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SUBSPACES OF L_p WITH MORE THAN ONE COMPLEX STRUCTURE

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Abstract. We propose a method of constructing explicit Banach spaces not isomorphic to their complex conjugates as subspaces of a natural class of Banach spaces. In particular, it is shown that L_p, for $1 \leq p < 2$, contains real subspaces with at least two non-isomorphic complex structures.

1. Introduction and notations

In considering real Banach spaces which admit complex structure, a natural question is whether we can construct at least two non-isomorphic complex structures. For a complex Banach space X we can define $\overline{X}$, the *complex conjugate* of X, to be the Banach space with the same elements and norm and the same addition of vectors, while the multiplication by scalars is given by $\lambda \odot x = \overline{\lambda} x$, for $\lambda \in \mathbf{C}$ and $x \in X$. Obviously X and $\overline{X}$ are identical as real spaces and, in many cases, the spaces X and $\overline{X}$ are in fact (complex) isomorphic. For example, when X has unconditional basis $\{e_j\}_j$, the natural map $J : X \longrightarrow \overline{X}$ given by $J(\sum_j t_j e_j) = \sum_j t_j \odot e_j$ is an isomorphism between X and $\overline{X}$.

There are two types of known examples of complex Banach spaces not isomorphic to their complex conjugates, one constructed by J. Bourgain [1] (with a variant by S. Szarek [12]) and the other by N. Kalton [4]. As a consequence, these examples provide the existence of a real Banach space which admits two non-isomorphic complex structures.

In Bourgain's example, the space X is an l_2-direct sum $X = (\sum_k \oplus X_k)_{l_2}$, where X_k are suitable finite dimensional spaces obtained by considering certain random norms on $\mathbf{C}^N$. Szarek's variant of this example has the finite dimensional spaces X_k obtained (again by random methods) as proportional dimensional subspaces of $l_{q_k}^{n_k}$, for certain $q_k \searrow 2$ and $n_k \nearrow \infty$. It should be noted that by this method it is not possible to obtain an example of the same type with $X_k \subset l_{q_k}^{n_k}$, for $q_k < 2$.

The space that Kalton constructed is a *twisted sum of Hilbert spaces* i.e., X has a closed subspace E so that E and X/E are Hilbertian, while X itself is not isomorphic to a Hilbert space. His example is a variant of the Kalton-Peck space [3] and is constructed with a complex twisting function.

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The purpose of this paper is to propose a method of constructing Banach spaces not isomorphic to their complex conjugates as subspaces of a natural class of Banach spaces. In particular we show that L_p, for $1 \leq p < 2$, and $(\sum_n \oplus\, l_{r_n})_{l_2}$, for some $r_n \nearrow 2$, contain real subspaces with at least two non-isomorphic complex structures (in fact we can easily get continuum of such structures). This latter example complements the results by Bourgain and Szarek. In addition, we can exhibit a constructive version of the Bourgain-Szarek example (see Theorem 3.1).

Our argument is based on the approach introduced by Johnson, Lindenstrauss and Schechtman in [2], then refined by Ketonen [8] and subsequently generalized in the work of Komorowski and Tomczak-Jaegermann [5], [6], [7], where a general method of obtaining subspaces without unconditional basis is presented. Comparing to this latter case, in our context we have less linear operators available (a criterium of the same type as in [5], [6], [7], [8] is not known for a space not to be isomorphic to its complex conjugate).

Let us recall a minimum of standard notations from the Banach space theory (see [10], [11] and [14], together with other terminology not explained here).

A sequence $\{e_j\}_j$ in a Banach space is called a *(Schauder) basis* if every vector $x \in X$ has a unique representation $x = \sum_j t_j e_j$ as a sum of a convergent series. A Banach space is said to have a *finite dimensional decomposition* $\{Z_k\}_k$ if every vector $x \in X$ has a unique representation $x = \sum_k z_k$ as a sum of a convergent series such that $z_k \in Z_k$, for $k = 1, 2, \ldots$, and $\dim\ Z_k < \infty$, for $k = 1, 2, \ldots$. In this case we define the *support* of x with respect to the decomposition $\{Z_k\}_k$ to be $\operatorname{supp} x = \{k \,|\, z_k \neq 0\}$.

Let W, V be Banach spaces having finite dimensional decompositions $\{W_k\}_k$ and $\{V_j\}_j$ respectively. Let $T : W \longrightarrow V$ be a bounded linear operator. We say that T is *block-diagonal* with respect to $\{W_k\}_k$ and $\{V_j\}_j$ if for every k there exists a finite set $B_k \subset \{1, 2, \ldots\}$ such that

$$\begin{cases} \max B_k < \min B_l \quad \forall k, l \in \{1, 2, \ldots\} \text{ with } k < l, \\ \operatorname{supp} T w_k \subset B_k \quad \forall w_k \in W_k,\ \forall k \in \{1, 2, \ldots\} \end{cases}$$

where $\operatorname{supp} T w_k$ is taken with respect to the decomposition $\{V_j\}_{j \in J}$.

The following general observation will often be used in the sequel.

