ON STABLE EQUIVALENCES OF MORITA TYPE 
FOR FINITE DIMENSIONAL ALGEBRAS

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Abstract. In this paper, we assume that algebras are finite dimensional algebras with 1 over a fixed field \( k \) and modules over an algebra are finitely generated left unitary modules. Let \( A \) and \( B \) be two algebras (where \( k \) is a splitting field for \( A \) and \( B \)) with no semisimple summands. If two bimodules \( AM \) and \( BN \) induce a stable equivalence of Morita type between \( A \) and \( B \), and if \( N \otimes A \) – maps any simple \( A \)-module to a simple \( B \)-module, then \( N \otimes A \) – is a Morita equivalence. This conclusion generalizes Linckelmann’s result for selfinjective algebras. Our proof here is based on the construction of almost split sequences.

1. Introduction

Given two finite dimensional \( k \)-algebras \( A \) and \( B \), suppose that \( M \) is an \( A-B \)-bimodule and \( N \) is a \( B-A \)-bimodule. Following [2] we say that \( M \) and \( N \) induce a stable equivalence of Morita type between \( A \) and \( B \) if \( M \) and \( N \) are projective both as left and right modules, and if

\[ M \otimes_B N \cong A \oplus P \]

as \( A-A \)-bimodules, where \( P \) is a projective \( A-A \)-bimodule, and

\[ N \otimes_A M \cong B \oplus Q \]

as \( B-B \)-bimodules, where \( Q \) is a projective \( B-B \)-bimodule.

Since the projective \( A-A \)-bimodule \( P \) tensoring any \( A \)-module is a projective \( A \)-module and since the projective \( B-B \)-bimodule \( Q \) tensoring any \( B \)-module is a projective \( B \)-module (see Lemma 2.1), the functor \( N \otimes_A \) – (defined by the above bimodule \( N \) induces a stable equivalence between the stable categories \( \text{mod} A \) and \( \text{mod} B \), and \( M \otimes_B \) – induces its quasi-inverse. Note that a stable equivalence \( \alpha : \text{mod} A \rightarrow \text{mod} B \) gives a one-to-one correspondence between the isomorphism classes of indecomposable non-projective modules in \( \text{mod} A \) and \( \text{mod} B \) (see [1, Proposition 1.1, p. 336]).

Important examples of stable equivalences of Morita type are selfinjective algebras which are derived equivalent [7, Corollary 5.5]. Linckelmann in [5] proved that, for two selfinjective algebras \( A \) and \( B \) having no simple projective modules, if there is an exact functor \( F \) which induces a stable equivalence between \( A \) and

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Let $A$ and $B$ be finite dimensional $k$-algebras (where $k$ is a splitting field for $A$ and $B$) with no semisimple summands. If two bimodules $AM_B$ and $BN_A$ induce a stable equivalence of Morita type between $A$ and $B$, and if $N \otimes_A -$ maps any simple $A$-module to a simple $B$-module, then $N \otimes_A -$ is a Morita equivalence.

2. Preliminaries

For an algebra $A$, we denote by $\text{mod} A$ and by $\underline{\text{mod}} A$ the category of finitely generated left $A$-modules and its stable category, respectively. Note that $\text{mod} A$ is a Krull-Schmidt category. For $X$ in $\text{mod} A$, we define the top of $X$ by $\text{top}(X) = X/\text{rad}(X)$, where $\text{rad}(X)$ is the radical of $X$. Recall that an $A$-module $X$ is called a generator if the regular module $\underline{A}A$ is a direct summand of a finite direct sum of $X$.

Suppose that two algebras $A$ and $B$ are stably equivalent of Morita type. We can define functors $T_M : \text{mod} B \to \text{mod} A \text{ by } X \mapsto M \otimes_B X$ and $T_N : \text{mod} A \to \text{mod} B \text{ by } Y \mapsto N \otimes_A Y$. Similarly, we have the functors $T_P$ and $T_Q$. Related to these functors, we have the following two lemmas.

Lemma 2.1 (see [10] Theorem 4.1). (1) $T_M$, $T_N$, $T_P$ and $T_Q$ are exact functors.
(2) $T_M \circ T_N \to \text{id}_{\text{mod} A} \oplus T_P$ and $T_N \circ T_M \to \text{id}_{\text{mod} B} \oplus T_Q$ are natural isomorphisms.
(3) The images of $T_P$ and $T_Q$ consist of projective modules.

