k(t) dt\ dx \\
= &\sum_{k\in\mathbb{Z}} \int_{y_{k-1}}^{y_k} \int_{x_k - \frac{\delta}{2}}^{x_k + \frac{\delta}{2}} \left| \int_0^{t-x} f'(x+s) ds \right|^2 u_k(t) dt\ dx \\
\le &\sum_{k\in\mathbb{Z}} \int_{y_{k-1}}^{y_k} \int_{x_k - \frac{\delta}{2}}^{x_k + \frac{\delta}{2}} \frac{\delta+\beta}{2} \int_{-\frac{\delta+\beta}{2}}^{\frac{\delta+\beta}{2}} |f'(x+s)|^2 u_k(t) ds\ dt\ dx \\
= &\frac{\delta+\beta}{2} \int_{-\frac{\delta+\beta}{2}}^{\frac{\delta+\beta}{2}} \sum_{k\in\mathbb{Z}} \int_{y_{k-1}}^{y_k} |f'(x+s)|^2 dx\ ds = \frac{(\delta+\beta)^2}{2} \|f'\|_2^2.
\end{aligned}$$

On the other hand, it can be shown (e.g., see [16]) that $\|f'\|_2^2 \le \frac{M_m}{A_m}\|f\|_2^2$ with

$$\begin{aligned}
M_m &= \sup_{|\omega|\le\frac{\pi}{2}} \sum_{k\in\mathbb{Z}} |2\omega + 2k\pi|^2 \left| \frac{\sin\omega}{\omega + k\pi} \right|^{2m+2} \le M_1 = 4, \\
A_m &= \inf_{|\omega|\le\frac{\pi}{2}} \sum_{k\in\mathbb{Z}} \left| \frac{\sin\omega}{\omega + k\pi} \right|^{2m+2} = \left(\frac{2}{\pi}\right)^{2m+2} \sum_{k\in\mathbb{Z}} \frac{1}{(2k+1)^{2m+2}}.
\end{aligned}$$

Hence

$$\sum_{k\in\mathbb{Z}} \frac{x_{k+1} - x_{k-1}}{2} |\langle f, u_k\rangle|^2 \le \left(1 + \frac{(\delta+\beta)\sqrt{M_m}}{\sqrt{2A_m}}\right)^2 \|f\|_2^2 := B\|f\|_2^2.$$

Therefore, we can choose $0 < \lambda < \frac{2}{B}$ to ensure Proposition 4.1 works. However, we cannot give the convergence rate due to the absence of the lower frame bound.

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