Proposition 1.1. *Let W, V be Banach spaces having decompositions into 2-dimensional spaces $\{W_k\}_k$ and $\{V_j\}_j$ respectively. Let $W_k = \operatorname{span}\{w_{1,k},\, w_{2,k}\}$, for $k = 1, 2, \ldots$, and $V_j = \operatorname{span}\{v_{1,j},\, v_{2,j}\}$, for $j = 1, 2, \ldots$, and suppose that $\{w_{1,k},\, w_{2,k}\}_k$ is a w-null normalized basis in W and $\{v_{1,j},\, v_{2,j}\}_j$ is a normalized basis in V. Let $T : W \longrightarrow V$ be a bounded linear operator.*

Then, for every $\epsilon > 0$, there exist a subsequence $I_0 \subset \{1, 2, \ldots\}$ and $T_0 : W^0 = \overline{\operatorname{span}}\{W_k\}_{k \in I_0} \longrightarrow V$, a block-diagonal operator with respect to $\{W_k\}_{k \in I_0}$ and $\{V_j\}_j$, such that

$$\|T_{|\,W^0} - T_0\| < \epsilon.$$

Proof. Denote by $\{w^*_{1,k},\, w^*_{2,k}\}_k$ and $\{v^*_{1,j},\, v^*_{2,j}\}_j$ the biorthogonal functionals associated to $\{w_{1,k},\, w_{2,k}\}_k$ and $\{v_{1,j},\, v_{2,j}\}_j$ respectively. Then

$$\lim_j v^*_{1,j}(T w_{1,k}) = \lim_j v^*_{1,j}(T w_{2,k}) = 0, \quad \text{for all } k = 1, 2, \ldots,$$

$$\lim_k v^*_{1,j}(T w_{1,k}) = \lim_k v^*_{1,j}(T w_{2,k}) = 0, \quad \text{for all } j = 1, 2, \ldots,$$

and similarly for $v^*_{2,j}$. Now, by a classical gliding-hump argument we can find a subsequence I_0 of I and a block-diagonal operator $T_0 : W^0 \longrightarrow V$ such that the columns of T_0 are approximated by the correspondent columns of T. $\square$

2. Subspaces of $l_{p_1} \oplus l_{p_2} \oplus l_{p_3} \oplus l_{p_4} \oplus l_{p_5}$ not well isomorphic to their complex conjugates

We will present a construction of infinite dimensional subspaces of $l_{p_1} \oplus l_{p_2} \oplus l_{p_3} \oplus l_{p_4} \oplus l_{p_5}$ $(1 < p_5 < \ldots < p_1 < \infty)$ whose Banach-Mazur distance to their complex conjugates is arbitrarily large.

Let $\wp = \{(p_1, p_2, ..., p_5) \mid 1 < p_5 < \ldots < p_1 < \infty\}$.

For every $\eta = (p_1, ..., p_5) \in \wp$ and $N \in \mathbf{N}$ we will construct a Banach space $X_{N,\eta}$ as follows: we will define 2-dimensional subspaces Z_k of $l_{p_1} \oplus \ldots \oplus l_{p_5}$ (depending on N and η) which will form an unconditional decomposition for $X_{N,\eta} = \overline{\text{span}}\ [Z_k]_{k\geq 1}$.

Fix $\eta \in \wp$ and $N \in \mathbf{N}$. For $i = 2, \ldots, 5$ set $\alpha_i = 1/p_i - 1/p_{i-1}$, and let $\alpha = \min\{\alpha_2, \ldots, \alpha_5\}$. Fix a positive integer $\lambda > 2\,\alpha_3/\alpha_4 + 5$.

Denote by $\{f_{j,k}\}_k$ the natural basis of l_{p_j} $(j = 1, \ldots, 5)$. Define the vectors x_k and y_k spanning Z_k $(k = 1, 2, \ldots)$ by the formulas

$$\begin{array}{rcllll} x_k & = & f_{1,k} & +\gamma_1 f_{3,k} & +\gamma_2 f_{4,k} & +\gamma_3 f_{5,k} \\ y_k & = & f_{2,k} & & +\gamma_2 f_{4,k} & +i\gamma_3 f_{5,k} \end{array}$$

where $\gamma_1 = N^{-2\alpha_3}$, $\gamma_2 = N^{-4(\alpha_3+\alpha_4)}$ and $\gamma_3 = N^{-\lambda(\alpha_3+\alpha_4+\alpha_5)}$.

It is easy to see that the decomposition $\{Z_k\}_{k\geq 1}$ is 1-unconditional and $x_1, y_1, x_2, y_2, \ldots$ form a Schauder basis in $X_{N,\eta}$ (and also in $\overline{X}_{N,\eta}$). This is a shrinking basis (and hence w-null), since otherwise we can find $\delta > 0$ and disjointly supported normalized blocks $\{w_l\}_l$ (with respect to the decomposition $\{Z_k\}_{k\geq 1}$) such that for all $\{a_l\}_l \in c_{00}$

$$\|\sum_l a_l w_l\| \geq \delta \sum_l |a_l|,$$

contradicting the fact that $\{w_l\}_l$ satisfy an upper p_5-estimate.

The next result, concerning the behavior of the linear operators acting from $X_{N,\eta}$ to $\overline{X}_{N,\eta}$, will be essential for the proof of Theorem 3.1.

Proposition 2.1. *Let $\eta \in \wp$ and N be a positive integer.*

Let $I \subset \{1, 2, \ldots\}$ be an infinite set and consider Y to be the subspace of $X_{N,\eta}$ defined by $Y = \overline{\text{span}}[Z_k]_{k\in I}$. Consider $T : Y \longrightarrow \overline{X}_{N,\eta}$ to be a block-diagonal operator (with respect to $\{Z_k\}_{k\in I}$ and $\{\overline{Z_k}\}_{k\geq 1}$) with $\|T\| \leq 1$. Then

(i) *There exists a finite set $J \subset I$ such that*

$$\max\{\|Tx_k\|, \|Ty_k\|\} \leq 24N^{-\alpha}, \quad \textit{for all } k \in I \setminus J.$$

(ii) *Let $\{I_l\}_{l\geq 1}$ be a family of disjoint subsets of I with the property that $|I_l| = N$, for all $l \geq 1$. Let $\widetilde{x_l} = \sum_{k\in I_l} a_l(k)x_k$, $\widetilde{y_l} = \sum_{k\in I_l} a_l(k)y_k$ satisfy $\sum_{k\in I_l} |a_l(k)|^{p_2} = 1$, for $l = 1, 2, \ldots$. Then there exists a finite subset $J_0 \subset \{1, 2, ...\}$ such that*

$$\max\{\|T\widetilde{x_l}\|, \|T\widetilde{y_l}\|\} \leq 70N^{-\alpha}, \quad \textit{for all } l \in \{1, 2, ...\} \setminus J_0.$$

Combining Proposition 1.1 and Proposition 2.1 (i) we obtain

Corollary 2.2. *Let $\eta \in \wp$ and N be a positive integer. Then*

$$d(X_{N,\eta}, \overline{X}_{N,\eta}) \geq \frac{1}{100} N^{\alpha}.$$

Remark. In the same circle of problems, we should mention the result of Szarek [13] showing that in the finite dimensional case one may have the extremal situation $d(Y, \overline{Y}) = O(\dim Y)$.

