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ON A CHARACTERIZATION OF THE MAXIMAL IDEAL SPACES OF ALGEBRAICALLY CLOSED COMMUTATIVE C^*-ALGEBRAS

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Abstract. Let $C(X)$ be the algebra of all complex-valued continuous functions on a compact Hausdorff space X. We say that $C(X)$ is algebraically closed if each monic polynomial equation over $C(X)$ has a continuous solution. We give a necessary and sufficient condition for $C(X)$ to be algebraically closed for a locally connected compact Hausdorff space X. In this case, it is proved that $C(X)$ is algebraically closed if each element of $C(X)$ is the square of another. We also give a characterization of a first-countable compact Hausdorff space X such that $C(X)$ is algebraically closed.

1. Introduction

Let X be a compact Hausdorff space and $C(X)$ the commutative Banach algebra of all complex-valued continuous functions on X. Let $P(x,z)$ be a monic polynomial over $C(X)$. That is, $P(x,z) = z^n + a_{n-1}(x)z^{n-1} + \cdots + a_1(x)z + a_0(x)$ $(x \in X)$ for some natural number n and some $a_0, a_1, \cdots, a_{n-1} \in C(X)$. We say that $C(X)$ is algebraically closed, if $P(x,z) = 0$ has a root in $C(X)$ for every monic polynomial $P(x,z)$ over $C(X)$; that is, there exists an $f \in C(X)$ such that $P(x, f(x)) = 0$ on X. If to every $a_0 \in C(X)$ there corresponds a $g \in C(X)$ so that $a_0(x) = g^2(x)$ for every $x \in X$, then we say that $C(X)$ is square-root closed. By definition, if $C(X)$ is algebraically closed then the algebra is square-root closed.

Deckard and Pearcy [3] prove that $C(X)$ is algebraically closed if X is a Stonian space: a compact Hausdorff space which is also extremely disconnected. In [4] they consider two spaces: a totally disconnected compact Hausdorff space; a linearly ordered and order-complete topological space. In both cases, algebraic closedness is proved. It is also noted that $C(X)$ need not be square-root closed. Indeed, there is no continuous function on S^1, the unit circle in the complex plane $\mathbb{C}$, whose square is the identity function on S^1 (cf. [2, Lemma 2.1]). On the other hand, $C([0,1])$ is algebraically closed by [3, Theorem 2]. Here $[0,1]$ denotes the closed unit interval.

Countryman [2] gives some necessary and sufficient conditions for a first-countable compact Hausdorff space X in order that $C(X)$ be algebraically closed. In this case, $C(X)$ is algebraically closed if and only if X is hereditarily unicoherent

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and almost locally connected; intuitively X contains neither S^1 nor the closure of $\bigcup_{n\in\mathbb{N}}(\{1/n\}\times[0,1])$ in $\mathbb{R}^2$ with its usual topology. Here and after, $\mathbb{N}$ and $\mathbb{R}$ denote the space of all natural numbers and that of real numbers, respectively. It is also proved, for such X, that $C(X)$ is algebraically closed if the algebra is square-root closed. On the other hand, this is not the case unless X is first-countable. In fact, he gives a compact Hausdorff space X that is not first-countable with the following property: each function of $C(X)$ has a continuous 2^nth root for every $n \in \mathbb{N}$, while some function has no continuous fifth root ([2, Remark (3)]).

Recall that a uniform algebra on a compact Hausdorff space X is a uniformly closed subalgebra of $C(X)$ which contains the constants and separates the points of X. Let A be a uniform algebra on a locally connected compact Hausdorff space X. Čirka [1] proved that $A = C(X)$, if each function of A is the square of another. As noted above, $C(X)$ need not be square-root closed even if X is locally connected. In [6], a characterization is given of a locally connected compact Hausdorff space X such that $C(X)$ be square-root closed: A necessary and sufficient condition in order that $C(X)$ be square-root closed is as follows.