Lemma 2.2. The bimodules $AM_B$ and $BN_A$ are projective generators both as left and right modules. Therefore $T_M$ and $T_N$ are faithful functors.

Proof. We prove that $\underline{A}M$ is a generator. Since $BN$ is projective, there is a natural number $n$ such that $BN$ is a direct summand of $BB^n$. It follows that $\underline{A}M \otimes_B BN \cong_{\underline{A}M} (\underline{A}M \oplus P)$ is a direct summand of $\underline{A}M \otimes_B BB^n \cong_{\underline{A}M} M^n$. Therefore $\underline{A}M$ is a direct summand of $\underline{A}M^n$; this implies that $\underline{A}M$ is a generator in $\text{mod} A$.

For an $A$-module $X$, we have a unique (up to isomorphism) decomposition $X = X_1 \oplus X'$, where $X_1$ has no nonzero projective summands and $X'$ is projective. We call $X_1$ the non-projective part of $X$. The following lemma is obvious.

Lemma 2.3. If $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is exact in $\text{mod} A$, then there is an exact sequence $0 \to X \xrightarrow{f_1} Y_0 \xrightarrow{g_1} Z_1 \to 0$, where
where $Z \cong Z_1 \oplus Z'$, $Y \cong Y_0 \oplus Z'$, $f = \begin{bmatrix} f_1 \\ 0 \end{bmatrix}$, $g = \begin{bmatrix} g_1 & g_2 \\ 0 & g_3 \end{bmatrix}$, and $g_3$ is an isomorphism. Moreover, $g$ is a split epimorphism if and only if $g_1$ is a split epimorphism. 

Lemma 2.4. Let $0 \to X \xrightarrow{i} Y \to Z \to 0$ be an exact sequence in $\text{modA}$. If $X$ is a simple module and $Z$ has simple top, then either $Y$ has simple top or $Y \cong X \oplus Z$.

Proof. If $i(X)$ lies in $\text{rad}(Y)$, then $\text{top}(Y) = \text{top}(Z)$; therefore $Y$ has simple top.

If $i(X)$ is not in $\text{rad}(Y)$, then $i$ induces an isomorphism $X \to S$ which we also denote by $i$, where $S$ is a direct summand of $\text{top}(Y)$. We define $j : Y \to X$ to be the composition $Y \xrightarrow{\rho} S \xrightarrow{i^{-1}} X$ where $\rho$ is the canonical projection. Then we have $j \circ i = \text{id}_X$. This implies that $i$ is split, and therefore $Y \cong X \oplus Z$. \qed 

Let $A$ be a finite dimensional algebra. Suppose that $\{S_1, \ldots, S_n\}$ is a complete set of non-isomorphic simple $A$-modules and $\{P_1, \ldots, P_n\}$ is the corresponding projective covers. Recall that the Cartan matrix $C(A)$ of the algebra $A$ is an $n \times n$-matrix, with its $i$-$j$-entry given by the number of composition factors of $P_j$ which are isomorphic to $S_i$.

The following lemma is well-known (see, for example, [3, §54.16]).

Lemma 2.5. Let $A$ be a finite dimensional $k$-algebra where $k$ is a splitting field for $A$, and let $X$ be an $A$-module. Then for any $1 \leq i \leq n$, the number of composition factors of $X$ which are isomorphic to $S_i$ is $\dim_k \text{Hom}(P_i, X)$. \qed 

Remark. $k$ is a splitting field for $k$-algebra $A$ if and only if every $A$-endomorphism ring of simple $A$-module is isomorphic to the base field $k$. In particular, any algebraically closed field $k$ is a splitting field for algebras over $k$ (see [3, §29]).

3. Proof of Theorem 1.1

In this section, we shall give a proof of the generalization of Linckelmann’s theorem. Our proof here is based on the construction of almost split sequences. For the basic theory of almost split sequences, we refer to [1].

Proposition 3.1. Let $A$ and $B$ be finite dimensional $k$-algebras with no semisimple summands, and let $\{S_1, \ldots, S_n\}$ be a complete set of non-isomorphic simple $A$-modules and $\{P_1, \ldots, P_n\}$ be the corresponding projective covers. If two bimodules $A_MB$ and $BN_A$ induce a stable equivalence of Morita type between $A$ and $B$, and if $N \otimes_A -$ maps any simple $A$-module to a simple $B$-module, then we have the following:

1. $\{T_N(S_i) \mid i = 1, \ldots, n\}$ is a complete set of non-isomorphic simple $B$-modules;
2. $\{T_N(P_i) \mid i = 1, \ldots, n\}$ is a complete set of non-isomorphic indecomposable projective $B$-modules.