Proof of Proposition 2.1. Because T is block-diagonal with respect to $\{Z_k\}_{k\in I}$ and $\{\overline{Z_k}\}_{k\geq 1}$, for every $k \in I$ there exist a finite set $B_k \subset \{1, 2, \ldots\}$ and sequences of scalars $u_k = \left(\overline{u_k(j)}\right)_j$, $v_k = \left(\overline{v_k(j)}\right)_j$, $w_k = \left(\overline{w_k(j)}\right)_j$, $s_k = \left(\overline{s_k(j)}\right)_j$ (we start off with complex conjugate sequences for convenience only, since this will later produce some simplification of writing) such that

$$\begin{cases} \max B_k < \min B_l, \quad \forall k, l \in I \text{ with } k < l, \\ Tx_k = \sum_{j\in B_k} \left(\overline{u_k(j)} \odot x_j + \overline{v_k(j)} \odot y_j\right) = \sum_{j\in B_k} \left(u_k(j)x_j + v_k(j)y_j\right), \\ Ty_k = \sum_{j\in B_k} \left(\overline{w_k(j)} \odot x_j + \overline{s_k(j)} \odot y_j\right) = \sum_{j\in B_k} \left(w_k(j)x_j + s_k(j)y_j\right). \end{cases}$$

Thus, for all $k \in I$

$$\begin{aligned} Ty_k &= \sum_{j\in B_k} w_k(j) f_{1,j} + \sum_{j\in B_k} s_k(j) f_{2,j} + \sum_{j\in B_k} \gamma_1 w_k(j) f_{3,j} \\ &+ \sum_{j\in B_k} \gamma_2 \left(w_k(j) + s_k(j)\right) f_{4,j} + \sum_{j\in B_k} \gamma_3 \left(w_k(j) + i s_k(j)\right) f_{5,j}. \end{aligned} \tag{1}$$

We will only prove the estimates in **(i)** and **(ii)** involving y_k's (the others can be obtained similarly). The proof of **(i)** is presented in a few steps.

Let $Q_l : l_{p_1} \oplus \ldots \oplus l_{p_5} \longrightarrow l_{p_l}$ be the canonical projection ($l = 1, \ldots, 5$).

Step 1. We show first that there exists a set $A_1 \subset I$, $|A_1| < N$ such that

$$\gamma_1 \| \sum_{j\in B_k} w_k(j) f_{3,j} \| \leq 3N^{-\alpha}, \quad \text{for all } k \in I \setminus A_1. \tag{2}$$

Indeed, let A_1 be the set of all $k \in I$ such that $\gamma_1 \| \sum_{j\in B_k} w_k(j) f_{3,j} \| > 3N^{-\alpha}$, and assume that $|A_1| \geq N$. Then choose a subset $\tilde{A}$ of A_1 of cardinality N and consider the vector $y = \sum_{k\in\tilde{A}} y_k$. We have

$$\|y\| = N^{\frac{1}{p_2}} + N^{\frac{1}{p_4} - 4(\alpha_3+\alpha_4)} + N^{\frac{1}{p_5} - \lambda(\alpha_3+\alpha_4+\alpha_5)} \leq 3N^{\frac{1}{p_2}}$$

and

$$\|Ty\| \geq \|Q_3 Ty\| = \| \sum_{k\in\tilde{A}} \sum_{j\in B_k} \gamma_1 w_k(j) f_{3,j} \| > 3N^{-\alpha} N^{\frac{1}{p_3}}.$$

Since $\|Ty\| \leq \|T\| \, \|y\| \leq \|y\|$, we get the contradiction.

In a similar manner as above we can obtain $A_2, A_3 \subset I$, $|A_2|, |A_3| < N$ satisfying

$$\gamma_2 \| \sum_{j\in B_k} \left(w_k(j) + s_k(j)\right) f_{4,j} \| \leq 3N^{-\alpha}, \quad \text{for all } k \in I \setminus A_2, \tag{3}$$

$$\gamma_3 \| \sum_{j\in B_k} \left(w_k(j) + i s_k(j)\right) f_{5,j} \| \leq 3N^{-\alpha}, \quad \text{for all } k \in I \setminus A_3. \tag{4}$$

Combining (1), (2), (3) and (4) we find a set $A \subset I$, with $|A| < 3N$ such that

$$\|Ty_k\| \leq \|\sum_{j\in B_k} w_k(j) f_{1,j}\| + \|\sum_{j\in B_k} s_k(j) f_{2,j}\| + 9N^{-\alpha}, \quad \forall k \in I \setminus A. \tag{5}$$

Step 2. By considering elements of the form $x = \sum_{k\in\tilde{A}} x_k$, with $\tilde{A} \subset I$, $|\tilde{A}| = N$, we can obtain in an analogous way as (2) a set $A_4 \subset I$, with $|A_4| < N$ such that

$$\|\sum_{j\in B_k} v_k(j) f_{2,j}\| \leq \ 4N^{-\alpha}, \quad \text{for all } k \in I \setminus A_4. \tag{6}$$