(♯) The covering dimension of X is less than or equal to 1, and the first Čech cohomology group with integer coefficient is trivial.

The condition (♯) need not be necessary nor sufficient for $C(X)$ being square-root closed, unless X is locally connected. On the other hand, if X satisfies (♯) then $C(X)$ is "almost" square-root closed, even if X is not locally connected, in the following sense: For every $\varepsilon > 0$ and every $f \in C(X)$ with $\|f\|_\infty \leq 1$, there exist $g, h \in C(X)$ such that $f = gh$ and $\|g-h\|_\infty \leq \varepsilon$. Here $\|\cdot\|_\infty$ denotes the supremum norm on X (cf. [8, Theorem 2.1]).

In this paper we consider a locally connected compact Hausdorff space X which need not be first-countable. In a way similar to the arguments in [2], we show that $C(X)$ is algebraically closed if and only if $C(X)$ is square-root closed. They are also equivalent to the condition that X is hereditarily unicoherent. When these conditions are satisfied, then X has a base for the topology whose elements have at most finitely many boundary points. In terms of the covering dimension and the Čech cohomology group, we also give a characterization of a first-countable compact Hausdorff space X for which $C(X)$ is algebraically closed.

2. Preliminaries

We state characterization theorems obtained in [2] and [6]. To do this we need some terminology.

We say that a point p of a topological space T is of finite order, if for every open neighborhood V of p there exists an open set V_0 such that $p \in V_0 \subset V$ and that V_0 has at most finitely many boundary points. In this paper we say that a compact Hausdorff space X is an A-space, if each point of X is of finite order. Note that the definition of an A-space is not the same in [2]. However, for compact Hausdorff spaces, the two definitions are shown to be equivalent; see [2, p. 440].

A C-space is a compact Hausdorff space X such that $C(X_\lambda)$ is algebraically closed for every connected component X_λ. For simplicity we say that a compact Hausdorff space X is an AC-space, if X is both an A-space and a C-space.

We say that a topological space T is almost locally connected, if T contains no mutually disjoint connected closed subsets C_n $(n \in \mathbb{N})$, which are open in the

closure of $\bigcup_{n\in\mathbb{N}} C_n$, with the following property:

$$x_n, y_n \in C_n \text{ and } \{x_n\}_{n\in\mathbb{N}} \text{ and } \{y_n\}_{n\in\mathbb{N}} \text{ converge to distinct points.}$$

A topological space T is hereditarily unicoherent, if $M \cap N$ is connected for all connected closed subsets M, N of T.

For example, the unit interval $[0,1]$ is an AC-space, almost locally connected and hereditarily unicoherent. The closure of $\bigcup_{n\in\mathbb{N}}(\{1/n\} \times [0,1])$ is neither an A-space nor almost locally connected. Note that a locally connected topological space need not be almost locally connected. On the other hand, any topological space that contains a homeomorphic image of the unit circle S^1 is neither a C-space nor hereditarily unicoherent.

Let Y be a normal space, and n a non-negative integer. We say that the covering dimension $\dim Y$ of Y is less than or equal to n, if for every finite open covering $\mathfrak{A}$ of Y there exists a refinement $\mathfrak{B}$ of $\mathfrak{A}$ such that each $y \in Y$ is in at most $(n+1)$ elements of $\mathfrak{B}$. It is well known that $\dim Y \leq n$ if and only if for every closed subset F of Y and every S^n-valued continuous function f on F, there exists an S^n-valued continuous function $\tilde{f}$ on Y such that $\tilde{f}|_F = f$ (cf. [9]). Here S^n denotes the n-sphere. Let m be a natural number. We say that $\dim Y = m$, if $\dim Y \leq m$ and if $\dim Y \not\leq m-1$. For example, the covering dimension of $\mathbb{R}^m$ is m.