Proof. (1) First, we show that every projective $B$-module $E$ lies in $\text{add}(\bigoplus_{i=1}^n T_N(P_i))$.

Since $T_M(E)$ is a projective $A$-module, we have $T_M(E) \in \text{add}(\bigoplus_{i=1}^n P_i)$. It follows from $T_N \circ T_M(E) \cong E \oplus T_Q(E) \in \text{add}(\bigoplus_{i=1}^n T_N(P_i))$ that $E$ lies in $\text{add}(\bigoplus_{i=1}^n T_N(P_i))$. Since all composition factors of $T_N(P_i)(1 \leq i \leq n)$ occur in $\{T_N(S_i) \mid i = 1, \ldots, n\}$,
we know that \( \{ T_N(S_i) \}_{i = 1, \ldots, n} \) contains all isomorphism classes of simple \( B \)-modules. To finish the proof we need to show that \( T_N(S_i) \neq T_N(S_j) \) for all \( i \neq j \).

It suffices to prove this when \( T_i \) and \( T_j \) are projective modules. Note that \( S_i \) and \( S_j \) are non-injective modules since \( A \) has no semisimple summands. By [1 Proposition 2.6, p.151], we have an almost split sequence \( 0 \longrightarrow S_i \xrightarrow{f} T_i \xrightarrow{g} TrDS_i \longrightarrow 0 \), where \( T_i \) is projective. Applying \( T_N \) we get an exact sequence \( 0 \longrightarrow T_N(S_i) \xrightarrow{T_N(f)} T_N(T_i) \xrightarrow{T_N(g)} T_N(TrDS_i) \longrightarrow 0 \) in \( \text{mod}B \). By Lemma 2.3, we get an exact sequence

\[
0 \longrightarrow T_N(S_i) \xrightarrow{T_N(f)_i} T_N(T_i)_0 \xrightarrow{T_N(g)_i} T_N(TrDS_i)_1 \longrightarrow 0,
\]

where \( T_N(TrDS_i)_1 \) is the non-projective part of \( T_N(TrDS_i) \), \( T_N(TrDS_i) = T_N(TrDS_i)_1 \oplus T_N(TrDS_i)' \), \( T_N(T_i) \cong T_N(T_i)_0 \oplus T_N(TrDS_i)' \), \( T_N(f) = \begin{bmatrix} T_N(f)_1 & 0 \\ 0 & T_N(f)_2 \end{bmatrix} \), and \( T_N(g) = \begin{bmatrix} T_N(g)_1 & T_N(g)_2 \\ 0 & T_N(g)_3 \end{bmatrix} \). Since \( T_N(S_i) \) is simple, \( T_N(g)_1 \) is a projective cover.

We want to show that \((*)\) is an almost split sequence.

Clearly, there is an almost split sequence

\[
0 \longrightarrow X \longrightarrow E \xrightarrow{h} T_N(TrDS_i)_1 \longrightarrow 0.
\]

We claim that \( E \) is projective. Assume that \( E \) is not projective. Write \( E = E_1 \oplus E' \), where \( E_1 \neq 0 \) is the non-projective part of \( E \). Since \( T_N(TrDS_i)_1 \) is indecomposable non-projective, and since \( E = E_1 \oplus E' \longrightarrow T_N(TrDS_i)_1 \) is a minimal right almost split morphism, there exists a morphism \( F \longrightarrow T_M(T_N(TrDS_i)_1) \), with \( F \) projective in \( \text{mod}A \) such that \( T_M(E_1)_1 \oplus F \longrightarrow TrDS_i \) is a minimal right almost split morphism ([1 Proposition 1.3, p.337]), where \( T_M(E_1)_1 \neq 0 \) is the non-projective part of \( T_M(E_1) \), and \( T_M(T_N(TrDS_i)_1) \cong TrDS_i \) is the non-projective part of \( T_M(T_N(TrDS_i)_1) \). But \( T_i \longrightarrow TrDS_i \) is a minimal right almost split morphism with \( T_i \) projective; this contradicts the uniqueness of minimal right almost split morphism! Since \( E \) must be projective, it follows from [1 Theorem 3.3, p.154] that \( h \) is a projective cover. Hence the exact sequences \((*)\) and \((***)\) are isomorphic. This implies that \((*)\) is also an almost split sequence.