Step 3. We show that there exists a set $A_5 \subset I$, $|A_5| < N^\lambda$ such that

$$\|\sum_{j\in B_k} (u_k(j) + v_k(j) - w_k(j) - s_k(j))\, f_{4,j}\| \leq 5N^{-\alpha_4}, \quad \forall k \in I \setminus A_5. \tag{7}$$

Indeed, let A_5 be the set of all $k \in I$ such that

$$\|\sum_{j\in B_k} (u_k(j) + v_k(j) - w_k(j) - s_k(j))\, f_{4,j}\| > 5N^{-\alpha_4}$$

and assume that $|A_5| \geq N^\lambda$. Then pick a subset $\tilde{A}$ of A_5 of cardinality $K := N^\lambda$ and consider the vector $z = \sum_{k\in\tilde{A}}(x_k - y_k)$. We have

$$\|z\| = K^{\frac{1}{p_1}} + K^{\frac{1}{p_2}} + N^{-2\alpha_3} K^{\frac{1}{p_2}+\alpha_3} + N^{-\lambda(\alpha_3+\alpha_4+\alpha_5)}\sqrt{2} K^{\frac{1}{p_2}+\alpha_3+\alpha_4+\alpha_5}$$

while

$$\begin{aligned} \|Tz\| \quad &\geq \quad \|Q_4 Tz\| = \gamma_2 \|\sum_{k\in\tilde{A}} \sum_{j\in B_k} (u_k(j) + v_k(j) - w_k(j) - s_k(j))\, f_{4,j}\| \\ &> \quad N^{-4(\alpha_3+\alpha_4)}\ 5N^{-\alpha_4} K^{\frac{1}{p_2}+\alpha_3+\alpha_4}. \end{aligned}$$

But this contradicts $\|Tz\| \leq \|z\|$ since, by the choice of λ,

$$\begin{aligned} &N^{-4(\alpha_3+\alpha_4)} N^{-\alpha_4} K^{\frac{1}{p_2}+\alpha_3+\alpha_4} \\ &\geq \max\{K^{\frac{1}{p_2}}, N^{-2\alpha_3} K^{\frac{1}{p_2}+\alpha_3}, N^{-\lambda(\alpha_3+\alpha_4+\alpha_5)} K^{\frac{1}{p_2}+\alpha_3+\alpha_4+\alpha_5}\}. \end{aligned}$$

Step 4. This is a stronger estimate than (2) (and could have been proved directly instead of (2)). We show that there is a subset $A_6 \subset I$, with $|A_6| < N^3$ such that

$$\|\sum_{j\in B_k} w_k(j) f_{3,j}\| \leq 3N^{-\alpha_3}, \quad \text{for all } k \in I \setminus A_6. \tag{8}$$

For the proof take a vector $y = \sum_{k\in\tilde{A}} y_k$, with $\tilde{A}$ a subset of cardinality N^3 of the set of all $k \in I$ such that $\|\sum_{j\in B_k} w_k(j) f_{3,j}\| > 3N^{-\alpha_3}$. Then estimate $\|y\|$ from above and $\|Ty\|$ from below by using the definition of γ_i's and $\tilde{A}$.

As a remark, note that since $\{f_{1,j}\}_j$ is dominated by $\{f_{3,j}\}_j$ we also have the estimate analogous to (8)

$$\|\sum_{j\in B_k} w_k(j) f_{1,j}\| \leq 3N^{-\alpha_3}, \quad \text{for all } k \in I \setminus A_6. \tag{9}$$

Step 5. By considering elements of the form $z = \sum_{k\in\tilde{A}}(x_k + iy_k)$ with $|\tilde{A}|$ large enough, and taking into account that

$$Tz = \sum_{k\in\tilde{A}} (Tx_k + i \odot Ty_k) = \sum_{k\in\tilde{A}} (Tx_k - i\, Ty_k),$$

one can find a finite set $A_7 \subset I$ such that

$$\| \sum_{j\in B_k} (u_k(j) + s_k(j) + iv_k(j) - iw_k(j))\, f_{5,j}\| \leq 5N^{-\alpha},\ \forall k \in I \setminus A_7. \tag{10}$$

Since $p_1 > p_2 > p_3 > p_4 > p_5$ it follows that $\{f_{2,j}\}_j$ is dominated by $\{f_{3,j}\}_j$, $\{f_{4,j}\}_j$, $\{f_{5,j}\}_j$. Combining this with (6), (7), (8), (10) we obtain a finite set $A'(= A_4 \cup A_5 \cup A_6 \cup A_7)$ such that

$$\| \sum_{j\in B_k} s_k(j)\ f_{2,j}\| \leq 12N^{-\alpha}, \quad \text{for all } k \in I \setminus A'. \tag{11}$$

Using (9) and (11) in (5) we get a finite set $J(= A \cup A')$ satisfying

$$\|Ty_k\| \leq 24N^{-\alpha}, \quad \text{for all } k \in I \setminus J.$$

(ii) By ignoring a finite number of sets from the family $\{I_l\}_{l\geq 1}$ we can suppose that $I_l \subset I \setminus J$ for all $l \geq 1$. In particular, for each $l \in \{1, 2, ...\}$ we have

$$24N^{-\alpha} \geq \|Ty_k\| \geq \max\{\|Q_1Ty_k\|, \|Q_2Ty_k\|\}, \quad \text{for all } k \in I_l.$$