Let X be a compact Hausdorff space. Then $\check{H}^1(X,\mathbb{Z})$ denotes the first Čech cohomology group of X with integer coefficients. Here we are not concerned with the definition of $\check{H}^1(X,\mathbb{Z})$. Let $C(X)^{-1}$ be the multiplicative group of all invertible elements of $C(X)$ and $\exp C(X) = \{e^f : f \in C(X)\}$. It is well known that $\check{H}^1(X,\mathbb{Z})$ is isomorphic to the quotient group $C(X)^{-1}/\exp C(X)$, by a theorem of Arens and Royden (cf. [5, Theorem 7.2 of Chapter III]).

This completes our preparation for stating characterization theorems for a compact Hausdorff space X in order that $C(X)$ be algebraically closed or square-root closed.

Theorem A ([2]). *Let X be a compact Hausdorff space. For the following conditions,* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) *holds:*

(i) *X is an AC-space.*
(ii) *$C(X)$ is algebraically closed.*
(iii) *$C(X)$ is square-root closed.*
(iv) *X is almost locally connected and hereditarily unicoherent.*

Moreover if X is first-countable, then the condition (iv) *implies* (i); *that is, the conditions from* (i) *to* (iv) *are all equivalent.*

Theorem B (Theorem 2.2, [6]). *Let X be a locally connected compact Hausdorff space. Then the following conditions are equivalent.*

(i) *$C(X)$ is square-root closed.*
(ii) $\dim X \leq 1$ *and* $\check{H}^1(X,\mathbb{Z}) = 0$.

To prove our main result, we need the following lemma.

Lemma C (Lemma 2.2, [3]). *Let X be a compact Hausdorff space, and $P(x,z)$ a monic polynomial over $C(X)$. Suppose that $x_0 \in X$ and that $z_0 \in \mathbb{C}$ is a root of $P(x_0,z) = 0$ of multiplicity m. Let $\varepsilon > 0$ so that $P(x_0,z) = 0$ has no root in $\{z \in \mathbb{C} : 0 < |z - z_0| \leq \varepsilon\}$. Then there exists an open neighborhood V_0 of x_0 such that $P(y,z) = 0$ has exactly m roots, counting multiplicities, in $\{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$ for every $y \in V_0$.*

3. Main results

It seems natural to expect that the conditions of Theorem A have deep connections with those of Theorem B. In fact, the following result holds.

Lemma 3.1. *Let X be a compact Hausdorff space with* $\dim X \leq 1$. *If* $\check{H}^1(X, \mathbb{Z}) = 0$ *then X is hereditarily unicoherent.*

Proof. Suppose that X is not hereditarily unicoherent. We show that $\check{H}^1(X, \mathbb{Z}) \neq 0$. In a way similar to the proof of [2, Lemma 2.1] we see that there exist a closed subset F of X and an $h \in C(F)^{-1}$, the set of all invertible elements of $C(F)$, such that $h \neq g^2$ for all $g \in C(F)^{-1}$. Put $h_1 = h/|h| \in C(F)^{-1}$. Since $\dim X \leq 1$, we can find an $\tilde{h_1} \in C(X)^{-1}$ such that $|\tilde{h_1}| = 1$ and $\tilde{h_1}|_F = h_1$. Then there is no function $f \in C(X)$ such that $\tilde{h_1} = f^2$. In particular, $\tilde{h_1}$ does not belong to $\exp C(X)$. Therefore, we have $C(X)^{-1} \setminus \exp C(X) \neq \emptyset$. This implies $\check{H}^1(X, \mathbb{Z}) \neq 0$, by a theorem of Arens and Royden. This completes the proof. □

Let T be a connected topological space and p a point of T. Recall that p separates the distinct points a and b of $T \setminus \{p\}$ in T, if there exist disjoint open sets A and B such that $a \in A$, $b \in B$ and $T \setminus \{p\} = A \cup B$. In this case both $A \cup \{p\}$ and $B \cup \{p\}$ are connected. If p belongs to every connected closed subset that contains both a and b, then we say that p cuts T between a and b. If X is a locally connected and connected compact Hausdorff space, then by [7, Theorem 3-6], a point $p \in X$ separates the distinct points a and b of $X \setminus \{p\}$ in X, if and only if p cuts X between a and b.