Similarly, we have an almost split sequence

\[
0 \longrightarrow T_N(S_j) \longrightarrow T_N(T_j)_0 \longrightarrow T_N(TrDS_j)_1 \longrightarrow 0,
\]

where \( T_N(TrDS_j)_1 \) is the non-projective part of \( T_N(TrDS_j) \). Suppose that \( T_N(S_i) \cong T_N(S_j) \). Then \( T_N(TrDS_i)_1 \cong T_N(TrDS_j)_1 \) by the basic properties of almost split sequences. But \( TrDS_i \) and \( TrDS_j \) are non-isomorphic, indecomposable non-projective modules. This contradicts the fact that \( T_N \) induces a stable equivalence, and therefore (1) follows.

(2) By the previous proof, we only need to show that \( T_N(P_i) \) is indecomposable for all \( 1 \leq i \leq n \). In fact, we shall prove the following more general result: if an \( A \)-module \( X \) has simple top, then the \( B \)-module \( T_N(X) \) also has simple top. We prove this by induction on the length of \( X \). For \( l(X) = 1 \), the module \( X \) is simple, and \( T_N(X) \) is simple by assumption. For \( l(X) = m > 1 \), take an exact sequence \( 0 \longrightarrow S \longrightarrow X \longrightarrow X/S \longrightarrow 0 \), where \( S \) is a simple submodule of \( X \) and therefore \( X/S \) has simple top. Applying \( T_N \) we get an exact sequence \( 0 \longrightarrow T_N(S) \longrightarrow T_N(X) \longrightarrow T_N(X/S) \longrightarrow 0 \), where \( T_N(S) \) is simple, and \( T_N(X/S) \) has simple top by induction. By Lemma 2.4, either \( T_N(X) \) has simple top or \( T_N(X) \cong T_N(S) \oplus T_N(X/S) \). Suppose that \( T_N(X) \cong T_N(S) \oplus T_N(X/S) \). Applying
we have \( X \cong S \oplus X/S \) since \( X/S \) is an indecomposable non-projective module. This contradicts the indecomposability of \( X \), and completes our proof.

\[ \square \]

**Proof of Theorem 1.1.** By Lemma 2.2, \( T_N \) is a faithful functor and induces a monomorphism between algebras: \( \operatorname{End}_A(A) \longrightarrow \operatorname{End}_B(T_N(A)) \). By Proposition 3.1, \( T_N \) induces a bijection between the sets of isomorphism classes of indecomposable projective modules over \( A \) and \( B \). On the other hand, \( T_N \) is an exact functor which gives a one-to-one correspondence between the sets of isomorphism classes of simple modules over \( A \) and \( B \). It follows that \( T_N \) preserves the Cartan matrix. By Lemma 2.5, Cartan matrix is given by \( k \)-dimensions of homomorphism spaces between indecomposable projective modules. Therefore we have

\[
\dim_k \operatorname{Hom}_A(P_i, P_j) = \dim_k \operatorname{Hom}_B(T_N(P_i), T_N(P_j))
\]

for all \( 1 \leq i, j \leq n \). Assume that \( A \cong \bigoplus_{i=1}^n P_i^{m_i} \). Then we have

\[
\dim_k \operatorname{End}_A(A) = \sum_{i,j=1}^n \dim_k \operatorname{Hom}_A(P_i, P_j) m_i m_j
\]

and

\[
\dim_k \operatorname{End}_B(T_N(A)) = \sum_{i,j=1}^n \dim_k \operatorname{Hom}_B(T_N(P_i), T_N(P_j)) m_i m_j.
\]

Thus \( \operatorname{End}_A(A) \longrightarrow \operatorname{End}_B(T_N(A)) \) is an isomorphism. Since \( A \cong \operatorname{End}_A(A)^{\text{op}} \) and \( T_N(A) \cong N \) is a projective generator as \( B \)-module, we know that \( A \cong \operatorname{End}_B(N)^{\text{op}} \). Hence \( T_N \) is a Morita equivalence between \( A \) and \( B \) by the Morita theorem (see, for example, [4, Theorem 8.4.5]).

\[ \square \]

**Remark.** If \( k \) is a perfect field, then by [3, §54.19] we know that the condition \( k \) is a splitting field for \( A \) and \( B \) can be weakened as follows: \( \dim_k \operatorname{End}_A(S) = \dim_k \operatorname{End}_B(T_N(S)) \) for any simple \( A \)-module \( S \).

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**References**


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