Looking at $T\widetilde{y_l}$ we can write, for each $l \in \{1, 2, ...\}$,

$$\begin{aligned}
\|T\widetilde{y_l}\| &= \| \sum_{k\in I_l} a_l(k)Q_1Ty_k\| + \| \sum_{k\in I_l} a_l(k)Q_2Ty_k\| + \|(Q_3 + Q_4 + Q_5)T\widetilde{y_l}\| \\
&\leq 24N^{-\alpha} \left((\sum_{k\in I_l} |a_l(k)|^{p_1})^{\frac{1}{p_1}} + (\sum_{k\in I_l} |a_l(k)|^{p_2})^{\frac{1}{p_2}} \right) + \|(Q_3 + Q_4 + Q_5)T\widetilde{y_l}\| \\
&\leq 48N^{-\alpha} + \|Q_3T\widetilde{y_l}\| + \|Q_4T\widetilde{y_l}\| + \|Q_5T\widetilde{y_l}\|.
\end{aligned}$$

We show that there exists a subset $A \subset \{1, 2, ...\}$, $|A| < N$ such that

$$\|Q_3T\widetilde{y_l}\| \leq 3N^{-\alpha}, \quad \forall l \in \{1, 2, ...\} \setminus A.$$

Indeed, let A be the set of all $l \in \{1, 2, ...\}$ satisfying $\|Q_3T\widetilde{y_l}\| > 3N^{-\alpha}$, and assume that $|A| \geq N$. Choose a subset A_0 of A of cardinality N and consider the vector $y = \sum_{l\in A_0} \widetilde{y_l}$. We have

$$\begin{aligned}
&\|y\| = \| \sum_{l\in A_0} \sum_{k\in I_l} a_l(k)\ y_k\| \\
= &\ (\sum_{l\in A_0} \sum_{k\in I_l} |a_l(k)|^{p_2})^{\frac{1}{p_2}} + \gamma_2(\sum_{l\in A_0} \sum_{k\in I_l} |a_l(k)|^{p_4})^{\frac{1}{p_4}} + \gamma_3(\sum_{l\in A_0} \sum_{k\in I_l} |a_l(k)|^{p_5})^{\frac{1}{p_5}} \\
\leq &\ N^{\frac{1}{p_2}} + \left(\gamma_2\,(N^2)^{\frac{1}{p_4}-\frac{1}{p_2}} + \gamma_3\,(N^2)^{\frac{1}{p_5}-\frac{1}{p_2}}\right) (\sum_{l\in A_0} \sum_{k\in I_l} |a_l(k)|^{p_2})^{\frac{1}{p_2}} \\
= &\ N^{\frac{1}{p_2}} + N^{-2(\alpha_3+\alpha_4)}N^{\frac{1}{p_2}} + N^{-(\lambda-2)(\alpha_3+\alpha_4+\alpha_5)}N^{\frac{1}{p_2}}.
\end{aligned}$$

This contradicts $\|T\| \leq 1$ since

$$\|Ty\| \geq \|Q_3Ty\| = \| \sum_{l\in A_0} Q_3T\widetilde{y_l}\| > 3N^{-\alpha}N^{\frac{1}{p_3}}.$$

Arguing similarly for $Q_4T\widetilde{y_l}$ and $Q_5T\widetilde{y_l}$ we obtain the conclusion. □

Remark. Proposition 2.1 will still be true if we consider $X_{N,\eta}$ as a subspace of $l_{p_1} \oplus_q l_{p_2} \oplus_q \ldots \oplus_q l_{p_5}$, for some $q \geq 1$ (note that, in this case, $X_{N,\eta}$ is the same vector space as before endowed with an equivalent norm).

3. Subspaces of L_p, $1 \le p < 2$, with at least two non-isomorphic complex structures

Theorem 3.1. *Let $(r_n)_{n\ge1}$ be a strictly decreasing sequence of real numbers, with $r_n > 1$ for all n, and let $q \in [1, \lim_{n\to\infty} r_n]$. There exists a subspace X of $(\sum_{n\ge1} \oplus\ l_{r_n})_{l_q}$ which is not isomorphic to its complex conjugate. Furthermore, we can construct the subspace X such that, as a real space, it has continuum non-isomorphic complex structures.*

Proof. For each $m = 1, 2, \ldots$ we will define X_m as one of the spaces $X_{N,\eta}$ discussed before for the following choice of the parameters involved. Let $\eta_m = (r_{5m+1}, r_{5m+2}, \ldots, r_{5m+5}) \in \wp$. Set

$$\alpha_m = \min\left\{\frac{1}{r_{5m+2}} - \frac{1}{r_{5m+1}}, \ldots, \frac{1}{r_{5m+5}} - \frac{1}{r_{5m+4}}\right\}.$$

(This definition of α_m corresponds to α from the main construction in Section 2, and it hopefully will not get confused with the notation $\alpha_2, \ldots, \alpha_5$ used there.) Finally, fix a natural number $N_m \ge (560 \cdot m)^{2/\alpha_m}$. Now, let $X_m = X_{N_m,\eta_m}$ be the space defined in Section 2, treated as a subspace of $l_{r_{5m+1}} \oplus_q l_{r_{5m+2}} \oplus_q \ldots \oplus_q l_{r_{5m+5}}$ (see the Remark after Proof of Proposition 2.1).

We will show that the space $X = (\sum_{m\ge1} \oplus X_m)_{l_q}$ is not isomorphic to its complex conjugate $\overline{X} = (\sum_{m\ge1} \oplus \overline{X}_m)_{l_q}$.

Suppose that $T : X \longrightarrow \overline{X}$ is an isomorphism with $\|T\| \le 1/4$. Denote by $a = \|T^{-1}\|$ and by $P_j : \overline{X} \longrightarrow \overline{X}_j$ the projection of $\overline{X}$ onto its j-th term.

Let $m \ge 1$ be arbitrarily fixed. Recall that $X_m = \overline{\text{span}}[Z_k]_{k\ge1}$.