Let X be a connected compact Hausdorff space and $a, b \in X$. It is well known that there exists a minimal connected closed subset of X, with respect to the set inclusion, that contains both a and b (cf. [7, Theorem 2-10]). If X is also hereditarily unicoherent, then we see that such a set is uniquely determined. In this case, $E[a, b]$ denotes the smallest connected closed subset containing both a and b. Therefore, each point of $E[a, b] \setminus \{a, b\}$ cuts X between a and b, and hence separates a and b in X by [7, Theorem 3-6], if X is a locally connected and connected compact Hausdorff space which is also hereditarily unicoherent.

The separation order $\leq$ in $E[a, b]$ is defined as follows: for every distinct points $p, q \in E[a, b]$ we define $p < q$ if $p = a$ or if p separates a and q in X. Then $x \leq y$ means that $x = y$ or $x < y$. It is well known that the separation order in $E[a, b]$ is a total order [7, Theorem 2-21]. Then the order topology in $E[a, b]$ is defined by a base for the topology: the full space $E[a, b]$; for every $x \in E[a, b]$ the set of all y with $y < x$ and the set of y with $x < y$; for every $x < y$ the set of all z with $x < z < y$. By [7, Theorem 2-25], the order topology defined by the total order $\leq$ in $E[a, b]$ is the same as the relative topology in $E[a, b]$.

By [7, Theorem 2-26], we see that each non-empty subset of $E[a, b]$, which is bounded above, has a least upper bound with respect to the separation order. That is, $E[a, b]$ is order-complete. These facts will be used later.

Lemma 3.2. *Let X be a locally connected compact Hausdorff space which is also hereditarily unicoherent. Then X is an A-space.*

Proof. Since X is locally connected, each connected component of X is open. Thus we may assume that X is connected. Let x_0 be any point of X and V any open neighborhood of x_0. It is enough to consider the case that $X \setminus V \neq \emptyset$. For every $x \in X \setminus V$ let $E[x_0, x]$ be the smallest connected closed subset containing x_0 and x.

Fix any point of $V \cap (E[x_0, x] \setminus \{x_0, x\})$, say $y(x)$. Then $y(x)$ separates x_0 and x in X as noted above. That is, there exist disjoint open sets A_x, B_x such that $x_0 \in A_x$, $x \in B_x$ and $X \setminus \{y(x)\} = A_x \cup B_x$. Note that $y(x)$ is the only boundary point of A_x.

Since $X \setminus V$ is compact, there are finitely many points $x_1, x_2, \cdots, x_n \in X \setminus V$ so that $X \setminus V \subset \bigcup_{j=1}^n B_{x_j}$. Put $V_0 = \bigcap_{j=1}^n A_{x_j}$. Then V_0 is an open set with $x_0 \in V_0 \subset V$. Since V_0 has at most n boundary points, the point x_0 is of finite order. That is, X is an A-space and this completes the proof. □

Theorem 3.3. *Let X be a locally connected compact Hausdorff space. Then the following conditions are equivalent.*

(i) *X is an AC-space.*
(ii) *X is a C-space.*
(iii) *$C(X)$ is algebraically closed.*
(iv) *$C(X)$ is square-root closed.*
(v) *$\dim X \leq 1$ and $\check{H}^1(X, \mathbb{Z}) = 0$.*
(vi) *X is hereditarily unicoherent.*

Proof. Since X is locally connected, each connected component of X is open. Therefore, $C(X)$ is algebraically closed if X is a C-space. That is, (ii) implies (iii). By Theorem B and Lemma 3.1, it is enough to show that (vi) implies (i). By Lemma 3.2, it suffices to prove that X is a C-space. To do this, without loss of generality we may assume that X is a compact Hausdorff space which is locally connected and connected.