Let $s > m$. We will show that

$$(*) \qquad \begin{aligned} &\forall L \subset \{1, 2, \ldots\} \text{ infinite set } \forall \epsilon_s > 0\ \exists k \in L \text{ such that} \\ &\qquad \|P_s T z_k\| < \epsilon_s \|z_k\|, \qquad \forall z_k \in Z_k. \end{aligned}$$

If not we can find $\epsilon_s > 0$, an infinite set $\{k_j\}_{j\ge1}$ and, for each $j \ge 1$, normalized elements $z_j \in Z_{k_j}$ satisfying

$$\epsilon_s \le \|P_s T z_j\| \ \left(= (\|Q_1 P_s T z_j\|^q + \ldots + \|Q_5 P_s T z_j\|^q)^{\frac{1}{q}}\right).$$

By passing to a subsequence (apply Proposition 1.1 to $P_s T_{|\overline{\text{span}}[Z_{k_j}]_{j\ge1}}$) we may assume that $(P_s T z_j)_{j\ge1}$ are successive in $\overline{X}_s$ and also

$$\|Q_1 P_s T z_j\| \ge \frac{\epsilon_s}{10}, \quad \text{for all } j \ge 1.$$

The contradiction occurs when we observe that, for all positive integers M,

$$5M^{\frac{1}{r_{5m+5}}} \ge \|\sum_{j=1}^{M} z_j\|_{X_m} \ge \|P_s T(\sum_{j=1}^{M} z_j)\|_{\overline{X}_s} \ge \|Q_1(\sum_{j=1}^{M} P_s T z_j)\| \ge \frac{\epsilon_s}{10} M^{\frac{1}{r_{5s+1}}}.$$

Relation $(*)$, applied successively, now easily implies, for $s > m$,

$$(**) \qquad \begin{aligned} &\forall L \subset \{1, 2, \ldots\} \text{ infinite set } \forall \epsilon_s > 0\ \exists L_s \subseteq L \text{ infinite set such that} \\ &\qquad \|P_s T x\| < \epsilon_s \|x\|, \qquad \forall x \in \overline{\text{span}}[Z_k]_{k\in L_s}. \end{aligned}$$

Applying (∗∗) inductively we obtain that for every sequence $\epsilon_s \searrow 0$ there exist infinite sets of positive integers $L_{m+1} \supset L_{m+2} \supset ... \supset L_s \supset ...$ such that

$$\|P_s T_{\mid \overline{\text{span}}[Z_k]_{k\in L_s}}\| < \epsilon_s, \quad \text{for all } s > m.$$

Denoting by $I = \{k_j\}_{j\geq m+1}$ the diagonal sequence we have

$$\|P_s T_{\mid \overline{\text{span}}[Z_{k_j}]_{j\geq s}}\| \ < \epsilon_s, \quad \text{for all } s > m.$$

By a perturbation argument we obtain a subspace of X_m, namely $Y_m = \overline{\text{span}}[Z_k]_{k\in I}$, and an operator (denoted again by) $T : Y_m \longrightarrow \overline{X}$ satisfying

$$\begin{cases} P_s T_{\mid \overline{\text{span}}[Z_{k_j}]_{j\geq s}} = 0, \quad \text{for all } s > m, \\ \frac{1}{2a}\|x\| \leq \|Tx\| \leq \frac{1}{2}\|x\|, \quad \text{for all } x \in Y_m. \end{cases}$$

Let us denote by $R_m : \overline{X} \longrightarrow (\sum_{s>m} \oplus \overline{X}_s)_{l_q}$ the natural projection.

Suppose that we can find $\delta > 0$, an infinite set $L \subset I$ and, for each $k \in L$, normalized elements $z_k \in Z_k$ such that

$$\|R_m T z_k\| > \delta.$$

By passing to a subsequence (and perturbing the operator $R_m T$, similarly as in Proposition 1.1) we may assume that $(R_m T z_k)_{k\in L}$ are succesive in $(\sum_{s>m} \oplus \overline{X}_s)_{l_q}$ (with respect to the decomposition $\{\overline{X}_s\}_{s>m}$). Since $r_{5m+1} > ... > r_{5m+5} > q$ it is now clear that we can find real scalars $\{a_k\}_{k\in L}$ such that $z = \sum_{k\in L} a_k z_k$ is convergent in X_m while $R_m T z = \sum_{k\in L} a_k R_m T z_k$ is divergent in $\overline{X}$, showing that the above assumption is false. Hence, for all $\delta > 0$ and every infinite set $L \subset I$ we can find $k \in L$ with the property

$$\|R_m T z_k\| \leq \delta \|z_k\|, \quad \text{for all } z_k \in Z_k.$$

Thus there exists a subsequence $\tilde{I}$ of I such that, after some perturbations, we get an operator (denoted again by) $T : \overline{\text{span}}[Z_k]_{k\in\tilde{I}} \longrightarrow \overline{X}$ satisfying

$$\begin{cases} R_m T_{\mid \overline{\text{span}}[Z_k]_{k\in\tilde{I}}} = 0, \\ \frac{1}{4a}\|x\| \leq \|Tx\| \leq \|x\|, \quad \text{for all } x \in \overline{\text{span}}[Z_k]_{k\in\tilde{I}} \end{cases}$$

together with the following properties (by applying succesively Proposition 1.1):

$$\begin{cases} P_1 T : \overline{\text{span}}[Z_k]_{k\in\tilde{I}} \longrightarrow \overline{X}_1 \text{ is block-diagonal}, \\ \cdots \\ P_m T : \overline{\text{span}}[Z_k]_{k\in\tilde{I}} \longrightarrow \overline{X}_m \text{ is block-diagonal}. \end{cases} \tag{12}$$

Applying Proposition 2.1(ii) to $P_m T : \overline{\text{span}}[Z_k]_{k\in\tilde{I}} \longrightarrow \overline{X}_m$ we find $I_0 \subset \tilde{I}$, $|I_0| = N_m$ with the property that, considering $y = \sum_{k\in I_0} y_k$,

$$\|P_m T y\| \leq 70 N_m^{-\alpha_m} N_m^{\frac{1}{r_{5m+2}}} \leq 70 N_m^{-\frac{\alpha_m}{2}} N_m^{\frac{1}{r_{5m+2}}}.$$

Thus we can write

$$\|(P_1 + ... + P_{m-1})Ty\| \geq \|Ty\| - \|P_m T y\| \geq (\frac{1}{4a} - 70 N_m^{-\frac{\alpha_m}{2}}) N_m^{\frac{1}{r_{5m+2}}}.$$

Assume $1/(8a) - 70 N_m^{-\frac{\alpha_m}{2}} \geq 0$. There exists $s \in \{1, ..., m-1\}$ such that

$$\|P_s T y\| \geq \frac{1}{m-1}\|(P_1 + ... + P_{m-1})Ty\| \geq \frac{1}{m-1}\ \frac{1}{8a} N_m^{\frac{1}{r_{5m+2}}}.$$

Since $1/(m-1) \geq N_m^{-\frac{\alpha_m}{2}}$ and, by our assumption, $1/(8a) \geq 70 N_m^{-\frac{\alpha_m}{2}}$, the last quantity from above is larger than or equal to $70 {N_m}^{1/r_{5m+1}}$. This is a contradiction since

$$\begin{aligned} \|P_s T y\| &= \|\sum_{k\in I_0} P_s T y_k\| \leq \|\sum_{k\in I_0} Q_1 P_s T y_k\| + \ldots + \|\sum_{k\in I_0} Q_5 P_s T y_k\| \\ &\leq 2 {N_m}^{\frac{1}{r_{5s+1}}} + \ldots + 2 {N_m}^{\frac{1}{r_{5s+5}}} \end{aligned}$$

where, at the last inequality, we used (12) and

$$\|Q_l P_s T y_k\| \leq \|P_s T y_k\| \leq \|y_k\| \leq 2, \quad \forall k \in I_0, \quad \forall l = \overline{1,5}.$$

Hence we must have $a \geq 1/560\ N_m^{\alpha_m/2} \geq m$, for all $m \geq 1$, proving that X is not isomorphic to its complex conjugate.

We will indicate how we can obtain continuum non-isomorphic complex structures on X. For a set $A \subset \{1, 2, \ldots\}$ denote by $X^{(A)}$ the Banach space defined by $X^{(A)} = (\sum_{m\geq 1} \oplus \widetilde{X}_m)_{l_q}$ where $\widetilde{X}_m = \overline{X}_m$, if $m \in A$, or $\widetilde{X}_m = X_m$, if $m \notin A$.

It is well known that there exists a family of infinite sets of positive integers $\{A_t\}_{t\in\mathbf{R}}$ such that $|A_t \cap A_s| < \infty$, for $t \neq s$. Indeed, identifying $\mathbf{N}$ with the set of all rational numbers, we let A_t be an arbitrarily fixed infinite sequence of rational numbers converging to t, for every $t \in \mathbf{R}$.

It should be noted that any two Banach spaces from the family $\{X^{(A_t)}\}_{t\in\mathbf{R}}$ are not isomorphic. Indeed, let $A, B \in \{A_t\}_{t\in\mathbf{R}}$ and let T be an isomorphism between $X^{(A)}$ and $X^{(B)}$. Denoting $A^c \cap B = \{n_1, n_2, ..., n_l, ...\}$ we can repeat the whole argument for $T_{|X_{n_l}}$ and get $\|T^{-1}\| \geq n_l$, for all $l \geq 1$. □

Corollary 3.2. *For $1 \leq p < 2$, the space L_p contains a real subspace having continuum non-isomorphic complex structures.*

Proof. Let $\{r_n\}_{n\geq 1}$ be a strictly decreasing sequence of real numbers such that $p < r_n < 2$, for all n. Since, in this setting, L_p contains an isomorphic copy of $(\sum_{n\geq 1} \oplus\, l_{r_n})_{l_p}$, the conclusion follows from Theorem 3.1. □

4. Another Banach space with at least two non-isomorphic complex structures

The following fact will be used in the proof of Theorem 4.1: if $\{q_n\}_{n\geq 1}$ is a sequence of real numbers such that $1 \leq q_n < \infty$, for all n, and E is an M-dimensional subspace of $(\sum_{n\geq 1} \oplus l_{q_n})_{l_2}$, then $d(E, l_2^M) \leq M^{\sup_n |1/q_n - 1/2|}$.

Indeed, denoting by P_j the natural projection of $(\sum_{n\geq 1} \oplus l_{q_n})_{l_2}$ onto its j-th term, we have $E \subset (\sum_{n\geq 1} \oplus P_n E)_{l_2}$ and, by the result of Lewis [9],

$$d(P_n E, l_2^{\dim P_n E}) \leq (\dim P_n E)^{|\frac{1}{q_n} - \frac{1}{2}|} \leq M^{|\frac{1}{q_n} - \frac{1}{2}|}, \quad \text{for all } n \geq 1.$$

Thus

$$d(E, l_2^M) \leq d((\sum_{n\geq 1} \oplus P_n E)_{l_2}, l_2) \leq M^{\sup_n |\frac{1}{q_n} - \frac{1}{2}|}.$$

We can now prove the main result of this section.

Theorem 4.1. *There exists a sequence $r_n \nearrow 2$ such that the space $\left(\sum_{n\geq 0} \oplus l_{r_n}\right)_{l_2}$ contains a real subspace with continuum non-isomorphic complex structures.*

Proof. The sequence $\{r_n\}_{n\geq 0}$ will be defined inductively. We will also construct inductively a sequence of positive integers $\{N_m\}_{m\geq 1}$. Denoting $\eta_m = (r_{5m-1}, r_{5m-2}, \ldots, r_{5m-5})$ for all $m \geq 1$, we will then define X_{N_m,η_m} as one of the spaces discussed in Section 2. Set $\alpha_m = \min\{1/r_{5m-2} - 1/r_{5m-1}, \ldots, 1/r_{5m-5} - 1/r_{5m-4}\}$ (this definition of α_m corresponds to α from the construction in Section 2).