Let $P(x, z)$ be any monic polynomial over $C(X)$. Let $\mathfrak{D}$ be the set of all pairs (D, f) of $D \subset X$ and a complex-valued continuous function f on D with the following properties:

$$E[a, b] \subset D \text{ for every } a, b \in D \text{ and } P(x, f(x)) = 0 \quad (x \in D).$$

Note that each such set D is connected since $E[a, b] \subset D\,(a, b \in D)$. For every $(D_1, f_1), (D_2, f_2) \in \mathfrak{D}$ we define $(D_1, f_1) \preceq (D_2, f_2)$, if $D_1 \subset D_2$ and $f_2|_{D_1} = f_1$. Then $\preceq$ is a partial order in $\mathfrak{D}$. We denote $(D_1, f_1) \prec (D_2, f_2)$, if $(D_1, f_1) \preceq (D_2, f_2)$ and $D_1 \subsetneq D_2$.

We show that $\mathfrak{D}$ has a maximal element. To do this, let $\{(D_\lambda, f_\lambda)\}_{\lambda \in \Lambda}$ be any chain of $\mathfrak{D}$. Put $D_0 = \bigcup_{\lambda \in \Lambda} D_\lambda$; then it is elementary that $a, b \in D_0$ implies $E[a, b] \subset D_0$. The function f_0 on D_0 defined by $x \mapsto f_\lambda(x)$ is well defined, where $\lambda \in \Lambda$ so that $x \in D_\lambda$. By the definition of D_0 and f_0, we have $P(x, f_0(x)) = 0$ for every $x \in D_0$. We show that f_0 is continuous on D_0. Assume to the contrary that f_0 is not continuous on D_0. That is, there exist an $x_0 \in D_0$ and an $\varepsilon_0 > 0$ such that $f_0(D_0 \cap V) \not\subset \{z \in \mathbb{C} : |z - f_0(x_0)| < \varepsilon_0\}$ for every open neighborhood V of x_0. Let $z_1, z_2, \cdots, z_k$ be all the distinct roots of $P(x_0, z) = 0$. Put $2\varepsilon_1 = \min\{|z_s - z_t| : 1 \leq s < t \leq k\}$ and $\varepsilon = \min\{\varepsilon_0, \varepsilon_1\}$. If we apply Lemma C to each z_i for $1 \leq i \leq k$, we can find a connected open neighborhood $V(x_0)$ of x_0 such that $P(y, w) = 0$ implies $w \in \bigcup_{i=1}^k \{z \in \mathbb{C} : |z - z_i| < \varepsilon\}$ for every $y \in V(x_0)$. Since $f_0(D_0 \cap V(x_0)) \not\subset \{z \in \mathbb{C} : |z - f_0(x_0)| < \varepsilon_0\}$, there is a $y_0 \in D_0 \cap V(x_0)$ such that $|f_0(y_0) - f_0(x_0)| \geq \varepsilon_0$. Let μ be an element of Λ such that $x_0, y_0 \in D_\mu$. Since each point of $E[x_0, y_0] \setminus \{x_0, y_0\}$ separates x_0 and y_0 in X, we see that $E[x_0, y_0] \subset D_\mu \cap V(x_0)$. Therefore, we obtain $f_0(E[x_0, y_0]) = f_\mu(E[x_0, y_0]) \subset \bigcup_{i=1}^k \{z \in \mathbb{C} : |z - z_i| < \varepsilon\}$. On the other hand, $f_0(E[x_0, y_0])$ meets at least two disks of $\bigcup_{i=1}^k \{z \in \mathbb{C} : |z - z_i| < \varepsilon\}$, since

$|f_0(y_0) - f_0(x_0)| \geq \varepsilon_0$. This contradicts that the range $f_\mu(E[x_0, y_0])$ is connected. Hence, f_0 is continuous on D_0. By Zorn's lemma we see that $\mathfrak{D}$ has a maximal element.