We start the inductive construction with $\eta_1 = (r_4, r_3, \ldots, r_0)$ such that $2 > r_4 > \ldots > r_0 \geq 1$. Having defined $\eta_1, N_1, \ldots, \eta_{m-1}, N_{m-1}$ and $\eta_m = (r_{5m-1}, r_{5m-2}, \ldots, r_{5m-5})$ we take $N_m \in \{1, 2, \ldots\}$ such that

$$[N_m^{\alpha_m}]^{\frac{1}{r_{5m-2}} - \frac{1}{2}} \geq 100m. \tag{13}$$

Denoting by $M_m = [N_m^{\alpha_m}]$ we can then choose $\eta_{m+1} = (r_{5m+4}, \ldots, r_{5m})$ such that $2 > r_{5m+4} > \ldots > r_{5m} > r_{5m-1} > \ldots > r_{5m-5}$ and

$$M_m^{\frac{1}{r_{5m}} - \frac{1}{2}} \leq 2. \tag{14}$$

Let $X_m = X_{N_m,\eta_m}$ be the space defined in Section 2 and treated as a subspace of $l_{r_{5m-1}} \oplus_2 l_{r_{5m-2}} \oplus_2 \ldots \oplus_2 l_{r_{5m-5}}$. We will show that the space $X = (\sum_{m\geq 1} \oplus X_m)_{l_2}$ is not isomorphic to its complex conjugate.

Suppose that $T : X \longrightarrow \overline{X}$ is an isomorphism with $\|T\| \leq 1/2$. Denote by $a = \|T^{-1}\|$ and by $P_j : \overline{X} \longrightarrow \overline{X}_j$ the projection of $\overline{X}$ onto its j-th term.

Let $m \geq 1$ be arbitrarily fixed. Recall that $X_m = \overline{\text{span}}[Z_k]_{k\geq 1}$. Arguing similarly as in Theorem 3.1 (see $(**)$) we can find an infinite set of positive integers $I (= I_m)$ and an operator (denoted again by) $T : Y_m = \overline{\text{span}}[Z_k]_{k\in I} \longrightarrow \overline{X}$ such that

$$\begin{cases} P_s T = 0, \quad \forall s = \overline{1, m-1}, \\ \frac{1}{2a}\|x\| \leq \|Tx\| \leq \|x\|, \quad \forall x \in Y_m. \end{cases}$$

We may also assume that $P_m T : \overline{\text{span}}[Z_k]_{k\in I} \longrightarrow \overline{X}_m$ is a block-diagonal operator (see Proposition 1.1). By Proposition 2.1(i) we get a set $K \subseteq I, |K| = M_m$ such that

$$\|P_m T y_k\| \leq 24 N_m^{-\alpha_m}, \quad \text{for all } k \in K.$$

Let us denote by $R_m : \overline{X} \longrightarrow (\sum_{s>m} \oplus \overline{X}_s)_{l_2}$ the natural projection. For every choice of signs $\{\epsilon_k\}_{k\in K}$ we can write

$$\begin{aligned} \|R_m T(\sum_{k\in K} \epsilon_k y_k)\| &\geq \frac{1}{2a}\|\sum_{k\in K} \epsilon_k y_k\| - \|P_m T(\sum_{k\in K} \epsilon_k y_k)\| \\ &\geq \frac{1}{2a}\|\sum_{k\in K} y_k\| - 24 M_m N_m^{-\alpha_m}. \end{aligned} \tag{15}$$

We have two cases. Assume first that $1/4a\,\|\sum_{k\in K} y_k\| \geq 24 M_m N_m^{-\alpha_m}$. Since any M_m-dimensional subspace E of $(\sum_{s>m} \oplus \overline{X}_s)_{l_2} \subset (\sum_{s\geq 5m} \oplus \overline{l}_{r_s})_{l_2}$ satisfies

$$d(E, l_2^{M_m}) = d(E, \overline{l}_2^{M_m}) \leq M_m^{\frac{1}{r_{5m}} - \frac{1}{2}} \leq 2,$$

using the paralelogramic identity in l_2, estimate (15) and our assumption we obtain

$$\sum_{k\in K} \|R_m T y_k\|^2 \geq \frac{1}{4}\,\frac{1}{2^{M_m}} \sum_{(\epsilon_k)_{k\in K}} \|\sum_{k\in K} \epsilon_k R_m T y_k\|^2 \geq \frac{1}{64a^2}\|\sum_{k\in K} y_k\|^2.$$

Since $\|R_m T y_k\| \le \|T y_k\| \le \|y_k\| \le 2$ for all $k \in K$, we get

$$4M_m \ge \frac{1}{64a^2} \|\sum_{k\in K} y_k\|^2 \ge \frac{1}{64a^2} M_m^{\frac{2}{r_{5m-2}}}.$$

Thus, by (13),

$$a \ge \frac{1}{16} M_m^{\frac{1}{r_{5m-2}} - \frac{1}{2}} \ge m.$$

In the other case, that is, $1/4a \|\sum_{k\in K} y_k\| < 24 M_m N_m^{-\alpha_m} \le 24 M_m N_m^{-\alpha_m/2}$, we have

$$a > \frac{1}{96} M_m^{-1} N_m^{\frac{\alpha_m}{2}} M_m^{\frac{1}{r_{5m-2}}} \ge \frac{1}{96} M_m^{\frac{1}{r_{5m-2}} - \frac{1}{2}} \ge m.$$

The fact that X, as a real space, has continuum non-isomorphic complex structures follows in the same manner as in Theorem 3.1. □

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