Let (D^*, f^*) be a maximal element of $\mathfrak{D}$. Then we show that $D^* = X$. Assume to the contrary that $X \setminus D^* \neq \emptyset$. Then there exists a $b \in X \setminus D^*$. Fix any element $a \in D^*$. Put m be the least upper bound of $E[a, b] \cap D^*$ with respect to the separation order in $E[a, b]$. Clearly, we see that $E[a, m] \setminus \{m\} \subset D^*$ and $E[m, b] \setminus \{m\} \subset X \setminus D^*$. We show that $m \in D^*$. Suppose $m \in X \setminus D^*$; then we see that there is no continuous extension of f^* to $D^* \cup \{m\}$. Suppose that there exists a continuous extension of f^*, say $\tilde{f}^*$. It follows that $P(x, \tilde{f}^*(x)) = 0$ for every $x \in D^* \cup \{m\}$, since the function $x \mapsto P(x, \tilde{f}^*(x))$ is continuous on the connected set $D^* \cup \{m\}$ and is identically 0 on D^*. Therefore, we have $(D^*, f^*) \prec (D^* \cup \{m\}, \tilde{f}^*)$. This contradicts the maximality of (D^*, f^*). Hence we have that there is no continuous extension of f^* to $D^* \cup \{m\}$. Therefore, there exists a $\delta_0 > 0$ with the following property:

> To every open neighborhood V of m there correspond $p, q \in V \cap D^*$ so that $|f^*(p) - f^*(q)| \geq \delta_0$.

Let $\eta_1, \eta_2, \cdots, \eta_l$ be all the distinct roots of $P(m, z) = 0$. Put $2\delta_1 = \min\{|\eta_s - \eta_t| : 1 \leq s < t \leq l\}$ and $\delta = \min\{\delta_0, \delta_1\}$. Then by Lemma C, there exists a connected open neighborhood $V(m)$ of m such that $P(y, w) = 0$ implies $w \in \bigcup_{j=1}^{l} \{z \in \mathbb{C} : |z - \eta_j| < \delta\}$ for every $y \in V(m)$. Therefore, $f^*(D^* \cap V(m))$ is contained in $\bigcup_{j=1}^{l} \{z \in \mathbb{C} : |z - \eta_j| < \delta\}$. Since each point of $E[x, y] \setminus \{x, y\}$ separates x and y in X, we have $E[x, y] \subset V(m)$ if $x, y \in V(m)$. Hence $E[x, y] \subset D^* \cap V(m)$ for every $x, y \in D^* \cap V(m)$. This implies that $D^* \cap V(m)$ is connected. Since f^* is continuous, the range $f^*(D^* \cap V(m))$ is also connected. Recall that there is no continuous extension of f^*, thus we can find $p, q \in D^* \cap V(m)$ with $|f^*(p) - f^*(q)| \geq \delta_0$. Therefore, $f^*(D^* \cap V(m))$ meets at least two disks of $\bigcup_{j=1}^{l} \{z \in \mathbb{C} : |z - \eta_j| < \delta\}$, though $f^*(D^* \cap V(m))$ is connected. We arrived at a contradiction, hence $m \in D^*$ is proved.

Finally, since $E[m, b]$ is a totally ordered and order-complete space, there exist $f_1, f_2, \cdots, f_n \in C(E[m, b])$ such that

$$P(x, z) = (z - f_1(x))(z - f_2(x)) \cdots (z - f_n(x)) \quad (x \in E[m, b]),$$

by [4, Theorem 3]. Without loss of generality we may assume $f^*(m) = f_1(m)$. Put $\tilde{D} = D^* \cup E[m, b]$; then the function

$$\tilde{f}(x) = \begin{cases} f^*(x), & (x \in D^*) \\ f_1(x), & (x \in E[m, b]) \end{cases}$$

is well defined. Note that both $D^* \setminus \{m\}$ and $E[m, b] \setminus \{m\}$ are open subsets of $\tilde{D}$, since m separates a and b in X if $m \neq a$. As a result we see that $\tilde{f}$ is continuous on $\tilde{D}$. Hence we obtain $(D^*, f^*) \prec (\tilde{D}, \tilde{f})$. This contradicts the maximality of (D^*, f^*). Therefore, we have $D^* = X$ and this completes the proof. □

Theorem 3.4. *Let X be a first-countable compact Hausdorff space. Then the following conditions are equivalent.*

(i) *X is an AC-space.*
(ii) *$C(X)$ is algebraically closed.*

(iii) *$C(X)$ is square-root closed.*
(iv) *X is almost locally connected and hereditarily unicoherent.*
(v) *X is almost locally connected and for every connected component X_λ of X, X_λ is locally connected,* $\dim X_\lambda \leq 1$ *and* $\check{H}^1(X_\lambda, \mathbb{Z}) = 0$.

Proof. It is enough to show that (v) is equivalent to (iv). Suppose that (iv) is true. Let X_λ be an arbitrary connected component of X. Note that X_λ is almost locally connected and hereditarily unicoherent. Therefore, we see that X_λ is locally connected by [2, Proof of Lemma 2.5]. Recall that $C(X_\lambda)$ is square-root closed by Theorem A. Therefore, we have $\dim X_\lambda \leq 1$ and $\check{H}^1(X_\lambda, \mathbb{Z}) = 0$ by Theorem B. This implies that (iv) $\Rightarrow$ (v).

Conversely, suppose that (v) holds. Then each connected component of X is hereditarily unicoherent, by Lemma 3.1. Thus X is also hereditarily unicoherent. Therefore we have that (v) implies (iv). This completes the proof. □

Example 3.1. $C(X)$ being algebraically closed need not imply $\check{H}^1(X, \mathbb{Z}) = 0$ unless X is locally connected. In fact, let X be the Stone-Čech compactification of the space $\mathbb{N} \times [0, 1]$ with its usual topology. Then $\dim X \leq 1$ and $\check{H}^1(X, \mathbb{Z}) \neq 0$ (cf. [6, p. 1188, (i)]). By a simple calculation, we see that $C(X)$ is algebraically closed.

Note that X is hereditarily unicoherent by Theorem A. Therefore, this example also shows that the converse need not be true in Lemma 3.1.

Example 3.2. Lemma 3.2 need not be true, unless X is locally connected. In fact, let X be the closure of $\bigcup_{n\in\mathbb{N}}(\{1/n\} \times [0, 1])$ with its usual topology. Since each connected component is hereditarily unicoherent, so is X. On the other hand, we see that no point of $\{0\} \times [0, 1]$ is of finite order. Hence X is not an A-space.

Example 3.3. In Theorem 3.4, the condition (ii) need not imply (v) unless X is first-countable. In fact, let X be the Stone-Čech compactification of $\mathbb{R}$. Then by a simple calculation, we see that $C(X)$ is algebraically closed (cf. [4, Corollary]). By [9, Theorem 5.14] we have $\dim X = 1$. On the other hand, X is connected but is not locally connected (cf. [10, p. 221, Theorem]). Also $\check{H}^1(X, \mathbb{Z}) \neq 0$ since the function e^{ix} $(x \in \mathbb{R})$ is extended to X.

Example 3.4. As stated above, $C(X)$ is algebraically closed, if and only if $C(X)$ is square-root closed for a compact Hausdorff space X that is first-countable or locally connected. On the other hand, by the example in [2, Remark (3)], we see that it is not the case if X is neither first-countable nor locally connected.

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